ANNALS OF COMMUNICATIONS IN MATHEMATICS
Volume 6, Number 4 (2023), 266-281
ISSN: 2582-0818

ANTI FUZZY K-IDEALS OF ORDERED SEMIRINGS

M. MURALI KRISHNA RAO

ABSTRACT. In this paper we introduce the notion of anti fuzzy ideals, anti fuzzy $k-$ideals of ordered semirings and we study the properties of anti fuzzy ideals, anti fuzzy $k-$ideals, homomorphic and anti homomorphic image and pre-image of fuzzy ideals, anti fuzzy ideals and anti fuzzy $k-$ideals of an ordered semiring. We characterize the ideals of an ordered semiring in terms of anti fuzzy $k-$ideals.

1. INTRODUCTION

The notion of a semiring was introduced by Vandiver [21] in 1934. Semiring is a well known universal algebra. A universal algebra $(S, +, \cdot)$ is called a semiring if and only if $(S, +), (S, \cdot)$ are semigroups which are connected by distributive laws, $i.e., a(b + c) = ab + ac,\ (a + b)c = ac + bc,$ for all $a, b, c \in S$. Though semiring is a generalization of a ring, ideals of semiring do not coincide with ring ideals. For example an ideal of semiring need not be the kernel of some semiring homomorphism. To solve this problem Herniksen [5] defined $k-$ideals and Iizuka [10] defined $h-$ideals in semirings to obtain analogous of ring results for semiring. Semiring have been used for studying optimization theory, graph theory, matrices, determinants, theory of automata, coding theory, analysis of computer programmes, etc.

The notion of a Γ-ring was introduced by Nobusawa as a generalization of ring in 1964. Sen [19] introduced the notion of Γ- semigroup in 1981. The notion of a ternary algebraic system was introduced by Lehmer in 1932, Lister introduced ternary ring. In 1995, Murali Krishna Rao [16] introduced the notion of a Γ- semiring which is a generalization of Γ-ring, ternary semiring and semiring. The fuzzy set theory was developed by Zadeh [22] in 1965. The fuzzification of an algebraic structures was introduced by Rosenfeld [20] in 1971. Biswas [4] introduced the concept of anti fuzzy subgroups. Kim and Jun [12] introduced the concept of anti fuzzy ideals in near rings. Mandal [15] studied fuzzy ideals and fuzzy interior ideals in an ordered $\Gamma-$semiring. Biswas [4] introduced the concept of an anti fuzzy subgroup of a group and studied the basic properties of a group in terms of anti fuzzy subgroups. Akram and Dar [2] defined anti fuzzy $h-$ideals in hemirings and

2010 *Mathematics Subject Classification.* 16Y60; 03G25.

Key words and phrases. $\Gamma-$semiring; Anti fuzzy ideal; Anti level subset; Anti fuzzy $k-$ideal; Anti homomorphism.

Received: December 07, 2023. Accepted: December 27, 2023. Published: December 31, 2023.

discussed basic properties. Fuzzy sets in ordered semigroups were first studied by Kehayopulu and Tsingelis [11]. Hong and Jun [9] defined anti fuzzy ideal of a BCK-algebra. In this paper, we introduce the notion of anti fuzzy ideals, anti fuzzy $k-$ideals of ordered semirings and we study the properties of anti fuzzy ideals, anti fuzzy $k-$ideals, homomorphic and anti homomorphic image and pre-image of fuzzy ideals, anti fuzzy ideals and anti fuzzy $k-$ideals of an ordered semiring.

2. PRELIMINARIES

In this section we will recall some of the fundamental concepts and definitions, which are necessary for this paper.

Definition 2.1. [1] A set S together with two associative binary operations called addition and multiplication (denoted by $+$ and $\cdot$ respectively) will be called a semiring provided

(i) addition is a commutative operation.
(ii) multiplication distributes over addition both from the left and from the right.
(iii) there exists $0 \in S$ such that $x + 0 = x$ and $x \cdot 0 = 0 \cdot x = 0$, for all $x \in S$.

Definition 2.2. [1] A function $f : R \to M$ where R and M are semirings is said to be semiring homomorphism if $f(a + b) = f(a) + f(b)$ and $f(ab) = f(a)f(b)$, for all $a, b \in R$.

Definition 2.3. [22] Let M be a non-empty set. Then a mapping $f : M \to [0, 1]$ is called a fuzzy subset of M.

Definition 2.4. [22] Let f be a fuzzy subset of a non-empty set M. For $t \in [0, 1]$, the set $f_t = \{x \in M \mid f(x) \geq t\}$ is called a level subset of M with respect to f.

Definition 2.5. [5] Let M be a semiring. A fuzzy subset μ of M is said to be fuzzy subsemiring of M if it satisfies the following conditions
(i) $\mu(x + y) \geq min\ \{\mu(x), \mu(y)\}$
(ii) $\mu(xy) \geq min\ \{\mu(x), \mu(y)\}$, for all $x, y \in M$.

Definition 2.6. [5] A fuzzy subset μ of a semiring M is called a fuzzy left (right) ideal of M if it satisfies the following conditions
(i) $\mu(x + y) \geq min\{\mu(x), \mu(y)\}$
(ii) $\mu(xy) \geq \mu(y)$ $(\mu(x))$, for all $x, y \in M$.

Definition 2.7. [5] A fuzzy subset μ of a semiring M is called a fuzzy ideal of M if it satisfies the following conditions
(i) $\mu(x + y) \geq min\{\mu(x), \mu(y)\}$
(ii) $\mu(xy) \geq max\ \{\mu(x), \mu(y)\}$, for all $x, y \in M$.

Definition 2.8. [5] An ideal I of a semiring M is called a $k-$ideal if for all $x, y \in M,\ x + y \in I, y \in I \Rightarrow x \in I$.

Definition 2.9. [5] Let f and g be fuzzy subsets of semiring M. Then $f \circ g, f + g, f \cup g, f \cap g$, are defined by

$$f \circ g(z) = \begin{cases} \sup\limits_{z=xy} \{min\{f(x), g(y)\}\}, \\ 0, \text{ otherwise.} \end{cases} ; f + g(z) = \begin{cases} \sup\limits_{z=x+y} \{min\{f(x), g(y)\}\}, \\ 0, \text{ otherwise} \end{cases}$$

$$f \cup g(z) = max\{f(z), g(z)\}\ ;\ f \cap g(z) = min\{f(z), g(z)\}$$

$x, y \in M,$ for all $z \in M$.

Definition 2.10. [5] Let A be a non-empty subset of M. The characteristic function of A is a fuzzy subset of M, defined by

$$\chi_{_A}(x) = \begin{cases} 1, & \text{if } x \in A; \\ 0, & \text{if } x \notin A. \end{cases}$$

Definition 2.11. [15] A semiring M is called an ordered semiring if it admits a compatible relation $\leq$. i.e. $\leq$ is a partial ordering on M satisfies the following conditions. If $a \leq b$ and $c \leq d$ then

(i) $a + c \leq b + d$ (ii) $ac \leq bd$ (iii) $ca \leq db$, for all $a, b, c, d \in M$.

Example 2.12. Let $M = [0, 1]$, $+$, $\cdot$ binary operations be defined as $x+y = \max\{x, y\}$, $x \cdot y = \min\{x, y\}$ for all $x, y \in M$. Then M is an ordered semiring with respect to usual ordering.

Definition 2.13. [15] Let M be an ordered semiring and A be a non-empty subset of M. A is called a subsemiring of an ordered semiring M if A is a sub-semigroup of $(M, +)$ and $AA \subseteq A$.

Definition 2.14. [15] Let M be an ordered semiring. A non-empty subset A of M is called a left (right) ideal of an ordered semiring M if A is closed under addition and $MA \subseteq A$ $(AM \subseteq A)$ and for any $a \in M, b \in A, a \leq b$ then $a \in A$. A is called an ideal of M if it is both a left ideal and a right ideal.

Definition 2.15. [15] Let M be an ordered semiring. A fuzzy subset μ of M is called a fuzzy subsemiring of M if

(i) $\mu(x + y) \geq min\{\mu(x), \mu(y)\}$ (ii) $\mu(xy) \geq min\{\mu(x), \mu(y)\}$
(iii) $x \leq y \Rightarrow \mu(x) \geq \mu(y)$, for all $x, y \in M$.

Definition 2.16. [15] Let μ be a non-empty fuzzy subset of an ordered semiring M. Then μ is called a fuzzy left (right) ideal of M if

(i) $\mu(x + y) \geq min\{\mu(x), \mu(y)\}$ (ii) $\mu(xy) \geq \mu(y)\big(\mu(x)\big)$
(iii) $x \leq y \Rightarrow \mu(x) \geq \mu(y)$, for all $x, y \in M$.

Definition 2.17. [15] Let a function $\phi : M \to N$ be a (anti) homomorphism of ordered semirings M, N and μ be a fuzzy subset of M. Then μ is said to be ϕ homomorphism invariant if $\phi(a) = \phi(b)$ then $\mu(a) = \mu(b)$, for $a, b \in M$.

3. Anti fuzzy ideal and anti fuzzy k-ideal of ordered semirings

In this section we introduce the notion of anti fuzzy ideals and anti fuzzy $k-$ideals of ordered semirings and study some of their properties. Through out this section M is an ordered semiring we mean M is an ordered semiring with zero 0.

Definition 3.1. A fuzzy subset μ of an ordered semiring M is called an anti fuzzy left(right) ideal of M if it satisfies $\mu(x+y) \leq max\{\mu(x), \mu(y)\}$, $\mu(xy) \leq \mu(y)\big(\mu(xy) \leq \mu(x)\big)$, for all $x, y \in M$, and $x \leq y \Rightarrow \mu(x) \leq \mu(y)$.

Definition 3.2. A fuzzy subset μ of an ordered semiring M is called an anti fuzzy ideal of M if μ is both an anti fuzzy left ideal and an anti fuzzy right ideal of M.

Definition 3.3. Let M be an ordered semiring. An anti fuzzy ideal μ of M is said to be an anti fuzzy-$k-$ideal of M if $\mu(x) \leq max\{\mu(x + y), \mu(y)\}$, for all $x, y \in M$.

Example 3.4. Let M be the set of all non negative integers. $+, \cdot$ binary operations defined by usual addition and usual multiplication of integers Then M is an ordered semiring with respect to usual ordering $\leq$.

$$\text{Define } \mu(x) = \begin{cases} \frac{1}{4}, & \text{if } x = 0; \\ \frac{1}{3}, & \text{if } x \in \{2, 4, 6, \cdots\}; \\ 1, & \text{otherwise.} \end{cases}$$

for all $x \in M$. Then μ is an anti fuzzy $k-$ideal, of M.

Theorem 3.1. *If μ is a $k-$anti fuzzy ideal of an ordered semiring M then $M_\mu = \{x \mid x \in M, \mu(x) = 0\}$ is either empty or a $k-$ideal of an ordered semiring M.*

Proof. Suppose μ is a $k-$anti fuzzy ideal of an ordered semiring M, $M_\mu = \{x \mid x \in M, \mu(x) = 0\}$, $M_\mu \neq \phi, x, y \in M_\mu$. Then

$$\mu(x) = 0, \mu(y) = 0, \mu(x + y) \leq \max\{\mu(x), \mu(y)\} = 0$$
$$\Rightarrow \mu(x + y) = 0.$$

Therefore $x + y \in M_\mu$.

Now $x \in M_\mu, y \in M$

$$\Rightarrow \mu(x) = 0 \text{ and } \mu(xy) \leq \min\{\mu(x), \mu(y)\} = 0$$
$$\Rightarrow xy \in M_\mu.$$

Let $x, x + y \in M_\mu$

$$\Rightarrow \mu(x) = 0, \mu(x + y) = 0$$
$$\Rightarrow \mu(y) = 0.$$

Therefore $y \in M_\mu$. Hence M_μ is a $k-$ideal of an ordered semiring M.

Let $x \in M, y \in M_\mu$ and $x \leq y$. Then $\mu(x) \leq \mu(y) = 0 \Rightarrow \mu(x) = 0$. Therefore $x \in M_\mu$. Hence the theorem. □

Definition 3.5. Let M be an ordered semiring. If μ is an anti fuzzy-ideal of an ordered semiring M, for any $t \in [0, 1]$, μ_t is defined by $\mu_t = \{x \in M \mid \mu(x) \leq t\}$ then μ_t is called an anti level subset.

Theorem 3.2. *Let μ be an anti fuzzy ideal of an ordered semiring M. Then μ is an anti fuzzy $k-$ ideal of M if and only if non-empty set μ_t is a $k-$ideal of M, for any $t \in (0, 1]$.*

Proof. Suppose μ is an anti fuzzy ideal of an ordered semiring M. Let $x, y \in \mu_t$. Then
$\mu(x) \leq t, \mu(y) \leq t, \mu(x + y) \leq \max\{\mu(x), \mu(y)\} \leq t$
$\Rightarrow x + y \in \mu_t$.
$\mu(xy) \leq \min\{\mu(x), \mu(y)\} \leq t$

Let $y, x + y \in \mu_t$. Then $\mu(y) \leq t,\ \mu(x + y) \leq t$. Since μ is an anti fuzzy $k-$ideal of M. We have $\mu(x) \leq \max\{\mu(x + y), \mu(y)\} \leq t \Rightarrow \mu(x) \leq t \Rightarrow x \in \mu_t$.
Hence μ_t is a $k-$ideal of M. Suppose $a \in M, b \in \mu_t$ and $a \leq b$. Then $\mu(a) \leq \mu(b) \leq t \Rightarrow a \in \mu_t$. Hence μ_t is an ideal of an ordered semiring M.

Conversely suppose that μ_t is a $k-$ideal of an ordered semiring M, for any $t \in (0, 1]$. Let $x, a \in M$ and $\mu(a) = t_1, \mu(x + a) = t_2$, put $t = max\{t_1, t_2\}$. Then $a \in \mu_t$ and $x+a \in \mu_t$. Since μ_t is a $k-$ideal, we have $x \in \mu_t$, therefore $\mu(x) \leq \max\{\mu(x+a), \mu(a)\}$ for all $x, a \in M$. Hence μ is an anti fuzzy $k-$ideal of an ordered semiring M.

Let $x, y \in M$ and $x \leq y$. Suppose $\mu(x) > \mu(y) = t, (say)$. Then $y \in \mu_t$ and $x \notin \mu_t$. This is a contradiction to the fact μ_t is an ideal of ordered semiring M, for all $t \in [0, 1]$. Therefore $\mu(x) \leq \mu(y)$. Hence μ is an anti fuzzy $k-$ideal of an ordered semiring M. □

Theorem 3.3. *Let μ be a fuzzy subset of an ordered semiring M. μ is an anti fuzzy $k-$ideal of M if and only if μ^c is a fuzzy $k-$ideal of M.*

Proof. Let M be an ordered semiring and μ be an anti fuzzy $k-$ideal of M and $x, y \in M$.

$$\mu^c(x+y) = 1-\mu(x+y) \geq 1-\max\{\mu(x),\mu(y)\} = \min\{1-\mu(x), 1-\mu(y)\}$$
$$= \min\{\mu^c(x),\mu^c(y)\}$$
$$\mu^c(xy) = 1-\mu(xy) \geq \max\{\mu^c(x),\mu^c(y)\}.$$

Therefore μ^c is a fuzzy ideal of M.

$$\mu^c(x) = 1-\mu(x) \geq 1-\max\{\mu(x+y),\mu(y)\} = \min\{1-\mu(x+y), 1-\mu(y)\}$$
$$= \min\{\mu^c(x+y),\mu^c(y)\}.$$

Suppose $x \leq y$ and $x, y \in M$.

$$\Rightarrow \mu(x) \leq \mu(y)$$
$$\Rightarrow 1-\mu(x) \geq 1-\mu(y)$$
$$\Rightarrow \mu^c(x) \geq \mu^c(y).$$

Hence μ^c is a fuzzy $k-$ideal of an ordered semiring M.

Conversely suppose that μ^c is a fuzzy $k-$ideal of an ordered semiring M. Let $x, y \in M$.

$$\mu(x+y) = 1-\mu^c(x+y) \leq 1-\min\{\mu^c(x),\mu^c(y)\} = \max\{\mu(x),\mu(y)\}$$
$$\mu(xy) = 1-\mu^c(xy) \leq 1-\max\{\mu^c(x),\mu^c(y)\} = \min\{\mu(x),\mu(y)\}.$$

Therefore μ is an anti fuzzy ideal of an ordered semiring M.

$$\mu(x) = 1-\mu^c(x) \leq 1-\min\{\mu^c(x+y),\mu^c(y)\} = \max\{1-\mu^c(x+y), 1-\mu^c(y)\}$$
$$= \max\{\mu(x+y),\mu(y)\}.$$

Let $x, y \in M$ and $x \leq y$.

$$\Rightarrow \mu^c(x) \geq \mu^c(y)$$
$$\Rightarrow \mu(x) \leq \mu(y).$$

Hence μ is an anti fuzzy $k-$ideal of an ordered semiring M. □

Theorem 3.4. *Let A be a non-empty subset of an ordered semiring M, $t \in [0,1)$ and a fuzzy subset μ of M such that*

$$\mu(x) = \begin{cases} t, & \text{if } x \in A; \\ 1, & \text{if } x \notin A. \end{cases}$$

Then μ is an anti fuzzy ideal of an ordered semiring M if and only if $\mu_t = A$ is an ideal of an ordered semiring M.

Proof. Suppose μ is an anti fuzzy ideal of an ordered semiring M and $x, y \in A$.
Then $\mu(x) = t,\ \mu(y) = t,\ \mu(x+y) \leq \max\{\mu(x),\mu(y)\} = t \Rightarrow \mu(x+y) = t \Rightarrow x+y \in A$. $\mu(xy) \leq \min\{\mu(x),\mu(y)\} = t \Rightarrow xy \in A$.
Let $x \in M, y \in A$ and $x \leq y$. Then $\mu(x) \leq \mu(y) = t$, since $y \in A$.
Therefore $x \in A$. Hence $\mu_t = A$ is an ideal of M.

Conversely, let A be an ideal of M.

case(i) If $x, y \in A, x+y, xy \in A \Rightarrow \mu(xy) = t$.
Then $\mu(x+y) \leq \max\{\mu(x),\mu(y)\}, \mu(xy) \leq \min\{\mu(x),\mu(y)\}$.

case(ii) If $x \in A, y \notin A, x+y \notin A, xy \in A \Rightarrow \mu(x) = t, \mu(y) = 1, \mu(x+y) = 1, \mu(xy) = t$. Then
$\mu(x+y) \leq \max\{\mu(x), \mu(y)\}, \mu(xy) \leq \min\{\mu(x), \mu(y)\}$.

case(iii) If $x \in A, y \notin A, x+y \notin A, \Rightarrow \mu(x) = t, \mu(y) = 1, \mu(x+y) = 1, xy \in A \Rightarrow \mu(xy) = t$. Then
$\mu(x+y) \leq \max\{\mu(x), \mu(y)\}, \mu(xy) \leq \min\{\mu(x), \mu(y)\}$.

case(iv) If $y \in A, x \notin A, xy \in A$ and $x+y \notin A \Rightarrow \mu(x) = 1, \mu(y) = t, \mu(x+y) = t, \mu(xy) = 1$. Then
$\mu(x+y) \leq \max\{\mu(x), \mu(y)\}, \mu(xy) \leq \min\{\mu(x), \mu(y)\}$.

Suppose $x, y \in M$ and $x \leq y$. We can show $\mu(x) \leq \mu(y)$.

Therefore μ is an anti fuzzy ideal of the ordered semiring M. □

Corollary 3.5. *I is an ideal of an ordered semiring M if and only if χ_I is a fuzzy ideal of an ordered semiring M.*

Lemma 3.6. *If μ is an anti fuzzy ideal of an ordered semiring M and $\mu(x+y) = 1, x, y \in M$ then $\mu(x) = 1$ or $\mu(y) = 1$.*

Proof. Suppose μ is an anti fuzzy ideal of an ordered semiring $M, x, y \in M$ and $\mu(x+y) = 1$. $\mu(x+y) \leq \max\{\mu(x), \mu(y)\} \Rightarrow 1 \leq \max\{\mu(x), \mu(y)\} \Rightarrow \mu(x) = 1$ or $\mu(y) = 1$. □

Theorem 3.7. *If f and g be anti fuzzy $k-$ideals of an ordered semiring M then $f \cap g$ is also an anti fuzzy $k-$ideal of an ordered semiring M.*

Proof. Suppose f and g are anti fuzzy $k-$ideals of the ordered semiring M and $x, y \in M$.

$$\begin{aligned} f \cap g(x+y) &= \min\{f(x+y), g(x+y)\} \\ &\leq \min\{\max\{f(x), f(y)\}, \max\{g(x), g(y)\}\} \\ &= \max\{\min\{f(x), g(x)\}, \min\{f(y), g(y)\}\} \\ &= \max\{f \cap g(x), f \cap g(y\} \end{aligned}$$

$$\begin{aligned} f \cap g(xy) &= \min\{f(xy), g(xy)\} \\ &\leq \min\{\min\{f(x), f(y)\}, \min\{g(x), g(y)\}\} \\ &= \min\{\min\{f(x), g(x)\}, \min\{f(y), g(y)\}\} \\ &= \min\{f \cap g(x), f \cap g(y\} \\ f \cap g(x) &= \min\{f(x), g(x)\} \\ &\leq \min\{\max\{f(x+y), f(y)\}, \max\{g(x+y), g(y)\}\} \\ &= \max\{\min\{f(x+y), g(x+y)\}, \min\{f(y), g(y)\}\} \\ &= \min\{f \cap g(x), f \cap g(y)\}. \end{aligned}$$

Suppose $x, y \in M$ and $x \leq y$.

$$\Rightarrow f(x) \leq f(y) \text{ and } g(x) \leq g(y).$$

$$\begin{aligned} f \cap g(x) &= \min\{f(x), g(x)\} \\ &\leq \min\{f(y), g(y)\} \\ &= f \cap g(y). \end{aligned}$$

Hence $f \cap g$ is also an anti fuzzy $k-$ideal of the ordered semiring M. □

Theorem 3.8. *If f and g are anti fuzzy $k-$ideals of an ordered semiring M then $f \cup g$ is also an anti fuzzy $k-$ideal of an ordered semiring M.*

Proof. Let f and g be anti fuzzy $k-$ideals of the ordered semiring M and $x, y \in M$.

$$\begin{aligned} f \cup g(x+y) &= \max\{f(x+y), g(x+y)\} \\ &\leq \max\{\max\{f(x), f(y)\}, \max\{g(x), g(y)\}\} \\ &= \max\{\max\{f(x), g(x)\}, \max\{f(y), g(y)\}\} \\ &= \max\{f \cup g(x), f \cup g(y)\} \end{aligned}$$

$$\begin{aligned} f \cup g(xy) &= \max\{f(xy), g(xy)\} \\ &\leq \max\{\min\{f(x), f(y)\}, \min\{g(x), g(y)\}\} \\ &= \min\{\max\{f(x), g(x)\}, \max\{f(y), g(y)\}\} \\ &= \min\{f \cup g(x), f \cup g(y)\} \end{aligned}$$

$$\begin{aligned} f \cup g(x) &= \max\{f(x), g(x)\} \\ &\geq \max\{\max\{f(x+y), f(y)\}, \max\{g(x+y), g(y)\}\} \\ &= \max\{\max\{f(x+y), g(x+y)\}, \max\{f(y), g(y)\}\} \\ &= \max\{f \cup g(x+y), f \cup g(y)\}. \end{aligned}$$

Suppose $x, y \in M$ and $x \leq y$.

$\Rightarrow f(x) \leq f(y)$ and $g(x) \leq g(y)$.

$$\begin{aligned} f \cup g(x) &= \max\{f(x), g(x)\} \\ &\leq \max\{f(y), g(y)\} \\ &= f \cup g(y). \end{aligned}$$

Hence $f \cup g$ is also an anti fuzzy $k-$ideal of the ordered semiring M. □

Definition 3.6. An anti fuzzy $k-$ideal μ of an ordered semiring M is said to be a normal if $\mu(0) = 0$

Theorem 3.9. *Let μ be an anti fuzzy $k-$ideal of an ordered semiring M. If μ^+ be a fuzzy subset of M is defined by $\mu^+(x) = \mu(x) - \mu(0)$ for all $x \in M$ then μ^+ is a normal anti fuzzy $k-$ideal of M and $\mu^+ \subseteq \mu$.*

Proof. Let μ be an anti fuzzy $k-$ideal of the ordered semiring M and μ^+ be a fuzzy subset of M, defined by $\mu^+(x) = \mu(x) - \mu(0)$ for all $x \in M$. For any $x, y \in M$

$$\begin{aligned} \mu^+(x+y) &= \mu(x+y) - \mu(0) \\ &\leq \max\{\mu(x), \mu(y)\} - \mu(0) \\ &= \max\{\mu(x) - \mu(0), \mu(y) - \mu(0)\} \\ &= \max\{\mu^+(x), \mu^+(y)\} \end{aligned}$$

$$\begin{aligned} \mu^+(xy) &= \mu(xy) - \mu(0) \\ &\leq \min\{\mu(x), \mu(y)\} - \mu(0) \\ &= \min\{\mu(x) - \mu(0), \mu(y) - \mu(0)\} \\ &= \min\{\mu^+(x), \mu^+(y)\} \end{aligned}$$

We have $\mu(x) \leq \max\{\mu(x+y), \mu(y\}$

$\Rightarrow \mu(x) - \mu(0) \leq \max\{\mu(x+y) - \mu(0), \mu(y) - \mu(0)\}$

$\Rightarrow \mu^+(x) \leq \max\{\mu^+(x+y), \mu^+(y)\}.$

$\mu^+(x) = \mu(x) - \mu(0)$

$\Rightarrow \mu^+(0) = 0$

Suppose $x, y \in M$ and $x \leq y$.

$\Rightarrow \mu(x) \leq \mu(y),$ since μ is an anti fuzzy $k-$ideal

$\Rightarrow \mu(x) - \mu(0) \leq \mu(y) - \mu(0)$

$\Rightarrow \mu^+(x) \leq \mu^+(y).$

Hence μ^+ is an anti fuzzy $k-$ideal of M. Clearly μ contains μ^+ and μ^+ is a normal anti fuzzy $k-$ideal of M. Let μ be a fuzzy subset of M and $a \in M$. Then the set $\{b \in M \mid \mu(b) \leq \mu(a)\}$ is denoted by I_a. □

Theorem 3.10. *Let μ be an anti fuzzy $k-$left ideal of an ordered semiring M. If $a \in M$ then I_a is a left $k-$ideal of an ordered semiring M.*

Proof. Let μ be an anti fuzzy $k-$ideal of the ordered semiring M and $a \in M$. We have $\mu(0) \leq \mu(x)$ for all $x \in M$. Therefore $0 \in I_a$. Let $b, c \in I_a$. Then $\mu(b) \leq \mu(a)$ and $\mu(c) \leq \mu(a)$.

$$\begin{aligned}\mu(b+c) &\leq \max\{\mu(b), \mu(c)\} \\ &\leq \max\{\mu(a), \mu(a)\} \\ &= \mu(a).\end{aligned}$$

Then $b + c \in I_a$. Suppose $b \in I_a, c \in M$. Then $\mu(b) \leq \mu(a)$. Now $\mu(cb) \leq \mu(b) \leq \mu(a)$. Therefore $cb \in I_a$.
Suppose $x \in M, y \in I_a$ and $x \leq y$. Then $\mu(y) \leq \mu(a)$ and $\mu(x) \leq \mu(y) \Rightarrow \mu(x) \leq \mu(y) \leq \mu(a)$. Therefore $x \in I_a$.

Suppose $x \in I_a$ and $x + y \in I_a$.
$\Rightarrow \mu(x) \leq \mu(a), \mu(x+y) \leq \mu(a)$
$\Rightarrow \mu(y) \leq \max\{\mu(x+y), \mu(x)\} \leq \mu(a)$
$\Rightarrow y \in I_a$. Hence I_a is a left $k-$ideal of the ordered semiring M. □

Corollary 3.11. *Let μ be an anti fuzzy right $k-$ideal of an ordered semiring M and $a \in M$. Then I_a is a right $k-$ideal of an ordered semiring M.*

Corollary 3.12. *Let μ be an anti fuzzy $k-$ideal of an ordered semiring M and $a \in M$. Then I_a is a $k-$ideal of an ordered semiring M.*

Definition 3.7. A family of fuzzy subsets $\{\mu_i \mid i \in I\}$ of an ordered semiring M, then $\bigvee_{i \in I} \mu_i$ is defined by $\bigvee_{i \in I} \mu_i(x) = \sup\{\mu_i(x) \mid i \in I\},$ for all $x \in M$.

Theorem 3.13. *If $\{\mu_i \mid i \in I\}$ is a family of anti fuzzy ideals of an ordered semiring M then $\bigvee_{i \in I} \mu_i$ is an anti fuzzy ideal of an ordered semiring M.*

Proof. Let $\{\mu_i \mid i \in I\}$ be a family of anti fuzzy ideals of the ordered semiring M and $x, y \in M$. Then we have,

$$\left(\bigvee_{i\in I}\mu_i\right)(x+y) = \sup\{\mu_i(x+y) \mid i \in I\}$$
$$\leq \sup\{\max\{\mu_i(x),\ \mu_i(y)\} \mid i \in I\}$$
$$= \max\{\sup_{i\in I}\mu_i(x), \sup_{i\in I}\mu_i(y)\}$$
$$= \max\left\{\bigvee_{i\in I}\mu_i(x), \bigvee_{i\in I}\mu_i(y)\right\}$$
$$\left(\bigvee_{i\in I}\mu_i\right)(xy) = \sup\{\mu_i(xy) \mid i \in I\}$$
$$\leq \sup\{\min\{\mu_i(x)\mu_i(y)\} \mid i \in I\}$$
$$= \min\{\sup_{i\in I}\mu_i(x), \sup_{i\in I}\mu_i(y)\}$$
$$= \min\left\{\bigvee_{i\in I}\mu_i(x), \bigvee_{i\in I}\mu_i(y)\right\}.$$

Suppose $x, y \in M$ and $x \leq y$.Then

$$\mu_i(x) \leq \mu_i(y), \text{ for all } i \in I$$
$$\Rightarrow \bigvee_{i\in I}\mu_i(x) \leq \bigvee_{i\in I}\mu_i(y).$$

Hence $\bigvee_{i\in I}\mu_i$ is an anti fuzzy ideal of M. □

Definition 3.8. Let μ be a fuzzy subset of X and $\alpha \in [0, 1 - \sup\{\mu(x) \mid x \in X\}]$ The mapping $\mu^T_\alpha : X \to [0, 1]$ is called a fuzzy translation of μ if
$\mu^T_\alpha(x) = \mu(x) + \alpha$

Definition 3.9. Let μ be a fuzzy subset of X and $\beta \in [0, 1]$. Then mapping
$\mu^M_\beta : X \to [0, 1]$ is called a fuzzy multiplication of μ if $\mu^M_\beta(x) = \beta\mu(x)$.

Definition 3.10. Let μ be a fuzzy subset of X and $\alpha \in [0, 1 - \sup\{\mu(x) \mid x \in X\}],\ \beta \in [0, 1]$. Then mapping $\mu^{MT}_{\beta,\alpha} : X \to [0, 1]$ is called a magnified translation of μ if $\mu^{MT}_{\beta,\alpha}(x) = \beta\mu(x) + \alpha$, for all $x \in X$.

Theorem 3.14. *A fuzzy subset μ is an anti fuzzy $k-$ideal of an ordered semiring M if and only if μ^T_α is an anti fuzzy $k-$ideal of an ordered semiring M*

Proof. Suppose μ is an anti fuzzy $k-$ideal of the ordered semiring M and $x, y \in M$.

$$\mu^T_\alpha(x+y) = \mu(x+y) + \alpha$$
$$\leq \max\{\mu(x), \mu(y)\} + \alpha$$
$$= \max\{\mu(x) + \alpha, \mu(y) + \alpha\}$$
$$= \max\{\mu^T_\alpha(x), \mu^T_\alpha(y)\}$$

$$\begin{aligned}\mu_\alpha^T(xy) &= \mu(xy)+\alpha\\ &\le \min\{\mu(x),\mu(y)\}+\alpha\\ &= \min\{\mu(x)+\alpha,\mu(y)+\alpha\}\\ &= \min\{\mu_\alpha^T(x),\mu_\alpha^T(y)\}\end{aligned}$$

$$\begin{aligned}\mu_\alpha^T(x) &= \mu(x)+\alpha\\ &\le \max\{\mu(x+y),\mu(y)\}+\alpha\\ &= \max\{\mu(x+y)+\alpha,\mu(y)+\alpha\}\\ &= \max\{\mu_\alpha^T(x+y),\mu_\alpha^T(y)\}.\end{aligned}$$

Let $x \le y$.Then $\mu(x) \le \mu(y)$
$\Rightarrow \mu(x)+\alpha \le \mu(y)+\alpha$
$\Rightarrow \mu_\alpha^T(x) \le \mu_\alpha^T(y)$.

Hence μ_α^T is an anti fuzzy $k-$ideal of the ordered semiring M.

Conversely suppose that μ_α^T is an anti fuzzy $k-$ideal of the ordered semiring $M,\ x, y \in M$.

$$\begin{aligned}\mu(x+y)+\alpha &= \mu_\alpha^T(x+y)\\ &\le \max\{\mu_\alpha^T(x),\mu_\alpha^T(y)\}\\ &= \max\{\mu(x)+\alpha,\mu(y)+\alpha\}\\ &= \max\{\mu(x),\mu(y)\}+\alpha\end{aligned}$$

Therefore $\mu(x+y) \le \max\{\mu(x),\mu(y)\}$.

$$\begin{aligned}\mu(xy)+\alpha &= \mu_\alpha^T(xy)\\ &\le \min\{\mu_\alpha^T(x),\mu_\alpha^T(y)\}\\ &= \min\{\mu(x)+\alpha,\mu(y)+\alpha\}\\ &= \min\{\mu(x),\mu(y)\}+\alpha\end{aligned}$$

Therefore $\mu(xy) \le \min\{\mu(x),\mu(y)\}$.

$$\begin{aligned}\mu(x)+\alpha &= \mu_\alpha^T(x)\\ &\le \max\{\mu_\alpha^T(x+y),\mu_\alpha^T(y)\}\\ &= \max\{\mu(x+y)+\alpha,\mu(y)+\alpha\}\\ &= \max\{\mu(x+y),\mu(y)\}+\alpha\end{aligned}$$

Therefore $\mu(x) \ge \min\{\mu(x+y),\mu(y)\}$.

Let $x \le y$.Then $\mu_\alpha^T(x) \le \mu_\alpha^T(y)$.
$\Rightarrow \mu(x)+\alpha \le \mu(y)+\alpha$
$\Rightarrow \mu(x) \le \mu(y)$.

Hence μ is an anti fuzzy $k-$ideal of M. □

Theorem 3.15. *A fuzzy subset μ is an anti fuzzy $k-$ideal of an ordered semiring M if and only if μ_β^M is an anti fuzzy $k-$ideal of an ordered semiring M.*

Proof. Suppose μ is an anti fuzzy $k-$ideal of the ordered semiring M and $x, y \in M$. Then

$$\begin{aligned}\mu_\beta^M(x+y) &= \beta\mu(x+y)\\ &\leq \beta\max\{\mu(x),\mu(y)\}\\ &= \max\{\beta\mu(x),\beta\mu(y)\}\\ &= \max\{\mu_\beta^M(x),\mu_\beta^M(y)\}.\end{aligned}$$

$$\begin{aligned}\mu_\beta^M(xy) &= \beta\mu(xy)\\ &\leq \beta\min\{\mu(x),\mu(y)\}\\ &= \min\{\beta\mu(x),\beta\mu(y)\}\\ &= \min\{\mu_\beta^M(x),\mu_\beta^M(y)\}.\end{aligned}$$

$$\begin{aligned}\mu_\beta^M(x) &= \beta\mu(x)\\ &\leq \beta\max\{\mu(x+y),\mu(y)\}\\ &= \max\{\beta\mu(x+y),\beta\mu(y)\}\\ &= \max\{\mu_\beta^M(x+y),\mu_\beta^M(y)\}.\end{aligned}$$

$$\begin{aligned}&\text{Let } x \leq y. \text{Then } \mu(x) \leq \mu(y)\\ &\Rightarrow \beta\mu(x) \leq \beta\mu(y)\\ &\Rightarrow \mu_\beta^M(x) \leq \beta_\beta^M\mu(y).\end{aligned}$$

Hence μ_β^M is an anti fuzzy ideal of the ordered semiring M.

Conversely, suppose that μ_β^M is an anti fuzzy ideal of the ordered semiring M and $x, y \in M$. Then

$$\begin{aligned}\mu_\beta^M(x+y) &\leq \max\{\mu_\beta^M(x),\mu_\beta^M(y)\}\\ \Rightarrow\ \beta\mu(x+y) &\leq \max\{\beta\mu(x),\beta\mu(y)\}\\ &= \beta\max\{\mu(x),\mu(y)\}\\ \text{Therefore } \mu(x+y) &\leq \max\{\mu(x),\mu(y)\}.\end{aligned}$$

$$\begin{aligned}\mu_\beta^M(xy) &\leq \min\{\mu_\beta^M(x),\mu_\beta^M(y)\}\\ &= \beta\min\{\mu(x),\mu(y)\}\\ \beta\mu(xy) &= \beta\min\{\mu(x),\mu(y)\}\\ \text{Therefore } \mu(xy) &= \min\{\mu(x),\mu(y)\}.\end{aligned}$$

$$\begin{aligned}\mu_\beta^M(x) &\leq \max\{\mu_\beta^M(x+y),\mu_\beta^M(y)\}\\ &= \max\{\beta\mu(x+y),\beta\mu(y)\}\\ &= \beta\max\{\mu(x+y),\mu(y)\}\\ \beta\mu(x) &= \beta\max\{\mu(x+y),\mu(y)\}\\ \text{Therefore } \mu(x) &\leq \max\{\mu(x+y),\mu(y)\}.\end{aligned}$$

$$\begin{aligned}&\text{Let } x \leq y. \text{Then } \mu_\beta^M(x) \leq \beta_\beta^M\mu(y)\\ &\Rightarrow \beta\mu(x) \leq \beta\mu(y)\\ &\Rightarrow \mu(x) \leq \mu(y).\end{aligned}$$

Hence μ is an anti fuzzy $k-$ideal of M. □

Theorem 3.16. *A fuzzy subset μ is an anti fuzzy $k-$ideal of an ordered semiring M if and only if $\mu^{MT}_{\beta,\alpha} : X \to [0,1]$ is an anti fuzzy $k-$ideal of an ordered semiring M.*

Proof. Suppose μ is an anti fuzzy $k-$ideal of the ordered semiring M.
$\Leftrightarrow$ μ^{M}_{β} is an anti fuzzy $k-$ideal of M, by Theorem 3.15
$\Leftrightarrow$ $\mu^{MT}_{\beta,\alpha}$ is an anti fuzzy $k-$ideal of M, by Theorem 3.14
□

4. HOMOMORPHIC, ANTI HOMOMORPHIC IMAGE AND PRE-IMAGE OF FUZZY IDEALS AND ANTI FUZZY IDEALS OF AN ORDERED SEMIRING

In this section the concept of an anti homomorphism of ordered semirings has been introduced. The properties of homomorphic, anti homomorphic image and pre-image of fuzzy ideals and anti fuzzy ideals of an ordered semiring are studied.

Definition 4.1. A function $f : M \to N$ where M and N are ordered semirings is called a homomorphism of ordered semirings if
$f(a+b) = f(a) + f(b), f(ab) = f(a)f(b)$ and $a \le b \Rightarrow f(a) \le f(b),$ for all $a, b \in M$.

Definition 4.2. A function $f : M \to N$ where M and N are ordered semirings is called an anti homomorphism of ordered semirings if
$f(a+b) = f(a) + f(b), f(ab) = f(b)f(a)$ and $a \le b \Rightarrow f(a) \le f(b),$ for all $a, b \in M$.

Theorem 4.1. *Let M and N be semirings and $\phi : M \to N$ be an onto anti homomorphism. If f is a homomorphism ϕ invariant anti fuzzy ideal of M then $\phi(f)$ is an anti fuzzy ideal of N.*

Proof. Let M and N be semirings and $\phi : M \to N$ be an anti homomorphism and f be a homomorphism ϕ invariant anti fuzzy ideal of M. If $x = \phi(a) \Rightarrow \phi^{-1}(x) = a$. Let $t \in \phi^{-1}(x)$ then $\phi(t) = x = \phi(a)$, since f is ϕ invariant, $f(t) = f(a) \Rightarrow \phi(f)(x) = \inf\limits_{t\in\phi^{-1}(x)} f(t) = f(a)$. Hence $\phi(f)(x) = f(a)$.

Let $x, y \in N$. Then there exist $a, b \in M$ such that

$$\begin{aligned}\phi(a) = x, \phi(b) = y \Rightarrow \phi(a+b) &= x+y\\ \Rightarrow \phi(f)(x+y) = f(a+b) &\ge \max\{f(a), f(b)\}\\ &= \max\{\phi(f)(x), \phi(f)(y)\}\\ \phi(f)(xy) = f(ba) &\ge \min\{f(a), f(b)\}\\ &= \min\{\phi(f)(x), \phi(f)(y)\}.\end{aligned}$$

$$\begin{aligned}\text{Let } x \le y. \text{ Then } \phi(a) &\le \phi(b)\\ &\Rightarrow f(a) \ge f(b)\\ &\Rightarrow \phi(f)(x) \ge \phi(f)(y).\end{aligned}$$

Hence $\phi(f)$ is an anti fuzzy ideal of N. □

Theorem 4.2. *Let M and N be ordered semirings and $\phi : M \to N$ be an onto homomorphism. If f is a homomorphism ϕ invariant anti fuzzy $k-$ideal of M then $\phi(f)$ is an anti fuzzy $k-$ideal of N.*

Proof. Let M and N be ordered semirings, $\phi : M \to N$ be an onto homomorphism, f be a homomorphism ϕ invariant anti fuzzy ideal of M and $a \in M$. Suppose $x \in N, t \in \phi^{-1}(x)$

and $x = \phi(a)$. Then $a \in \phi^{-1}(x) \Rightarrow \phi(t) = x = \phi(a)$, since f is a ϕ invariant, $f(t) = f(a) \Rightarrow \phi(f)(x) = \inf_{t\in\phi^{-1}(x)} f(t) = f(a)$. Hence $\phi(f)(x) = f(a)$. Let $x, y \in N$. Then there exist $a, b \in M$ such that $\phi(a) = x, \phi(b) = y \Rightarrow \phi(a + b) = x + y \Rightarrow \phi(f)(x + y) = f(a + b) \leq \max\{f(a), f(b)\} = \min\{\phi(f)(x), \phi(f)(y)\}$.

$$\begin{aligned}\phi(ab) =&\phi(a)\phi(b)\\ =&xy\\ \phi f(xy) =&f(ab)\\ \leq&\min\{f(a), f(b)\}\\ =&\min\{\phi(f(x)), \phi(f(y)).\}\end{aligned}$$

We have $f(a) \leq \max\{f(a+b), f(b)\} \Rightarrow \phi(f)(x) = \max\{\phi(f)(x+y), \phi(f)(y)\}$, for all $x, y \in M$. Hence $\phi(f)$ is an anti fuzzy $k-$ ideal of N. □

Theorem 4.3. *Let $f : M \to N$ be a homomorphism of ordered semirings and η be an anti fuzzy ideal of N. If $\eta \circ f = \mu$ then μ is an anti fuzzy ideal of M.*

Proof. Let $f : M \to N$ be a homomorphism of ordered semirings, η be an anti fuzzy ideal of $N, \eta \circ f = \mu$ and $x, y \in M$.

$$\begin{aligned}\mu(x + y) &= \eta(f(x + y)) = \eta(f(x) + f(y))\\ &\leq \max\{\eta(f(x)), \eta(f(y))\}\\ &= \max\{\mu(x), \mu(y)\}\\ \mu(xy) &= \eta(f(xy)) = \eta(f(x)f(y))\\ &\leq \min\{\eta(f(x)), \eta(f(y))\}\\ &= \min\{\mu(x), \mu(y)\}.\end{aligned}$$

Suppose $x, y \in M$ and $x \leq y$. Since $f : M \to N$ be a homomorphism, we have

$$\begin{aligned}&f(x) \leq f(y)\\ \Rightarrow&\eta(f(x)) \leq \eta(f(y))\\ \Rightarrow&\mu(x) \leq \mu(y)\end{aligned}$$

Hence μ is an anti fuzzy ideal of M. □

Definition 4.3. Let M and N be two ordered semirings and f be a function from M into N. If μ is a fuzzy ideal of N then the pre-image of μ under f is the fuzzy subset of M, defined by $f^{-1}(\mu)(x) = \mu(f(x))$ for all $x \in M$.

Theorem 4.4. *Let $f : M \to N$ be an onto homomorphism of ordered semirings. If μ is an anti fuzzy $k-$ideal of N then $f^{-1}(\mu)$ is an anti fuzzy $k-$ideal of M.*

Proof. Suppose $f : M \to N$ is an onto homomorphism of ordered semirings and μ is an anti fuzzy $k-$ideal of N and $x_1, x_2 \in M$.

$$\begin{aligned}f^{-1}(\mu)(x_1 + x_2) &= \mu(f(x_1 + x_2) = \mu\big(f(x_1) + f(x_2)\big)\\ &\leq \max\{\mu(f(x_1)), \mu(f(x_2))\} = \max\{\mu(f^{-1}(x_1)), \mu(f^{-1}(x_2))\}\\ f^{-1}(\mu)(x_1x_2) &= \mu(f(x_1x_2)) \leq \min\{\mu(f(x_1)), \mu(f(x_2))\}\\ &= \min\{\mu(f^{-1}(x_1)), \mu(f^{-1}(x_2))\}\end{aligned}$$

$$f^{-1}(\mu)(x) = \mu(f(x)) \leq \max\{\mu(f(x+y)), \mu(f(y))\}$$
$$= \max\{(f^{-1}\mu(x+y)), (f^{-1}\mu(x+y))\}, \text{ for } x, y \in M.$$

Let $x_1, x_2 \in M$ and $x_1 \leq x_2$.

$$\Rightarrow f(x_1) \leq f(x_2)$$
$$\Rightarrow \mu(f(x_1)) \leq \mu(f(x_2))$$
$$\Rightarrow f^{-1}(\mu)(x_1) \leq f^{-1}(\mu)(x_2).$$

Let $x, y \in M,$ and $x \leq y$.

$$\Rightarrow f(x) \leq f(y)$$
$$\Rightarrow \mu(f(x)) \leq \mu(f(y))$$
$$\Rightarrow f^{-1}(\mu)(x) \leq f^{-1}(\mu)(y)$$

Hence $f^{-1}(\mu)$ is an anti fuzzy $k-$ideal of M. □

Theorem 4.5. *Let $f : M \to N$ be an onto anti homomorphism of an ordered semirings if η is a fuzzy left ideal of N and μ is the pre-image of η under f. Then μ is a fuzzy right ideal of M.*

Proof. Let $f : M \to N$ be an onto anti homomorphism of ordered semirings. If η is a fuzzy left ideal of$N$, μ is the pre-image of η under f, $x, y \in M$.

$$\mu(x+y) = \eta\big(f(x+y)\big) = \eta\big(f(x)+f(y)\big) \geq \min\{\eta(f(x)), \eta(f(y))\}$$
$$= \min\{\mu(x), \mu(y)\}$$
$$\mu(xy) = \eta\big(f(xy)\big) = \eta\big(f(y)f(x)\big) \geq \eta(f(x)) = \mu(x).$$

Let $x, y \in M$ and $x \leq y$.

$$\Rightarrow f(x) \leq f(y)$$
$$\Rightarrow \eta(f(x)) \leq \eta(f(y))$$
$$\Rightarrow \mu(x) \leq \mu(y).$$

Hence μ is a fuzzy right ideal of M. □

Corollary 4.6. *Let $f : M \to N$ be an onto anti homomorphism of an ordered semirings. if η is a fuzzy right ideal of N and μ is the pre-image of η under f. Then μ is a fuzzy left ideal of M.*

Corollary 4.7. *Let $f : M \to N$ be an onto anti homomorphism of an ordered semirings. if η is a fuzzy ideal of N and μ is the pre-image of η under f. Then μ is a fuzzy ideal of M.*

Theorem 4.8. *Let $f : M \to N$ be an onto anti homomorphism of ordered semirings. If μ is an anti fuzzy left $k-$ideal of N then $f^{-1}(\mu)$ is an anti fuzzy right $k-$ideal of M.*

Proof. Let $f : M \to N$ be an onto anti homomorphism of ordered semirings, μ be an anti fuzzy left $k-$ideal of N and $x_1, x_2, x, y \in M$.

$$f^{-1}(\mu)(x_1+x_2) = \mu(f(x_1+x_2)) = \mu\big(f(x_1)+f(x_2)\big)$$
$$\leq \max\{\mu(f(x_1)), \mu(f(x_2))\}$$
$$= \max\{f^{-1}(\mu)(x_1), f^{-1}(\mu)(x_2)\}$$

$$f^{-1}(\mu)(x_1x_2) = \mu(f(x_1x_2)) = \mu(f(x_2)f(x_1))$$
$$\leq \mu(f(x_1))$$
$$= f^{-1}(\mu)(x_1)$$
$$f^{-1}(\mu)(x) = \mu(f(x)) \leq \max\{\mu(f(x+y)), \mu(f(y))\}$$
$$= \max\{f^{-1}(\mu)(x+y), f^{-1}(\mu)(y)\}.$$

Let $x \leq y$. Then $f(x) \leq f(y)$

$\Rightarrow \mu(f(x)) \leq \mu(f(y))$

$\Rightarrow f^{-1}(\mu(x)) \leq f^{-1}(\mu(y))$.

Hence $f^{-1}(\mu)$ is an anti fuzzy $k-$right ideal of M. □

Theorem 4.9. *Let $f : M \to N$ be an onto anti homomorphism of ordered semirings. Then anti homomorphic image of an anti fuzzy left ideal of M is an anti fuzzy right ideal of N.*

Proof. Let $f : M \to N$ be an onto anti homomorphism of ordered semirings, μ be an anti fuzzy left ideal of M and η be a fuzzy subset of N such that $\eta \circ f = \mu$. Let $a, b \in N$, then there exist $x, y \in M$ such that $f(x) = a$ and $f(y) = b$.

$\eta(a + b) = \eta(f(x) + f(y)) = \eta(f(x + y))$

$= \mu(x + y) \leq \max\{\mu(x), \mu(y)\}$

$= \max\{\eta(f(x)), \eta(f(y))\}$.

$= \max\{\eta(a), \eta(b)\}$.

$\eta(ab) = \eta(f(x)f(y)) = \eta(f(yx))$

$= \mu(yx) \leq \mu(x)$

$= \eta(f(x)) = \eta(a)$.

Let $x, y \in M$ and $x \leq y$. Then $\mu(x) \leq \mu(y)$

$\Rightarrow \eta(f(x)) \leq \eta(f(y))$.

Hence η is an anti fuzzy right ideal of the ordered semiring N. □

Theorem 4.10. *Let $f : M \to N$ be an onto anti homomorphism of an ordered semiring. If η is an anti fuzzy left $k-$ideal of N and μ is the pre-image of η under f then μ is an anti fuzzy right $k-$ideal of M.*

Proof. Let $f : M \to N$ be an onto anti homomorphism of ordered semirings, η be an anti fuzzy left $k-$ideal of N, μ be the pre-image of η under f, $x, y \in M$.

$$\mu(x + y) = \eta(f(x + y)) = \eta(f(x) + f(y)) \leq \max\{\eta(f(x)), \eta(f(y))\}$$
$$= \max\{\mu(x), \mu(y)\}$$
$$\mu(xy) = \eta(f(xy)) = \eta(f(y)f(x)) \leq \eta(f(x)) = \mu(x)$$
$$\mu(x) = \eta(f(x)) \leq \max\{\eta(f(x + y)), \eta(f(x))\} = \max\{\mu(x + y), \mu(y)\}.$$

Let $x, y \in M$ and $x \leq y$. Then $f(x) \leq f(y)$, since f is an anti homomorphism.
$\Rightarrow \eta(f(x)) \leq \eta(f(y))$, since η is an anti fuzzy left $k-$ideal
$\Rightarrow \mu(x) \leq \mu(y)$. Hence μ is an anti fuzzy right $k-$ideal of the ordered semiring M. □

5. CONCLUSION:

In this paper, we introduced the notion of anti fuzzy ideals, anti fuzzy $k-$ideals of ordered semirings and we studied the properties of anti fuzzy ideals, anti fuzzy $k-$ideals, homomorphic, anti homomorphic image and pre-image of fuzzy ideals, anti fuzzy ideals and anti fuzzy $k-$ideals of an ordered semiring. We studied if μ is a $k-$anti fuzzy ideal of an ordered semiring M, then $M_\mu = \{x \mid x \in M, \mu(x) = 0\}$ is either empty or a $k-$ideal of an ordered semiring M. If $\{\mu_i \mid i \in I\}$ is a family of anti fuzzy ideals of an ordered semiring M then $\bigvee_{i\in I} \mu_i$ is an anti fuzzy ideal of an ordered semiring M. If $f : M \to N$ be an onto homomorphism of ordered semirings, and μ is an anti fuzzy $k-$ideal of N then $f^{-1}(\mu)$ is an anti fuzzy $k-$ideal of M. In the continuation of this paper, we characterize ordered semiring, regular ordered semiring and simple ordered semiring in terms of their anti fuzzy bi-ideals and anti fuzzy quasi ideals.

6. Acknowledgements

The author is greatly thankful to the reviewers for their valuable suggestions, which enabled us to improve the presentation of the work.

References

[1] P. J. Allen. A fundamental theorem of homomorphism for semirings, Proc. Amer. Math. Sco., 21 (1969), 412–416.
[2] M. Akram and K. H. Dar. On Anti fuzzy left $h-$ideals in Hemirings, Int. Math. Forum, 46(2) (2007), 2295–2304.
[3] M. Akram and W. A. Dudek. Intuitionistic fuzzy left k-ideals of semirings, Soft Comput., 12 (2008), 881–890.
[4] R. Biswas. Fuzzy subgroups and anti fuzzy subgroups, Fuzzy Sets Systems, 44 (1990), 121 – 124.
[5] S. I. Baik and H.S. Kim. On fuzzy k-ideals in semirings, Kangweon Kyungki Math j., 8 (2000), 147–154.
[6] A. Borumand Saeid, M. Murali Krishna Rao, K. Rajendra Kumar , N Rafi. Fuzzy (Soft) Quasi-Interior Ideals of Semirings, Transactions on Fuzzy Sets and Systems , 1 (2), (2022), 129–141.
[7] A. Borumand Saeid, M. Murali Krishna Rao, K. Rajendra Kumar. Fuzzy Ideals In $\Gamma-$BCK-algebras, J. Korean Soc. Math. Educ. Ser. B: Pure Appl. Math. , 30 (4) (2023), 429–442.
[8] M. Henriksen. Ideals in semirings with commutative addition, Amer. Math. Soc. Notices, 5 (1958), 321.
[9] S. M. Hong and Y. B. Jun. Anti fuzzy ideals in BCK-algebras, Kyungpook Math. J., 38(1998), 145–150.
[10] K. Izuka. On the Jacobson radical of a semiring, Tohoku, Math. J., 11(2)(1959), 409–421.
[11] N.Kehayopulu and M. Tsingelis. Regular ordered semigroups in terms of fuzzy subsets, Inform. Sci., 176 (2006), 3675–3693.
[12] K. H . Kim and Y. B. Jun. On Anti Fuzzy Ideals in Near-Rings, Iranian J. of Fuzzy Systems, 2(2) (2005), 71 – 80.
[13] N. Kuroki. On fuzzy semigroups, Information Sciences, 53(3) (1991), 203–236.
[14] Kul hur, S. Kim and K. Pyung. Intuitionistic fuzzy k-ideals of a semiring, Int. J. of Fuzzy Logic and Intelligent Systems, 9(2), (2009) 110–114.
[15] D. Mandal. Fuzzy ideals and fuzzy interior ideals in ordered semirings, Fuzzy info. and Engg., 6 (2014), 101–114.
[16] M. Murali Krishna Rao. $\Gamma-$semirings-I, Southeast Asian Bull. Math., 19 (1) (1995), 49–54.
[17] M. Murali Krishna Rao. A study of r-ideals, k-ideals and (m-k)ideals of an ordered semirings, Asia Pac. J. Math., 7(2) 2020.
[18] M. Murali Krishna Rao, K. Rajendra Kumar and Arsham Borumand Saeid. Hesitant Fuzzy Ideals of (Ordered) Γ-Semirings, New Mathematics and Natural Computation (2024), 1–19.
[19] M. K. Sen. On $\Gamma-$semigroup, Proc. of International Conference of algebra and its application, (1981), Decker Publicaiton, New York, 301–308.
[20] A. Rosenfeld. Fuzzy groups, J. Math. Anal. Appl., 35(1971), 512–517.
[21] H. S. Vandiver. Note on a simple type of algebra in which cancellation law of addition does not hold, Bull. Amer. Math. Soc.(N.S.), 40 (1934), 914–920.
[22] L. A. Zadeh. Fuzzy sets, Information and control, 8 (1965), 338–353.

M. Murali Krishna Rao
Department of Mathematics, Sankethika Institute of Tech. and Management, Visakhapatnam - 530 041, A.P., India.
Email address: mmarapureddy@gmail.com