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APPROXIMATION BY SEQUENCES OF q-SZÁSZ-OPERATORS GENERATED BY DUNKL EXPONENTIAL FUNCTION

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ABSTRACT. The main purpose of this article is to introduce a modification of q-Dunkl generalization of Szász-operators. We obtain approximation results via well known Korovkin's type theorem. Moreover, we obtain the order of approximation, rate of convergence, functions belonging to the Lipschitz class and some direct theorems.

1. INTRODUCTION AND PRELIMINARIES

In 1950, for $x \geq 0$, Szász [24] introduced the operators

$$S_n(f;x) = e^{-nx}\sum_{k=0}^{\infty}\frac{(nx)^k}{k!}f\left(\frac{k}{n}\right),\ f \in C[0,\infty). \tag{1.1}$$

In the field of approximation theory, the application of q-calculus emerged as a new area in the field of approximation theory. The first q-analogue of the well-known Bernstein polynomials was introduced by Lupaş by applying the idea of q-integers [5]. In 1997, Phillips [22] considered another q-analogue of the classical Bernstein polynomials. Later on, many authors introduced q-generalizations of various operators and investigated several approximation properties [7, 8, 9, 10, 11, 13, 14, 15, 16, 17, 18, 19, 20, 21, 25, 26].

We now present some basic definitions and notations of the q-calculus which are used in this paper.

Definition 1.1. For $|q| < 1$, the q-number $[\lambda]_q$ is defined by

$$[\lambda]_q = \begin{cases} \dfrac{1-q^\lambda}{1-q} & (\lambda \in \mathbb{C}) \\ \displaystyle\sum_{k=0}^{n-1} q^k = 1+q+q^2+\cdots+q^{n-1} & (\lambda = n \in \mathbb{N}). \end{cases} \tag{1.2}$$

Definition 1.2. For $|q| < 1$, the q-factorial $[n]_q!$ is defined by

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$$[n]_q! = \begin{cases} 1 & (n=0) \\ \prod_{k=1}^{n} [k]_q & (n \in \mathbb{N}). \end{cases} \tag{1.3}$$

Sucu [23] defined a Dunkl analogue of Szász operators via a generalization of the exponential function as follows:

$$S_n^*(f;x) := \frac{1}{e_\mu(nx)} \sum_{k=0}^{\infty} \frac{(nx)^k}{\gamma_\mu(k)} f\left(\frac{k+2\mu\theta_k}{n}\right), \tag{1.4}$$

where $x \geq 0,\ f \in C[0,\infty), \mu \geq 0,\ n \in \mathbb{N}$.
Cheikh et al., [2] stated the q-Dunkl classical q-Hermite type polynomials and gave definitions of q-Dunkl analogues of exponential functions and recursion relations for $\mu > -\frac{1}{2}$ and $0 < q < 1$.

$$e_{\mu,q}(x) = \sum_{n=0}^{\infty} \frac{x^n}{\gamma_{\mu,q}(n)},\ x \in [0,\infty) \tag{1.5}$$

$$E_{\mu,q}(x) = \sum_{n=0}^{\infty} \frac{q^{\frac{n(n-1)}{2}} x^n}{\gamma_{\mu,q}(n)},\ x \in [0,\infty) \tag{1.6}$$

where

$$\gamma_{\mu,q}(n) = \frac{(q^{2\mu+1}, q^2)_{[\frac{n+1}{2}]} (q^2, q^2)_{[\frac{n}{2}]}}{(1-q)^n} \gamma_{\mu,q}(n),\ n \in \mathbb{N}, \tag{1.7}$$

In [4], Içöz gave the Dunkl generalization of Szász operators via q-calculus as:

$$D_{n,q}(f;x) = \frac{1}{e_{\mu,q}([n]_q x)} \sum_{k=0}^{\infty} \frac{([n]_q x)^k}{\gamma_{\mu,q}(k)} f\left(\frac{1-q^{2\mu\theta_k+k}}{1-q^n}\right), \tag{1.8}$$

for $\mu > \frac{1}{2},\ x \geq 0,\ 0 < q < 1$ and $f \in C[0,\infty)$.

Previous studies demonstrate that providing a better error estimation for positive linear operators plays an important role in approximation theory, which allows us to approximate much faster to the function being approximated.

Motivated essentially by Içöz [4] the recent investigation of Dunkl generalization of Szász-Mirakjan operators via q-calculus we show that our modified operators have better error estimation than [4]. We also prove several approximation results and successfully extend the results of [4]. Several other related results have also been discussed.

2. Construction of operators and moments estimation

Let $\{a_n\}$ and $\{b_n\}$; are increasing and unbounded sequences of positive numbers such that

$$\lim_{n\to\infty} \frac{1}{b_n} \to 0,\ \text{and}\ \frac{a_n}{b_n} = 1 + O\left(\frac{1}{b_n}\right). \tag{2.1}$$

For any $\frac{1}{2n} \leq x < \frac{1}{1-q^n},\ 0 < q < 1,\ \mu > \frac{1}{2n}$ and $n \in \mathbb{N}$ we define

$$D_{n,q}^{a_n,b_n}(f;x) = \frac{1}{e_{\mu,q}([n]_q x a_n)} \sum_{k=0}^{\infty} \frac{(a_n [n]_q x)^k}{\gamma_{\mu,q}(k)} f\left(\frac{1-q^{2\mu\theta_k+k}}{b_n(1-q^n)}\right), \tag{2.2}$$

where $e_{\mu,q}(x),\ \gamma_{\mu,q}$ are defined in (1.5), (1.7) by [23] and $f \in C_\zeta[0,\infty)$ with $\zeta \geq 0$ and

$$C_\zeta[0,\infty) = \{f \in C[0,\infty) : \mid f(t) \mid \leq M(1+t)^\zeta, \text{ for some } M>0,\ \zeta>0\}. \qquad (2.3)$$

Note that the parameters a_n and b_n have an important effect for a better approach of the operator $D_{n,q}^{a_n,b_n}$.

Lemma 2.1. *Let $D_{n,q}^{a_n,b_n}(.\,;\,.)$ be the operators given by* (2.2). *Then for each $\frac{1}{2n} \leq x < \frac{1}{b_n(1-q^n)}$, $n \in \mathbb{N}$, we have the following identities/ estimates:*

(1) $D_{n,q}^{a_n,b_n}(1;x) = 1$,

(2) $D_{n,q}^{a_n,b_n}(t;x) = \left(\frac{a_n}{b_n}\right) x$,

(3) $\left(\frac{a_n}{b_n}\right)^2 x^2 + \left(\frac{a_n}{b_n}\right) q^{2\mu}[1-2\mu]_q \frac{e_{\mu,q}(\frac{a_n}{b_n} q[n]_q x)}{e_{\mu,q}([n]_q x)} \frac{x}{\frac{a_n}{b_n}[n]_q} \leq D_{n,q}^{a_n,b_n}(t^2;x) \leq \left(\frac{a_n}{b_n}\right)^2 x^2 + \left(\frac{a_n}{b_n}\right)[1+2\mu]_q \frac{x}{[n]_q}$..

Proof. As $D_{n,q}^{a_n,b_n}(1;x) = \frac{1}{e_{\mu,q}(a_n[n]_q x)} \sum_{k=0}^{\infty} \frac{(a_n[n]_q x)^k}{\gamma_\mu(k)} = 1$, and □

$$\begin{aligned}
D_{n,q}^{a_n,b_n}(t;x) &= \frac{1}{e_{\mu,q}(a_n[n]_q x)} \sum_{k=0}^{\infty} \frac{(a_n[n]_q x)^k}{\gamma_\mu(k)} \left(\frac{1-q^{2\mu\theta_k+k}}{b_n(1-q^n)}\right) \\
&= \frac{1}{b_n[n]_q e_{\mu,q}(a_n[n]_q x)} \sum_{k=1}^{\infty} \frac{(a_n[n]_q x)^k}{\gamma_\mu(k-1)} \\
&= \left(\frac{a_n}{b_n}\right) x
\end{aligned}$$

then (1) and (2) hold. Similarly

$$\begin{aligned}
D_{n,q}^{a_n,b_n}(t^2;x) &= \frac{1}{e_{\mu,q}(a_n[n]_q x)} \sum_{k=0}^{\infty} \frac{(a_n[n]_q x)^k}{\gamma_\mu(k)} \left(\frac{1-q^{2\mu\theta_k+k}}{b_n(1-q^n)}\right)^2 \\
&= \frac{1}{b_n^2[n]_q^2 e_{\mu,q}(a_n[n]_q x)} \sum_{k=0}^{\infty} \frac{(a_n[n]_q x)^k}{\gamma_\mu(k-1)} \left(\frac{1-q^{2\mu\theta_k+k}}{1-q}\right) \\
&= \frac{1}{b_n^2[n]_q^2 e_{\mu,q}(a_n[n]_q x)} \sum_{k=0}^{\infty} \frac{(a_n[n]_q x)^{k+1}}{\gamma_\mu(k)} \left(\frac{1-q^{2\mu\theta_{k+1}+k+1}}{1-q}\right).
\end{aligned}$$

From [4] we know that

$$[2\mu\theta_{k+1}+k+1]_q = [2\mu\theta_k+k]_q + q^{2\mu\theta_k+k}[2\mu(-1)^k+1]_q, \qquad (2.4)$$

Now by separating to the even and odd terms and using (2.4), we get

$$\begin{aligned}
D_{n,q}^{a_n,b_n}(t^2;x) &= \frac{1}{b_n^2[n]_q^2 e_{\mu,q}(a_n[n]_q x)} \sum_{k=0}^{\infty} \frac{(a_n[n]_q x)^{k+1}}{\gamma_\mu(k)} \left(\frac{1-q^{2\mu\theta_{k+1}+k+1}}{1-q}\right) \\
&+ \frac{[1+2\mu]_q}{b_n^2[n]_q^2 e_{\mu,q}(a_n[n]_q x)} \sum_{k=0}^{\infty} \frac{(a_n[n]_q x)^{2k+1}}{\gamma_\mu(2k)} q^{2\mu\theta_{2k}+2k} \\
&+ \frac{[1-2\mu]_q}{b_n^2[n]_q^2 e_{\mu,q}(a_n[n]_q x)} \sum_{k=0}^{\infty} \frac{(a_n[n]_q x)^{2k+2}}{\gamma_\mu(2k)} q^{2\mu\theta_{2k+1}+2k+1}.
\end{aligned}$$

Since

$$[1-2\mu]_q \leq [1+2\mu]_q, \tag{2.5}$$

we have

$$\begin{aligned} D_{n,q}^{a_n,b_n}(t^2;x) &\geq (x\frac{a_n}{b_n})^2 + \frac{xa_n[1-2\mu]_q}{b_n[n]_q e_{\mu,q}(a_n[n]_q x)} \sum_{k=0}^{\infty} \frac{(qa_n[n]_q x)^{2k}}{\gamma_\mu(2k)} \\ &+ \frac{q^{2\mu} x a_n [1-2\mu]_q}{b_n[n]_q e_{\mu,q}(a_n[n]_q x)} \sum_{k=0}^{\infty} \frac{(qa_n[n]_q x)^{2k+1}}{\gamma_\mu(2k+1)} \\ &\geq (x\frac{a_n}{b_n})^2 + q^{2\mu}[1-2\mu]_q \frac{e_{\mu,q}(q\frac{a_n}{b_n}[n]_q x)}{e_{\mu,q}(\frac{a_n}{b_n}[n]_q x)} \frac{xa_n}{b_n[n]_q}. \end{aligned}$$

On the other hand we have

$$D_{n,q}^{a_n,b_n}(t^2;x) \leq (x\frac{a_n}{b_n})^2 + [1+2\mu]_q \frac{xa_n}{b_n[n]_q}.$$

This completes the proof.

Lemma 2.2. *Let the operators $D_{n,q}^{a_n,b_n}(.\,;\,.)$ be given by* (2.2). *Then for each $x \geq \frac{1}{2n}$, $n \in \mathbb{N}$, we have*

(1) $D_{n,q}^{a_n,b_n}(t-x;x) = \left(\frac{a_n}{b_n} - 1\right)x$,

(2) $D_{n,q}^{a_n,b_n}((t-x)^2;x) \leq \left(\frac{a_n}{b_n} - 1\right)^2 x^2 + \left(\frac{a_n}{b_n}\right)[1+2\mu]_q \frac{x}{[n]_q}$.

Proof. For proof of this lemma we use

$$D_{n,q}^{a_n,b_n}(t-x;x) = D_{n,q}^{a_n,b_n}(t;x) - D_{n,q}^{a_n,b_n}(1;x),$$

And

$$D_{n,q}^{a_n,b_n}((t-x)^2;x) = D_{n,q}^{a_n,b_n}(t^2;x) - 2xD_{n,q}^{a_n,b_n}(t;x) + x^2 D_{n,q}^{a_n,b_n}(1;x)$$

This ends the proof of (2). □

3. Main results

We obtain the Korovkin's type approximation properties for our operators (see [1], [6]).

Let $C_B(\mathbb{R}^+)$ be the set of all bounded and continuous functions on $\mathbb{R}^+ = [0,\infty)$, which is linear normed space with

$$\| f \|_{C_B} = \sup_{x \geq 0} | f(x) | .$$

Let

$$H := \{f : x \in [0,\infty), \frac{f(x)}{1+x^2} \text{ is convergent as } x \to \infty\}.$$

Theorem 3.1. *Let $D_{n,q}^{a_n,b_n}(.\,;\,.)$ be the operators defined by* (2.2). *Then for any function $f \in C_\zeta[0,\infty) \cap H$, $\zeta \geq 2$,*

$$\lim_{n\to\infty} D_{n,q}^{a_n,b_n}(f;x) = f(x)$$

is uniformly on each compact subset of $[0,\infty)$*, where* $x[0,\infty)$.

Proof. The proof is based on Lemma 2.1 and well known Korovkin's theorem regarding the convergence of a sequence of linear positive operators, so it is enough to prove the conditions

$$\lim_{n\to\infty} D_{n,q}^{a_n,b_n}((t^j;x) = x^j, \ j = 0,1,2, \ \{\text{as } n \to \infty\}$$

uniformly on $[0,1]$.
Clearly $\frac{1}{[n]_q} \to 0$ $(n\to\infty)$ we have

$$\lim_{n\to\infty} D_{n,q}^{a_n,b_n}(t;x)=x,\ \lim_{n\to\infty} D_{n,q}^{a_n,b_n}(t^2;x)=x^2.$$

This complete the proof. □

We recall the weighted spaces of the functions on $\mathbb{R}^+$, which are defined as follows:

$$\begin{aligned}
P_\rho(\mathbb{R}^+) &= \{f :\mid f(x)\mid \leq M_f\rho(x)\},\\
Q_\rho(\mathbb{R}^+) &= \left\{f : f\in P_\rho(\mathbb{R}^+)\cap C[0,\infty)\right\},\\
Q_\rho^k(\mathbb{R}^+) &= \left\{f : f\in Q_\rho(\mathbb{R}^+) \text{ and } \lim_{x\to\infty}\frac{f(x)}{\rho(x)}=k(k \text{ is a constant})\right\},
\end{aligned}$$

where $\rho(x)=1+x^2$ is a weight function and M_f is a constant depending only on f. Note that $Q_\rho(\mathbb{R}^+)$ is a normed space with the norm $\parallel f\parallel_\rho=\sup_{x\geq 0}\frac{|f(x)|}{\rho(x)}$.

4. RATE OF CONVERGENCE

Let $f\in C_B[0,\infty]$, the space of all bounded and continuous functions on $[0,\infty)$ and $x\geq\frac{1}{2n}$, $n\in\mathbb{N}$. Then for $\delta>0$, the modulus of continuity of f denoted by $\omega(f,\delta)$ gives the maximum oscillation of f in any interval of length not exceeding $\delta>0$ and it is given by

$$\omega(f,\delta)=\sup_{|t-x|\leq\delta}\mid f(t)-f(x)\mid,\ t\in[0,\infty). \tag{4.1}$$

It is known that $\lim_{\delta\to 0+}\omega(f,\delta)=0$ for $f\in C_B[0,\infty)$ and for any $\delta>0$ we have

$$\mid f(t)-f(x)\mid\leq\left(\frac{\mid t-x\mid}{\delta}+1\right)\omega(f,\delta). \tag{4.2}$$

Now we calculate the rate of convergence of operators (2.2) by means of modulus of continuity and Lipschitz type maximal functions.

Theorem 4.1. *Let $D_{n,q}^{a_n,b_n}(.\,;\,.)$ be the operators defined by* (2.2)*. Then for $f\in C_B[0,\infty),\ x\geq\frac{1}{2n}$ and $n\in\mathbb{N}$ we have*

$$\mid D_{n,q}^{a_n,b_n}(f;x)-f(x)\mid\leq 2\omega\left(f;\delta_{n,x}\right),$$

where

$$\delta_{n,x}=\sqrt{\left(\frac{a_n}{b_n}-1\right)^2x^2+\left(\frac{a_n}{b_n}\right)[1+2\mu]_q\frac{x}{[n]_q}}. \tag{4.3}$$

Proof. We prove it by using (4.1), (4.2) and Cauchy-Schwarz inequality. We can easily get

$$\mid D_{n,q}^{a_n,b_n}(f;x)-f(x)\mid\leq\left\{1+\frac{1}{\delta}\left(D_{n,q}^{a_n,b_n}(t-x)^2;x\right)^{\frac{1}{2}}\right\}\omega(f;\delta)$$

if we choose $\delta=\delta_{n,x}$ and by applying the result (2) of Lemma 2.2, we get the result. □

Remark. *For every $f\in C_B[0,\infty),\ x\geq 0$ and $n\in\mathbb{N}$, suppose $\{a_n\},\{b_n\}$ be the sequence satisfies* (2.1) *then the operators $D_{n,q}^{*}(.\,;\,.)$ defined by* **??** *reduced to $D_{n,q}^{a_n,b_n}(.\,;\,.)$.*

Now we give the rate of convergence of the operators $D_{n,q}^{a_n,b_n}(f;x)$ defined in (2.2) in terms of the elements of the usual Lipschitz class $Lip_M(\nu)$.

Let $f\in C_B[0,\infty)$, $M>0$ and $0<\nu\leq 1$. The class $Lip_M(\nu)$ is defined as

$$Lip_M(\nu)=\{f :\mid f(\zeta_1)-f(\zeta_2)\mid\leq M\mid\zeta_1-\zeta_2\mid^\nu\ \ (\zeta_1,\zeta_2\in[0,\infty))\} \tag{4.4}$$

Theorem 4.2. *Let* $D_{n,q}^{a_n,b_n}(.\ ;\ .)$ *be the operators defined in* (2.2)*.Then for each* $f \in Lip_M(\nu)$, $(M>0,\ 0<\nu\leq 1)$ *satisfying* (4.4) *we have*

$$\mid D_{n,q}^{a_n,b_n}(f;x)-f(x)\mid\leq M\left(\delta_{n,x}\right)^{\frac{\nu}{2}}$$

where $\delta_{n,x}$ *is given in Theorem 4.1.*

Proof. We prove it by using (4.4) and Hölder inequality. We have

$$\begin{aligned}\mid D_{n,q}^{a_n,b_n}(f;x)-f(x)\mid &\leq \mid D_{n,q}^{a_n,b_n}(f(t)-f(x);x)\mid\\ &\leq D_{n,q}^{a_n,b_n}\left(\mid f(t)-f(x)\mid;x\right)\\ &\leq MD_{n,q}^{a_n,b_n}\left(\mid t-x\mid^{\nu};x\right).\end{aligned}$$

Therefore

$$\begin{aligned}&\mid D_{n,q}^{a_n,b_n}(f;x)-f(x)\mid\\ &\leq M\frac{[n]_q}{e_{\mu,q}(a_n[n]_qx)}\sum_{k=0}^{\infty}\frac{(a_n[n]_qx)^k}{\gamma_{\mu,q}(k)}\left|\frac{1-q^{2\mu\theta_k+k}}{b_n(1-q^n)}-x\right|^{\nu}\\ &\leq M\frac{[n]_q}{e_{\mu,q}(a_n[n]_qx)}\sum_{k=0}^{\infty}\left(\frac{(a_n[n]_qx)^k}{\gamma_{\mu,q}(k)}\right)^{\frac{2-\nu}{2}}\\ &\times\left(\frac{(a_n[n]_qx)^k}{\gamma_{\mu,q}(k)}\right)^{\frac{\nu}{2}}\left|\frac{1-q^{2\mu\theta_k+k}}{b_n(1-q^n)}-x\right|^{\nu}\\ &\leq M\left(\frac{n}{e_{\mu,q}(a_n[n]_qx)}\sum_{k=0}^{\infty}\frac{(a_n[n]_qx)^k}{\gamma_{\mu,q}(k)}\right)^{\frac{2-\nu}{2}}\\ &\times\left(\frac{[n]_q}{e_{\mu,q}(a_n[n]_qx)}\sum_{k=0}^{\infty}\frac{(a_n[n]_qx)^k}{\gamma_{\mu,q}(k)}\left|\frac{1-q^{2\mu\theta_k+k}}{b_n(1-q^n)}-x\right|^{2}\right)^{\frac{\nu}{2}}\\ &= M\left(D_{n,q}^{a_n,b_n}(t-x)^2;x\right)^{\frac{\nu}{2}}.\end{aligned}$$

This completes the proof. □

Let

$$C_B^2(\mathbb{R}^+)=\{g\in C_B(\mathbb{R}^+):g',g''\in C_B(\mathbb{R}^+)\},\tag{4.5}$$

with the norm

$$\parallel g\parallel_{C_B^2(\mathbb{R}^+)}=\parallel g\parallel_{C_B(\mathbb{R}^+)}+\parallel g'\parallel_{C_B(\mathbb{R}^+)}+\parallel g''\parallel_{C_B(\mathbb{R}^+)},\tag{4.6}$$

also

$$\parallel g\parallel_{C_B(\mathbb{R}^+)}=\sup_{x\in\mathbb{R}^+}\mid g(x)\mid.\tag{4.7}$$

Theorem 4.3. *Let* $D_{n,q}^{a_n,b_n}(.\ ;\ .)$ *be the operators defined in* (2.2)*. Then for any* $g\in C_B^2(\mathbb{R}^+)$ *we have*

$$\mid D_{n,q}^{a_n,b_n}(f;x)-f(x)\mid\leq\left\{\left(\left(\frac{a_n}{b_n}-1\right)x\right)+\frac{\delta_{n,x}}{2}\right\}\parallel g\parallel_{C_B^2(\mathbb{R}^+)},$$

where $\delta_{n,x}$ *is given in Theorem 4.1.*

Proof. Let $g \in C_B^2(\mathbb{R}^+)$. Then by using the generalized mean value theorem in the Taylor series expansion we have

$$g(t) = g(x) + g'(x)(t-x) + g''(\psi)\frac{(t-x)^2}{2},\ \psi \in (x,t).$$

By applying linearity property on $D_{n,q}^{a_n,b_n}$, we have

$$D_{n,q}^{a_n,b_n}(g,x) - g(x) = g'(x) D_{n,q}^{a_n,b_n}\left((t-x);x\right) + \frac{g''(\psi)}{2} D_{n,q}^{a_n,b_n}\left((t-x)^2;x\right),$$

which implies that

$$\mid D_{n,q}^{a_n,b_n}(g;x) - g(x) \mid \leq \left(\left(\frac{a_n}{b_n}-1\right)x\right) \parallel g' \parallel_{C_B(\mathbb{R}^+)} + \left(\left(\frac{a_n}{b_n}-1\right)^2 x^2 + \left(\frac{a_n}{b_n}\right)[1+2\mu]_q \frac{x}{[n]_q}\right)\frac{\|g''\|_{C_B(\mathbb{R}^+)}}{2}.$$

From (4.6) we have $\parallel g' \parallel_{C_B[0,\infty)} \leq \parallel g \parallel_{C_B^2[0,\infty)}$.

$$\mid D_{n,q}^{a_n,b_n}(g;x) - g(x) \mid \leq \left(\left(\frac{a_n}{b_n}-1\right)x\right) \parallel g \parallel_{C_B^2(\mathbb{R}^+)} + \left(\left(\frac{a_n}{b_n}-1\right)^2 x^2 + \left(\frac{a_n}{b_n}\right)[1+2\mu]_q \frac{x}{[n]_q}\right)\frac{\|g\|_{C_B^2(\mathbb{R}^+)}}{2}.$$

The proof follows from (2) of Lemma 2.2. □

The Peetre's K-functional is defined by

$$K_2(f,\delta) = \inf_{C_B^2(\mathbb{R}^+)}\left\{\left(\parallel f-g \parallel_{C_B(\mathbb{R}^+)} + \delta \parallel g'' \parallel_{C_B^2(\mathbb{R}^+)}\right) : g \in \mathcal{W}^2\right\}, \qquad (4.8)$$

where

$$\mathcal{W}^2 = \left\{g \in C_B(\mathbb{R}^+) : g', g'' \in C_B(\mathbb{R}^+)\right\}. \qquad (4.9)$$

There exits a positive constant $C > 0$ such that $K_2(f,\delta) \leq C\omega_2(f,\delta^{\frac{1}{2}}),\ \delta > 0$, where the second order modulus of continuity is given by

$$\omega_2(f,\delta^{\frac{1}{2}}) = \sup_{0<h<\delta^{\frac{1}{2}}} \sup_{x\in\mathbb{R}^+} \mid f(x+2h) - 2f(x+h) + f(x) \mid. \qquad (4.10)$$

Theorem 4.4. *For* $x \geq \frac{1}{2n},\ n \in \mathbb{N}$ *and* $f \in C_B(\mathbb{R}^+)$ *we have*

$$\mid D_{n,q}^{a_n,b_n}(f;x) - f(x) \mid$$

$$\leq 2M\left\{\omega_2\left(f;\sqrt{\frac{(2(\frac{a_n}{b_n}-1)x)+\delta_{n,x}}{4}}\right) + \min\left(1, \frac{(2(\frac{a_n}{b_n}-1)x)+\delta_{n,x}}{4}\right) \parallel f \parallel_{C_B(\mathbb{R}^+)}\right\},$$

where M is a positive constant, $\delta_{n,x}$ is given in Theorem 4.3 and $\omega_2(f;\delta)$ is the second order modulus of continuity of the function f defined in (4.10).

Proof. We prove this by using the Theorem 4.3

$$\begin{aligned} \mid D_{n,q}^{a_n,b_n}(f;x) - f(x) \mid &\leq \mid D_{n,q}^{a_n,b_n}(f-g;x) \mid + \mid D_{n,q}^{a_n,b_n}(g;x) - g(x) \mid + \mid f(x) - g(x) \mid \\ &\leq 2 \parallel f-g \parallel_{C_B(\mathbb{R}^+)} + \frac{\delta_{n,x}}{2} \parallel g \parallel_{C_B^2(\mathbb{R}^+)} + \left(\left(\frac{a_n}{b_n}-1\right)x\right) \parallel g \parallel_{C_B(\mathbb{R}^+)} \end{aligned}$$

From (4.6) clearly we have $\parallel g \parallel_{C_B[0,\infty)} \leq \parallel g \parallel_{C_B^2[0,\infty)}$.
Therefore,

$$\mid D_{n,q}^{a_n,b_n}(f;x) - f(x) \mid \leq 2\left(\parallel f-g \parallel_{C_B(\mathbb{R}^+)} + \frac{\left(2\left(\frac{a_n}{b_n}-1\right)x\right)+\delta_{n,x}}{4} \parallel g \parallel_{C_B^2(\mathbb{R}^+)}\right),$$

where $\delta_{n,x}$ is given in Theorem 4.1.

By taking infimum over all $g \in C_B^2(\mathbb{R}^+)$ and by using (4.8), we get

$$| D_{n,q}^{a_n,b_n}(f;x) - f(x) | \leq 2K_2 \left(f; \frac{\left(2\left(\frac{a_n}{b_n}-1\right)x\right)+\delta_{n,x}}{4}. \right)$$

Now for an absolute constant $Q > 0$ in [3] we use the relation

$$K_2(f;\delta) \leq Q\{\omega_2(f;\sqrt{\delta}) + \min(1,\delta) \parallel f \parallel\}.$$

This complete the proof. □

Conclusion

Purpose of this paper is to provide a better error estimation of convergence of the q-Dunkl generalization of Szász-operators by initiating the increasing and unbounded sequences $\{a_n\}$, $\{b_n\}$ of positive numbers such that $\lim_{n\to\infty} \frac{1}{b_n} \to 0,$ and $\frac{a_n}{b_n} = 1 + O\left(\frac{1}{b_n}\right)$. Here we have defined a Dunkl generalization of these modified operators. This type of modifications enables better error estimation on the interval $[0,\infty)$ rather than the classical Dunkl Szász operators [4]. We obtained some approximation results via well known Korovkin's type theorem.We have also calculated the rate of convergence of operators by means of modulus of continuity and Lipschitz type maximal functions.

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