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SCHAUDER-TYCHONOFF FIXED POINT THEOREM ON SEQUENTIALLY COMPLETE HAUSDORFF STRONGLY CONVEX TOPOLOGICAL VECTOR SPACES

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ABSTRACT. In this paper, we study the schauder-tychonoff fixed point(STFP) on a subset A of a sequentially complete Hausdorff strongly convex topological vector space (SCHSCTVS) E (over the field R) with calibration Γ have a unique STFP in Topological Vector Space(TVS).

1. INTRODUCTION

The concept of topological vector spaces was introduced by Kolmogroff [2] in 1934. The topology on a vector space introduces the idea of open sets, where operations like addition and scalar multiplication are continuous. This leads to a rich interplay between algebraic operations and topological concepts. Its properties were further studied by different mathematicians[[1],[9],[10],[12]] and the development of topological vector spaces has deep historical roots, with contributions from mathematicians like Banach, Frechet, and Hausdorff. The theory has evolved over time, contributing to various branches of mathematics and its applications. Due to its large number of exciting properties, it has been used in different advanced branches of mathematics like fixed point theory, operator theory, differential calculus etc.

Let A be a subset of a sequentially complete Hausdorff strongly convex topological vector space E (over the field R) with calibration Γ. By the terminology of R.T. Moore [7], a calibration Γ for E means a collection of continuous seminorms p on E which induce the topology of E. Let $f,\ g$ be nonself mappings from A into E. Let a_p, b_p, c_p, d_p and e_p be nonnegative real numbers such that $a_p + b_p + c_p + d_p + e_p < 1$ and for any $x,\ y$ in A, and $p \in \Gamma$.

$$\begin{aligned} p(f(x) - g(y)) \le a_p p(x-y) + b_p p(x - f(x)) + c_p p(y - g(y)) \\ + d_p p(x - g(y)) + e_p p(y - f(x)). \end{aligned} \tag{1.1}$$

Wlodarczyk [14] proved that f has a unique fixed point if $f = g$. In section 3, we prove that $f,\ g$ have a unique common fixed point if $b_p = c_p$ and $d_p = e_p$. When $f = g$, because of $p(x-y) = p(y-x)$, one can, without loss of generality, assume $b_p = c_p$

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and $d_p = e_p$. So our result generalizes the result of Wlodarczyk [14]. Since our Theorem includes Theorem 3.3 of Wlodarczyk [14], it also includes the corresponding theorems in: Hardy and Rogers [5], Goebel, Kirk and Shimi[4], Kannan[6], Nova [8] and Wong [15].

2. Preliminaries

Definition 2.1. [3] Let L be a vector space over the field F (R or C). Let τ be a topology on L such that

(i) For each $x, y \in L$ and each open neighborhood W of $x+y$ in L there exist open neighborhoods U and V of x and y respectively, in L such that $U+V \subseteq W$,
(ii) For each $\lambda \in F,\ x \in L$ and each open neighborhood W of λx in L, there exist open neighborhoods U of λ in F and V of x in L such that $U, V \subseteq W$.

Definition 2.2. [7] Let $\Gamma_0 \subseteq \Gamma, \Gamma_0 \neq \{0\}$. A subset A of E is said to be of type Γ_0 with respect to $x_0 \in A$, if the inequality $p(y) \leq p(x)$, for some $x \in A - x_0$ and for all $p \in \Gamma_0$ implies that $y \in A - x_0$.

Definition 2.3. [11] **(Schauder's Theorem)** Let A be a compact convex subset of a Banach spacce and f a continuous map of A into itself. Then f has a fixed point.

Definition 2.4. [11] **(Tychonoff's Theorem)** Let A be a compact convex subset of a locally convex (linear topological) space and f a continuous map of A into itself Then f has a.fixed point.

Definition 2.5. [11] **(Mazur's Lemma)** The closed convex subset of a compact subset of a Banach space is compact.

Definition 2.6. [11] **(Strong version of Schauder's Theorem)** Let A be a closed convex subset of a Banach space and f a continuous map of A into a compact subset of A. Then f has a fixed point.

Theorem 2.1. [11] *Let A be a convex subset of a locally convex space and f a continuous map of A into a compact subset of A. Then f has a fixed point.*

Theorem 2.2. [11] ***(Strong version of the Tychonoff theorem)*** *Let f be a continuous mapping of a convex subset F of a Hausdorff locally convex linear topological space R into a compact subset A of F. Then there is at least one fixed point.*

Remark. [13] *A locally convex topological vector space is strongly convex in topological vector space.*

3. Main Result

Theorem 3.1. *Let E be a sequentially complete Hausdorff strongly convex topological vector space with calibration Γ, let A be a subset of E and let $f : A \to E, g : A \to E$ be two nonsetf mappings. Assume A is of type $\Gamma_0(\Gamma_0 \subset \Gamma)$, with respect to $x_0 \in A$, f and g satisfy (1.1), such that a_p, b_p, c_p, d_p, e_p are non-negative real-valued functions on $E \times E$ for $p \in \Gamma$. If*

(i) $\gamma \equiv \sup_{x,y \in E}\{a_p(x,\ y) + b_p(x,\ y) + c_p(x,\ y) + 2d_p(x,\ y)\} < 1$*; for* $p \in \Gamma$.
(ii) $b_p \equiv c_p, d_p \equiv e_p$ *for* $p \in \Gamma$,
(iii) $f(x_0) - x_0 \in \frac{1-a_p-b_p-c_p-2d_p}{1-c_p-d_p}(A - x_0)$ *, for all* $p \in \Gamma_0$,
(iv) $(g \circ f)(x_0) - x_0 \in \frac{1-a_p-b_p-c_p-2d_p}{1-c_p-d_p}(A - x_0)$ *for all* $p \in \Gamma_0$*, where* a_p, b_p, c_p *and* d_p *are evaluated at* $(x,\ y)$.

Then $x_n \to u$, and u is the fixed point of f or g in A. If both f and g have fixed points, then each of f, g has a unique fixed point and these two fixed points coincide.

Proof. Let the sequence $\{x_n\}$ be defined as follows

$$x_{2_n+1} = f(x_{2n}),\ x_{2n+2} = g(x_{2n+1}),\ n = 0, 1, 2, \ldots$$

We show that $x_n \in A, n \in \aleph$. Indeed, since A is of type Γ_0, the set $A - x_0$ is balanced and, since $\frac{1-a_p,-b_p-c_p-2d_p}{1-c_p-d_p} < 1,\ p \in \Gamma_0$, then

$$f(x_0) - x_0 \in \frac{1 - a_p - b_p - c_p - 2d_p}{1 - c_p - d_p}(A - x_0) \subset (A - x_0),$$

$$g(x_1) - x_0 \in \frac{1 - a_p - b_p - c_p - 2d_p}{1 - c_p - d_p}(A - x_{\mathrm{O}}) \subset (A - x_0)\ ,$$

for all $p \in \Gamma_0$. Consequently, $f(x_0) = x_1 \in A$, i.e., $x_n \in A$ for $n = 0, 1$. Suppose it is true for $n = k$. We show that it is true for $n = k + 1$.

Case 1. For x_{2n+1}, where $n = k + 1$,

$$\begin{aligned} p(x_{2(k+1)+1} - x_0) &= p(x_{2k+3} - x_0) \\ &\le \sum_{m=0}^{2k+2} p(x_{m+1} - x_m). \end{aligned} \tag{3.1}$$

If m is even then for all $p \in \Gamma$,

$$\begin{aligned} p(x_{m+1} - x_m) &= p(f(x_m) - g(x_{m-1})) \\ &\le a_p p(x_m - x_{m-1}) + b_p p(x_m - f(x_m)) + c_p p(x_{m-1} - g(x_{m-1})) \\ &\quad + d_p p(x_m - g(x_{m-1})) + e_p p(x_{m-1} - f(x_m)) \\ &= a_p p(x_m - x_{m-1}) + b_p p(x_m - x_{m+1}) + c_p p(x_{m-1} - x_m) \\ &\quad + d_p p(x_m - x_m) + e_p p(x_{m-1} - x_{m+1}) \\ &\le (a_p + c_p + e_p) p(x_{m-1} - x_m) + (b_p + e_p) p(x_m - x_{m+1})\ . \end{aligned}$$

It implies,

$$p(x_{m+1} - x_m) \le \frac{a_p + c_p + e_p}{1 - b_p - e_p} p(x_m - x_{m-1})$$

Also,
$p(x_m - x_{m-1}) = p(f(x_{m-2}) - g(x_{m-1}))$
$\le a_p p(x_{m-2} - x_{m-1}) + b_p p(x_{m-2} - f(x_{m-2}))$
$+c_p p(x_{m-1} - g(x_{m-1})) + d_p p(x_{m-2} - g(x_{m-1})) + e_p p(x_{m-1} - f(x_{m-2}))$
$= a_p p(x_{m-2} - x_{m-1}) + b_p p(x_{m-2} - x_{m-1}) + c_p p(x_{m-1} - x_m)$
$+d_p p(x_{m-2} - x_m) + e_p p(x_{m-1} - x_{m-1})$
$\le (a_p + b_p + d_p) p(x_{m-2} - x_{m-1}) + (c_p + d_p) p(x_{m-1} - x_m)$, for all $p \in \Gamma$. It further implies,

$$p(x_m - x_{m-1}) \le \frac{a_p + b_p + d_p}{1 - c_p - d_p} p(x_{m-1} - x_{m-2})\ .$$

Using (ii), we get,

$$p(x_{m+1} - x_m) \le (\frac{a_p + b_p + d_p}{1 - c_p - d_p})^2 p(x_{m-1} - x_{m-2})$$

for all $p \in \Gamma$.

So by induction, we obtain,

$$p(x_{m+1}-x_m) \leq (\frac{a_p+b_p+d_p}{1-c_p-d_p})^m p(x_1-x_0). \tag{3.2}$$

$||^{ly}$Similarly, if m is odd,

$$p(x_{m+1}-x_m) \leq (\frac{a_p+b_p+d_p}{1-c_p-d_p})^m p(x_1-x_0)\ .$$

Therefore,

$$p(x_{2(k+1)+1}-x_0) \leq \sum_{m-0}^{2k+2} p(x_{m+1}-x_m)$$

$$\leq \sum_{m=0}^{2k+2} p(x_{m+1}-X_m)(\frac{a_p+b_p+d_p}{1-c_p-d_p})^m p(x_1-x_0)$$

$$= \frac{1-(\frac{a_p+b_p+d_p}{1-c_p-d_p})^{2k+3}}{1-(\frac{a_p+b_p+d_p}{1-c_p-d_p})} p(x_1-x_0)$$

$\leq \frac{1-c_p-d_p}{1-a_p-b_p-c_p-2d_p} p(x_1-x_0)$ for all $p \in \Gamma$. Since A is of type Γ_0 with respect to x_0, hence

$x_{2(k+1)+1}-x_0 \in A-x_0$ and so $x_{2(k+1)+1} \in A$.

Case 2. For x_{2n+2}, where $n=k+1$,

$$P(x_{2(k+1)+2}-x_0) = p(x_{2k+4}-x_0) \leq \sum_{m=0}^{2k+3} p(x_{m+1}-x_m)\ .$$

Using (3.3), we get,

$$p(x_{2(k+1)+2}-x_0) \leq \sum_{m=0}^{2k+3} (\frac{a_p+b_p+d_p}{1-c_p-d_p})^m p(x_1-x_0),$$

$$\leq \frac{1-c_p-d_p}{1-a_p-b_p-c_p-2d_p} p(x_1-x_0),$$

since $a_p+b_p+d_p < 1-c_p-d_p$.

Since A is of type Γ_0 with respect to x_0, therefore $x_{2(k+1)+2} \in A$. By the induction argument $x_n \in A, (\forall) n \in N$.

The inequality (3.3), implies that $\{x_n\}$ is a Cauchy sequence. Hence it converges to some point u in E. Without loss of generality, we can assume that $x_{n+1} \neq x_n$ for each n, either $x_{2n-1} \neq u$ for infiniteIy many n or $x_{2n} \neq u$ for infinitely many n. By the symmetry we may assume that $x_{2n} \neq u$ for infinitely many n. Thus there is a subsequence $\{k(n)\}$ of $\{n\}$ such that $x_{2k(n)} \neq u$ for each n.

For any $n \geq 1$ and all $p \in \Gamma$ we have

$$p(u-f(u)) \leq p(u-x_{2k(n)}) + p(x_{2k(n)}-f(u)) \tag{3.3}$$

$$= p(u-x_{2k(n)}) + p(g(x_{2k(n)-1})-f(u))\ .$$

Now, $p(f(u)-g(x_{2k(n)-1})) \leq a_p p(u-x_{2k(n)-1}) + b_p p(u-f(u)) + c_p p(x_{2k(n)-1} - g(x_{2k(n)-1})) + d_p p(u-g(x_{2k(n)-1})) + e_p p(x_{2k(n)-1}-f(u)) = a_p p(x_{2k(n)-1}-u) + b_p p(u-f(u)) + c_p p(x_{2k(n)-1}-x_{2k(n)}) + d_p p(u-x_{2k(n)}) + e_p p(x_{2k(n)-1}-f(u)) \leq \gamma \max\{p(x_{2k(n)-1}-u), p(u-f(u)), p(x_{2k(n)-1}-x_{2k(n)}), p(u-x_{2k(n)}), p(x_{2k(n)-1} - f(u))\} \leq \gamma p(f(u)-u)$ as n is sufficently Iarge.

Thus

$$p(f(u)-g(x_{2k(n)-1})) \leq \gamma p(f(u)-u) \tag{3.4}$$

Since $\gamma < 1$. So $f(u) = u$.

Further we have to show that $u \in A$. But

$$p(u-x_0) = p(\lim_m x_m - x_0) = \lim_m p(x_m - x_0)$$

$\leq \lim_m \sum_{i=0}^{m-1} p(x_{i+1}-x_i) = \lim_m \sum_{i=0}^{m-1} p(x_{i+1}-x_i)(\frac{a_p+b_p+d_p}{1-c_p-d_p})^i p(x_1-x_0)$ for all $p \in \Gamma$ (using 3.3). So, by passing to the limit,

$$p(u-x_0) \leq \frac{1-b_p-d_p}{1-a_p-b_p-c_p-2d_p} p(x_1-x_0)$$

for all $p \in \Gamma$. Since A is of type Γ_0 with respect to x_0, so $u \in A$. Hence u is the fixed point of f in A. If $u,\ v$ are the fixed points of f and g respectively, such that $u \neq v$, then $p(u-v) = p(f(u)-g(v) \leq (a_p+2d_p)p(u-v) < p(u-v)$ for all $p \in \Gamma$, what is a contradiction. So $u = v$. □

Theorem 3.2. *Let E be a sequentially complete Hausdorff strongly convex topological vector spaces with calibration Γ. Consider two mappings $f,\ g$ from E into E satisfying a condition: for any given $\epsilon > 0$, there exits $\delta > 0$ such that the inequality*

$\epsilon \leq p(x-y) < \epsilon + \delta$ implies $p(f(x)-g(y)) < \epsilon$ for all $p \in \Gamma$.

If at least one of f and g is continuous then f or g has a fixed point, If both f and g have fixed points, then each of them has a unique fixed point and these two points coincide.

Proof. Fix $x_0 \in E$ and define $\{x_n\}$ by $x_{2n+1} = f(x_{2n}), x_{2n+2} = g(x_{2n+1})$. Then $\{x_n\}$ is a Cauchy sequence. Indeed, if otherwise, then there exists $\epsilon > 0$, such that $\limsup p(x_m - x_n) > 2\epsilon$, for all $p \in \Gamma$. By hypothesis, there exits $\delta > 0$, such that,

$$\epsilon \leq p(x-y) \leq \epsilon + \delta \text{ and so } p(f(x)-g(y)) \leq \epsilon \text{ for all } p \in F \tag{3.5}$$

Replace δ by $\delta' = \min\{\delta,\ \epsilon\}$. Firstly, we show that $\lim p(x_n - x_{n+1}) \downarrow o, (\forall) p \in \Gamma$. Let $C_n = p(x_n - x_{n+1})$. Since C_n, is a decreasing sequence, then it fails for $C_{m+1}, p \in \Gamma$, where C_m is chosen less than $\epsilon + \delta$. Hence

$$\lim_n C_n \downarrow 0 \text{ for all } p \in \Gamma. \tag{3.6}$$

By (3.7), we can find an M so that $C_M < \delta'/3$. Pick $m, n > M$, so that

$$p(x_m - x_n) > 2\epsilon,\ p \in \Gamma,\ |p(x_m - x_j) - p(x_m - x_{j+1})| \leq C_j < \frac{\delta'}{3} \tag{3.7}$$

for all $p \in$ F.

Since $C_m < \epsilon$ and $p(x_m - x_n) > \epsilon + \delta'$, for all $p \in \Gamma$, therefore there exists an integer $j \in [m,\ n]$ with $\epsilon + \frac{2\delta'}{3} < p(x_m - x_j) < \epsilon + \delta'$, for all $p \in \Gamma$. Indeed from (9), $p(x_m - x_{j+1}) - C_j \leq p(x_m - x_j)$. It gives, $\epsilon + \delta' - \frac{\delta'}{3} = \epsilon + \frac{2\delta'}{3} < p(x_m - x_j)$. AIso $p(x_m - x_j) < \epsilon + \delta'$ for all $p \in \Gamma$. Hence

$\epsilon + \frac{2\delta'}{3} < p(x_m - x_j) < \epsilon + \delta'$. Using (3.6), we conclude that for all m and j,

$$p(x_m - x_j) \leq p(x_m - x_{m+1}) + p(x_{m+1} - x_{j+1}) + p(x_{j+1} - x_j)$$

$\leq C_m + \epsilon + C_j < \frac{2\delta'}{3} + \epsilon$, for all $p \in \Gamma$.

Hence it is a contradiction. So $\{x_n\}$ is a Cauchy sequence. Since E is sequentially complete, $\{x_n\}$ converges to some point $x \in E$. Thus $f(x_{2n}) \to x$ and $g(x_{2n+1}) \to x$. If f is continuous, then

$$f(x) = f(\lim_{n\to\infty} g(x_{2n+1})) = \lim_{n\to\infty} f(x_{2n+2}) = x.$$

So x is a fixed point of f. Let u and v be the fixed points of f and g respectively such that $u \neq v$. Then by using (3.6), we have that $p(u-v) = p(f(u) - g(v)) < p(u-u)$, for all $p \in \Gamma$, a contradiction. Therefore $u = v$. □

Corollary 3.3. *Let E be a sequentially complete Hausdorff strongly convex topological vector space with calibration Γ. Let f be a mapping from E into E satisfying: for given $\epsilon > 0$, there exists $\delta > 0$ such that the condition $\epsilon \leq p(x-y) < \epsilon + \delta$ implies $p(f(x) - f(y)) < \epsilon$, for all $p \in \Gamma$. Then f has a unique fixed point.*

Corollary 3.4. *Let E be a sequentially complete Hausdorff strongly convex topological vector space with calibration Γ. Let f be a surjective mapping from E into E satisfying a condition: for given $\epsilon > 0$, there exists $\delta > 0$ such that, the inequality*

$$p(x-y) < \epsilon \text{ implies } \epsilon \leq p(f(x) - f(y)) < \epsilon + \delta, \tag{3.8}$$

for all $p \in \Gamma$.

Then f has a unique fixed point.

Proof. We shall show that f is a one-to-one mapping. Indeed, let $x \neq y$ and $p(x-y) < \epsilon$ but $f(x) = f(y)$. We obtain $0 \leq p(x-y) < p(f(x) - f(y)) = 0, p \in \Gamma$, what is impossible.

Let g be the inverse of f. Then (3.9) becomes $p(g(x) - g(y)) < \epsilon$, whenever $\epsilon \leq p(x-y) < \epsilon + \delta$. g has the unique fixed point u. Thus $g(u) = u = f(g(u)) = f(u)$. So x is the unique fixed point of f. □

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