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NEW DEFINITION OF A SINGULAR INTEGRAL OPERATOR

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ABSTRACT. Let D be a connected bounded domain in R^2, S be its boundary which is closed, connected and smooth or $S = (-\infty, \infty)$. Let $\Phi(z) = \frac{1}{2\pi i}\int_S \frac{f(s)ds}{s-z}$, $f \in L^1(S)$, $z = x+iy$. The singular integral operator $Af := \frac{1}{i\pi}\int_S \frac{f(s)ds}{s-t}$, $t \in S$, is defined in a new way. This definition simplifies the proof of the existence of $\Phi(t)$. Necessary and sufficient conditions are given for $f \in L^1(S)$ to be boundary value of an analytic in D function. The Sokhotsky-Plemelj formulas are derived for $f \in L^1(S)$. Our new definition allows one to treat singular boundary values of analytic functions.

1. INTRODUCTION

Let D be a connected bounded domain on the complex plane, S be its boundary, which is closed and $C^{1,a}$-smooth, $0 < a \leq 1$ or $S = (-\infty, \infty)$. The standard definition of the singular integral operator $Af = \frac{1}{i\pi}\int_S \frac{f(s)ds}{s-t}$ is:

$$Af = \lim_{\epsilon \to 0} \frac{1}{i\pi} \int_{|s-t|>\epsilon} \frac{f(s)}{s-t} ds. \tag{1.1}$$

We assume that $t \in S$ and $f \in L^1(S)$. The latter is *the basic new assumption*: in the literature it was assumed that $f \in H^\mu(S)$, where $H^\mu(S)$ is the space of Hölder-continuous functions, or $f \in L^p(S),\ p > 1$, see [2], [4]. In [1] there is a result for $f \in L^1(S)$, the existence of the limit (1.1) is proved, but the proof is not simple. Our goal is to give a new definition of the operator A. This definition makes the proof of the existence of Af for $f \in L^1(S)$ very simple. It is also of great interest to have a proof of the Sokhotsky formulas for $f \in L^1(S)$, see [6].

Definition 1.1.

$$(Af, \phi) := -(f, A\phi) \quad \forall \phi \in H^\mu(S),\ 0 < \mu < 1. \tag{1.2}$$

Here

$$(Af, \phi) = \frac{1}{i\pi}\int_S dt\phi(t) \int_S \frac{f(s)ds}{s-t}, \quad (f, A\phi) = -\int_S ds f(s) \frac{1}{i\pi}\int_S dt \frac{\phi(t)dt}{t-s}.$$

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By D^+ we denote D, by $D^- = D'$ we denote $\mathbb{R}^2 \setminus \bar{D}$, the $\bar{D}$ is the closure of D.

By D^+ we denote D, by $D^- = D'$

Lemma 1. *Formula* (1.2) *defines* $f \in L^1(S)$ *uniquely.*

Proof. Suppose that $f_1, f_2 \in L^1(S)$ satisfy (1.2). Then $q := f_1 - f_2$ satisfies the relation $(q, A\phi) = 0$ for all $\phi \in H^\mu(S)$. It is known [2] that the set $A\phi|_{\forall\phi\in H^\mu(S)} = H^\mu(S)$ if $0 < \mu < 1$. Therefore, $q \in L^1(S)$ is orthogonal to the set $H^\mu(S)$, which is dense in $L^1(S)$. Consequently, $q = 0$ and $f_1 = f_2$. Lemma 1 is proved. □

Let us check that the right side of formula (1.2) makes sense. This side can be written as $\frac{1}{i\pi}\int_S\int_S dsdtf(s)\phi(t)\frac{1}{s-t}$. The integrand here is absolutely integrable over $S \times S$. Therefore, the order of integration can be changed and formula (1.2) makes sense.

There are other advantages of Definition 1. For example, it is easy to prove that the operator A is closed.

Lemma 2. *The operator* A *in* $L^1(S)$ *is closed.*

Proof. One has to prove that the graph $\{f, Af\}$ is a closed set in $L^1(S) \times L^1(S)$. Let $f_n \to f$ and $Af_n \to h$, convergence is in $L^1(S)$. Then, by Definition 1,

$$(f_n, A\phi) \to (f, A\phi) = (Af, \phi)$$

and

$$(Af_n, \phi) \to (h, \phi).$$

Therefore, $(Af - h, \phi) = 0\ \forall\phi \in H^\mu(S)$. Since $H^\mu(S)$ is dense in $L^1(S)$, it follows that $Af = h$. Thus, A is closed. □

However, A is not continuous in $L^1(S)$.

Example 1.2. Let us show that there is an $f \in L^1(S)$ such that $Af \notin L^1(S)$. Let $S = (-\infty, \infty)$, $\mathcal{F}(f) := \tilde{f}$, $\tilde{f} := \int_S e^{i\xi s} f(s)ds$, $\mathcal{F}(Af) = \mathcal{F}(f)\mathcal{F}(s^{-1}) = \tilde{f} i\pi sgn(\xi)$. We have used the known formula, see [3]: $\mathcal{F}(s^{-1}) = i\pi sgn(\xi)$, where $sgn(\xi) = 1$ if $\xi > 0$, $sgn(\xi) = -1$ if $\xi < 0$. The Fourier transform of $f \in L^1(S)$ is a continuous uniformly bounded function. Therefore, $\tilde{f} sgn(\xi)$ is not, in general, a continuous function at $\xi = 0$. Thus, if $\tilde{f}|_{\xi=0} \neq 0$, then the function $Af \notin L^1(S)$.

In the next Section we prove the Sokhotsky-Plemelj formulas for $f \in L^1(S)$, see Theorem 1 there.

2. Other results

a) Consider the equation

$$(\star)\ Af = h,\ h \in L^1(S).$$

If equation $(\star)$ is solvable, then its solution is unique. Indeed, if f_1 and f_2 are solutions, then $q = f_1 - f_2$ solves the equation $Aq = 0$. Taking its Fourier transform leads to the relation $\tilde{q} sgn(\xi) = 0$. Therefore, $\tilde{q} = 0$ for $\xi \neq 0$. Since $q \in L^1(S)$, it follows that $\tilde{q} = 0$ so $q = 0$ and $f_1 = f_2$. □

b) *Let us prove the generalization of the Sokhotsky-Plemelj formulas to the case when* $f \in L^1(S)$.

Let $\Phi(z) = \frac{1}{2\pi i}\int_S \frac{f(s)ds}{s-z}$ and $c := \frac{1}{2\pi i}$. Then

$$\Phi(z) = f(t)\nu(z) + \Psi(z), \quad \Psi(z) := c\int_S \frac{f(s) - f(t)}{s - z} ds, \quad t \in S, \quad \nu(z) := c\int_S \frac{ds}{s - z}, \tag{2.1}$$

and

$$\nu(z) = \begin{cases} 1, & z \in D, \\ \frac{1}{2}, & z \in S, \\ 0, & z \in D'. \end{cases} \tag{2.2}$$

One has

$$\Phi^+(t) = \lim_{z \to t, z \in D} \Phi(z) = f(t) + \Psi^+(t), \tag{2.3}$$

where $\Psi^+(t) = \lim_{z \to t, z \in D} \Psi(z)$ and

$$\Phi^-(t) = \lim_{z \to t, z \in D'} \Phi(z) = \Psi^-(t). \tag{2.4}$$

If $t \in S$, then one gets (see equation (2.2), the line $z \in S$) :

$$\Phi(t) = \frac{f(t)}{2} + \Psi(t) := \frac{f(t)}{2} + c \int_S \frac{f(s) - f(t)}{s - t} ds. \tag{2.5}$$

The $\Psi(t)$ is the value of $\Psi(z)$ at $z = t$.

The $\Psi(t)$ and $\Phi(t)$ are understood as in Definition 1.

If some equation holds almost everywhere with respect to the Lebesgue measure on S, then we write that this equation holds $a.e.$

From formulas (2.1)–(2.4) one derives:

$$\Phi^+(t) - \Phi^-(t) = f(t) + \Psi^+(t) - \Psi^-(t) \ a.e., \quad \Phi^+(t) + \Phi^-(t) = f(t) + \Psi^+(t) + \Psi^-(t) \ a.e. \tag{2.6}$$

In Lemma 3 we prove that

$$\Psi^+(t) = \Psi^-(t) = \Psi(t) \ a.e.$$

Therefore, formula (2.6) can be rewritten as:

$$\Phi^+(t) - \Phi^-(t) = f(t) \ a.e., \quad \Phi^+(t) + \Phi^-(t) = f(t) + 2\Psi(t) \ a.e. \tag{2.7}$$

From equation (2.7), (2.3) and (2.5) it follows that

$$\Phi^+(t) = \Phi(t) + \frac{f(t)}{2}, \quad \Phi^-(t) = \Phi(t) - \frac{f(t)}{2}, \ a.e. \tag{2.8}$$

where $\Phi(t) = \Psi(t)$.

Formulas (2.8) are the Sokhotsky-Plemelj formulas for $f \in L^1(S)$.

Theorem 1. *For $f \in L^1(S)$ formulas* (2.8) *hold.*

To finish the proof of Theorem 1 it is sufficient to prove Lemma 3.

Lemma 3. If $f \in L^1(S)$ and S is $C^{1,a}-$smooth, $0 < a \le 1$, then

$$\Psi^+(t) = \Psi^-(t) = \Psi(t) \ \ a.e. \tag{2.9}$$

Before proving Lemma 3 we prove Lemma 4.

Lemma 4. One has

(2.10)

Proof. Formula (2.10) is understood according to Definition 1. Let $f \in L^1(S)$, $\phi \in H^\mu(S)$, N_t be a unit normal to S directed inside D. Then

$$\lim_{\epsilon \to +0} \int_S dt \phi(t) \int_S \frac{f(s) ds}{s - t - iN_t \epsilon} := \int_S dt \phi(t) \int_S \frac{f(s) ds}{s - t - i0}.$$

Furthermore,

$$\int_S dt\phi(t)\int_S \frac{f(s)ds}{s-t-i0} = \int_S dt\phi(t)\int_S \frac{f(s)ds}{s-t} + i\pi\int_S \phi(t)f(t)dt, \quad \forall\phi\in H^\mu(S), \tag{2.11}$$

and

$$\int_S dt\phi(t)\int_S \frac{f(s)ds}{s-t} = \int_S dsf(s)\int_S \frac{\phi(t)dt}{s-t}, \quad \forall\phi\in H^\mu(S). \tag{2.12}$$

We have proved formula (2.10) according to Definition 1 with the minus sign. Similarly one proves this formula with the plus sign. Lemma 4 is proved. □

In [3], p. 83, there is a formula $\frac{1}{x-i0} = \frac{1}{x} + i\pi\delta(x)$ understood in the sense of distributions. The formula in Lemma 4 is of the similar type. The Sokhotsky-Plemelj formulas (2.8) were derived in [2] and [5] under the assumption that $f\in H^\mu(S)$. Under such an assumption, these formulas hold everywhere, not almost everywhere.

Proof of Lemma 3. By Definition 1 one has (neglecting $\frac{1}{2\pi i}$ and denoting by $\int$ integration over $S\times S$):

$$\lim_{\epsilon\to 0}\int \frac{[f(s)-f(t)]\phi(t)}{s-t\pm iN_t\epsilon}dsdt = \int \frac{[f(s)-f(t)]\phi(t)}{s-t}dsdt \mp i\pi J = (\psi,\phi), \tag{2.13}$$

where

$$J := \int [f(s)-f(t)]\phi(t)\delta(s-t)dsdt = \int_S\int_S [f(s)-f(t)]\phi(t)\delta(s-t)dsdt = 0. \tag{2.14}$$

For $f\in H^\mu(S)$ formula (2.14) is trivial by the standard definition of the delta-function.

For $f\in L^1(S)$ we consider $\delta(s-t)$ as the kernel of the identity operator, so

$$\int_S f(s)\delta(s-t)ds = f(t)\ a.e. \quad f\in L^1(S).$$

Lemma 3 is proved. □

c) Let $z\in D$. The following question is of interest:

When is the boundary value $\Phi^+(t)$ of $\Phi(z)$ on S equal to f a.e.?

Equation (2.8) yields a necessary and sufficient condition for an answer to the above question:

$\Phi^+(t) = f(t)$ *iff* $\Phi(z) = 0,\ z\in D_-$, *and* $f(t) = \frac{1}{i\pi}\int_S \frac{f(s)}{s-t}ds \quad a.e.$

Indeed, $\Phi(z) = 0,\ z\in D_-$, is equivalent to $\Phi(t) = \frac{f(t)}{2}\ a.e.$, so $\Phi^+(t) = f(t)\ a.e.$

If one wants to formulate a necessary and sufficient condition for $f(s)\in L^1(S)$ to be a boundary value of an analytic in D_- function $\Phi(z),\ \Phi(\infty)=0$, then an argument, similar to the above yields the following conditions:

$$f(t) = -\frac{1}{i\pi}\int_S \frac{f(s)}{s-t}ds \quad a.e. \tag{2.15}$$

If equation (2.15) holds, then $\Phi^+(t) = 0$ and, consequently, $\Phi(z) = 0$ if $z\in D_+ = D$.

Remark 1. *If $\Phi^+(t) = f(t)\ a.e., f\in L^1(S)$, then $Af\in L^1(S)$, where $Af := \frac{1}{i\pi}\int_S \frac{f(s)}{s-t}ds\ a.e.$ If $-\Phi^-(t) = f(t)\ a.e., f\in L^1(S)$ and $\Phi(\infty) = 0$, then $Af\in L^1(S)$. Since for some $f\in L^1(S)$ one does not have $\Phi(z) = 0,\ z\in D_-$, it follows that* not every $f\in L^1(S)$ is a boundary value of an analytic function in D.

3. Conclusion

A new definition of singular integral operator in $L^1(S)$ is given. Sokhotsky-Plemelj formulas are derived for $f\in L^1(S)$. Other results are obtained.

4. CONFLICT OF INTEREST

There is no conflict of interest.

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