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TRIGONOMETRIC GENERATED L_p DEGREE OF APPROXIMATION BY SMOOTH PICARD SINGULAR INTEGRAL OPERATORS

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ABSTRACT. In this article we continue the study of smooth Picard singular integral operators that started in [3], see there chapters 10-14. This time the foundation of our research is a trigonometric Taylor's formula. We establish the L_p convergence of our operators to the unit operator with rates via Jackson type inequalities engaging the first L_p modulus of continuity. Of interest here is a residual appearing term. Note that our operators are not positive.

1. INTRODUCTION

We are motivated by [2], [3] chapters 10-14, and [4], [5]. We use a trigonometric new Taylor formula from [4], see also [5]. Here we consider some very general operators, the smooth Picard singular integral operators over the real line and we study further their L_p, $p \geq 1$, convergencnce properties quantitatively. We establish related inequalities involving the first L_p, $p \geq 1$, modulus of continuity with respect to L_p, $p \geq 1$, norm. We provide detailed proofs.

2. RESULTS

By [4], [5], for $f \in C^2(\mathbb{R})$ and $a, x \in \mathbb{R}$, we have by the trigonometric Taylor formula

$$f(x) - f(a) = f'(a)\sin(x-a) + 2f''(a)\sin^2\left(\frac{x-a}{2}\right) + \tag{1}$$

$$\int_a^x [(f''(t) + f(t)) - (f''(a) + f(a))]\sin(x-t)\,dt.$$

For $r \in \mathbb{N}$ and $n \in \mathbb{Z}^+$, we set

$$\alpha_j := \begin{cases} (-1)^{r-j}\binom{r}{j} j^{-n}, & j = 1, ..., r. \\ 1 - \sum_{j=1}^{r} (-1)^{r-j}\binom{r}{j} j^{-n}, & j = 0, \end{cases} \tag{2}$$

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that is

$$\sum_{j=0}^{r} \alpha_j = 1. \tag{3}$$

Here we consider both $f, f'' \in L_p(\mathbb{R}) \cap C(\mathbb{R})$, $1 \leq p < \infty$.

For $x \in \mathbb{R}$, $\xi > 0$ we consider the Lebesgue integrals, so called smooth Picard operators

$$P_{r,\xi}(f,x) = \frac{1}{2\xi} \int_{-\infty}^{\infty} \left(\sum_{j=0}^{r} \alpha_j f(x + jt) \right) e^{-\frac{|t|}{\xi}} dt, \tag{4}$$

see [2]; $P_{r,\xi}$ are not in general positive operators, see [3].

We notice by

$$\frac{1}{2\xi} \int_{-\infty}^{\infty} e^{-\frac{|t|}{\xi}} dt = 1, \tag{5}$$

that

$$P_{r,\xi}(c,x) = c, \text{ where } c \text{ is a constant}, \tag{6}$$

and

$$P_{r,\xi}(f,x) - f(x) = \frac{1}{2\xi} \left(\sum_{j=0}^{r} \alpha_j \int_{-\infty}^{\infty} (f(x + jt) - f(x)) e^{-\frac{|t|}{\xi}} dt \right). \tag{7}$$

Denote by

$$\omega_1(f,h)_p := \sup_{\substack{t \in \mathbb{R} \\ |t| \leq h}} \|f(x + t) - f(x)\|_{p,x}, \tag{8}$$

the first L_p modulus of smoothness of f, $1 \leq p < \infty$.

By (1) we get that

$$f(x + jt) - f(x) = f'(x) \sin(jt) + 2f''(x) \sin^2\left(\frac{jt}{2}\right) +$$
$$\int_x^{x+jt} [(f''(s) + f(s)) - (f''(x) + f(x))] \sin(x + jt - s) \, ds, \tag{9}$$

or better,

$$f(x + jt) - f(x) = f'(x) \sin(jt) + 2f''(x) \sin^2\left(\frac{jt}{2}\right) +$$
$$\int_0^{jt} [(f''(x + z) + f(x + z)) - (f''(x) + f(x))] \sin(jt - z) \, dz. \tag{10}$$

Furthermore, it holds

$$\sum_{j=0}^{r} \alpha_j [f(x + jt) - f(x)] =$$
$$f'(x) \sum_{j=0}^{r} \alpha_j \sin(jt) + 2f''(x) \sum_{j=0}^{r} \alpha_j \sin^2\left(\frac{jt}{2}\right) +$$
$$\sum_{j=0}^{r} \alpha_j \int_0^{jt} [(f''(x + z) + f(x + z)) - (f''(x) + f(x))] \sin(jt - z) \, dz, \tag{11}$$

or better

$$\sum_{j=0}^{r} \alpha_j [f(x + jt) - f(x)] =$$

$$f'(x)\sum_{j=0}^{r}\alpha_j\sin(jt)+2f''(x)\sum_{j=0}^{r}\alpha_j\sin^2\left(\frac{jt}{2}\right)+$$

$$\sum_{j=0}^{r}\alpha_j j\int_0^t\left[\left(f''(x+jw)+f(x+jw)\right)-\left(f''(x)+f(x)\right)\right]\sin j(t-w)\,dw. \quad (12)$$

Call

$$R:=R(t):=$$

$$\sum_{j=0}^{r}\alpha_j j\int_0^t\left[\left(f''(x+jw)+f(x+jw)\right)-\left(f''(x)+f(x)\right)\right]\sin j(t-w)\,dw, \quad (13)$$

$\forall\, t\in\mathbb{R}$.

We notice that

$$\Delta(x):=P_{r,\xi}(f,x)-f(x)-f'(x)\sum_{j=0}^{r}\alpha_j\frac{1}{2\xi}\left(\int_{-\infty}^{\infty}\sin(jt)\,e^{-\frac{|t|}{\xi}}dt\right) \quad (14)$$

$$-2f''(x)\sum_{j=0}^{r}\alpha_j\frac{1}{2\xi}\left(\int_{-\infty}^{\infty}\sin^2\left(\frac{jt}{2}\right)e^{-\frac{|t|}{\xi}}dt\right)$$

$$=\frac{1}{2\xi}\int_{-\infty}^{\infty}R(t)\,e^{-\frac{|t|}{\xi}}dt.$$

Next we simplify (14).

We observe that

$$\int_{-\infty}^{\infty}\sin(jt)\,e^{-\frac{|t|}{\xi}}dt=\int_{-\infty}^{0}\sin(jt)\,e^{-\frac{|t|}{\xi}}dt+\int_{0}^{\infty}\sin(jt)\,e^{-\frac{|t|}{\xi}}dt. \quad (15)$$

Notice $-\infty\leq t\leq 0\Rightarrow\infty\geq -t\geq 0$. So that

$$-\int_{-\infty}^{0}\sin(j(-(-t)))\,e^{-\frac{|t|}{\xi}}d(-t)=-\int_{-\infty}^{0}\left(-\sin(j(-t))\right)e^{-\frac{|t|}{\xi}}d(-t)= \quad (16)$$

$$\int_{-\infty}^{0}\left(\sin j(-t)\right)e^{-\frac{|t|}{\xi}}d(-t)=\int_{\infty}^{0}\sin j(t)\,e^{-\frac{|t|}{\xi}}dt=-\int_{0}^{\infty}\sin(jt)\,e^{-\frac{|t|}{\xi}}dt.$$

Therefore, it is

$$\int_{-\infty}^{\infty}\sin(jt)\,e^{-\frac{|t|}{\xi}}dt=0,\quad j=0,1,...,r. \quad (17)$$

Furthermore, we have that

$$\int_{-\infty}^{\infty}\sin^2\left(\frac{jt}{2}\right)e^{-\frac{|t|}{\xi}}dt=2\int_{0}^{\infty}\sin^2\left(\frac{jt}{2}\right)e^{-\frac{t}{\xi}}dt,\quad j=0,1,...,r. \quad (18)$$

The last follows by

$$\int_{-\infty}^{0}\sin^2\left(\frac{jt}{2}\right)e^{-\frac{|t|}{\xi}}dt=-\int_{-\infty}^{0}\left(-\sin\left(\frac{j(-t)}{2}\right)\right)^2e^{-\frac{|-t|}{\xi}}d(-t)\overset{(z=-t)}{=} \quad (19)$$

$$-\int_{\infty}^{0}\sin^2\left(\frac{jz}{2}\right)e^{-\frac{|z|}{\xi}}dz=\int_{0}^{\infty}\sin^2\left(\frac{jz}{2}\right)e^{-\frac{z}{\xi}}dz.$$

Next, we calculate

$$\int_{0}^{\infty}\sin^2\left(\frac{jt}{2}\right)e^{-\frac{|t|}{\xi}}dt=\xi\int_{0}^{\infty}\sin^2\left(\left(\frac{j\xi}{2}\right)\frac{t}{\xi}\right)e^{-\frac{t}{\xi}}d\frac{t}{\xi}=$$

(call $\frac{t}{\xi} =: x$ and $\frac{j\xi}{2} =: a$)

$$\xi \int_0^{\infty} \sin^2 (ax) e^{-x} dx = \tag{20}$$

(by Wolfram Alpha Computational Inteligence)

$$\xi \left(\frac{2a^2}{4a^2+1} \right) = \frac{j^2 \xi^3}{2(j^2\xi^2+1)}.$$

Thus, we find that

$$\int_{-\infty}^{\infty} \sin^2 \left(\frac{jt}{2} \right) e^{-\frac{|t|}{\xi}} dt = \frac{j^2\xi^3}{j^2\xi^2+1}, \quad j = 0, 1, ..., r. \tag{21}$$

Consequently, it holds ($\xi > 0$)

$$\frac{1}{2\xi} \int_{-\infty}^{\infty} \sin^2 \left(\frac{jt}{2} \right) e^{-\frac{|t|}{\xi}} dt = \frac{j^2\xi^2}{2(j^2\xi^2+1)} \to 0, \text{ as } \xi \to 0, \quad j = 0, 1, ..., r. \tag{22}$$

Eventually we obtain

$$-2f''(x) \sum_{j=0}^{r} \alpha_j \frac{1}{2\xi} \left(\int_{-\infty}^{\infty} \sin^2 \left(\frac{jt}{2} \right) e^{-\frac{|t|}{\xi}} dt \right) = -f''(x) \left(\sum_{j=0}^{r} \alpha_j \frac{j^2}{(j^2\xi^2+1)} \right) \xi^2. \tag{23}$$

Consequently, based on the above, it is

$$\Delta(x) = P_{r,\xi}(f,x) - f(x) - f''(x) \left(\sum_{j=0}^{r} \alpha_j \frac{j^2}{(j^2\xi^2+1)} \right) \xi^2, \quad \xi > 0. \tag{24}$$

We present our first result, L_p approximation, $p > 1$.

Theorem 2.1. *Let* $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$, $\xi > 0$, *and both* $f, f'' \in L_p(\mathbb{R}) \cap C(\mathbb{R})$, $x \in \mathbb{R}$. *Then*

$$\|\Delta(x)\|_p = \left\| P_{r,\xi}(f) - f - f'' \left(\sum_{j=0}^{r} \alpha_j \frac{j^2}{(j^2\xi^2+1)} \right) \xi^2 \right\|_p \leq$$

$$\left(\frac{4(r+1)}{q(q+1)} \right)^{\frac{1}{q}} \left(\frac{2}{p} \right)^{2+\frac{1}{p}} \omega_1 (f''+f, \xi)_p \, \xi^2$$

$$\left\{ \sum_{j=0}^{r} |\alpha_j|^p j^{2p} \left[\Gamma(qp-q+p+1) + \frac{(2j)^p}{p^p(p+1)} \Gamma(qp-q+2p+1) \right] \right\}^{\frac{1}{p}} \to 0, \tag{25}$$

as $\xi \to 0$.

Above Γ *stands for the gamma function.*

Proof. Let

$$I := \int_0^t \left[(f''(x+jw) + f(x+jw)) - (f''(x) + f(x)) \right] \sin j(t-w) \, dw, \ \forall \, t \in \mathbb{R}. \tag{26}$$

For $t < 0$, we have that

$$|I| = \left| \int_t^0 \left[(f''(x+jw) + f(x+jw)) - (f''(x) + f(x)) \right] \sin j(t-w) \, dw \right| \leq$$

$$\int_t^0 \left|\left(f''(x+jw)+f(x+jw)\right)-\left(f''(x)+f(x)\right)\right| \left|\sin j(t-w)\right| dw \leq \tag{27}$$

$$j\int_t^0 \left|\left(f''(x+jw)+f(x+jw)\right)-\left(f''(x)+f(x)\right)\right| (w-t)\, dw =$$

$$-j\int_t^0 \left|\left(f''(x-j(-w))+f(x-j(-w))\right)-\left(f''(x)+f(x)\right)\right| (-t-(-w))\, d(-w)$$

$(t \leq w \leq 0 \Rightarrow -t \geq -w =: \theta \geq 0)$

$$= -j\int_{-t}^0 \left|\left(f''(x-j\theta)+f(x-j\theta)\right)-\left(f''(x)+f(x)\right)\right| (-t-\theta)\, d\theta =$$

$$j\int_0^{-t} \left|\left(f''(x-j\theta)+f(x-j\theta)\right)-\left(f''(x)+f(x)\right)\right| (-t-\theta)\, d\theta = \tag{28}$$

$$j\int_0^{|t|} \left|\left(f''(x+sign(t)\, j\theta)+f(x+sign(t)\, j\theta)\right)-\left(f''(x)+f(x)\right)\right| (|t|-\theta)\, d\theta.$$

So, we have proved that

$$|I| \leq j\int_0^{|t|} \left|\left(f''(x+sign(t)\, j\theta)+f(x+sign(t)\, j\theta)\right)-\left(f''(x)+f(x)\right)\right| (|t|-\theta)\, d\theta, \tag{29}$$

$\forall\, t \in \mathbb{R}$,

and by (13),

$$|R(t)| \leq \sum_{j=0}^{r} |\alpha_j|\, j^2$$

$$\int_0^{|t|} \left|\left(f''(x+jsign(t)\,\theta)+f(x+jsign(t)\,\theta)\right)-\left(f''(x)+f(x)\right)\right| (|t|-\theta)\, d\theta, \tag{30}$$

$\forall\, t \in \mathbb{R}$.

By (14) we have

$$\Delta(x) = \frac{1}{2\xi}\int_{-\infty}^{\infty} R(t)\, e^{-\frac{|t|}{\xi}}\, dt. \tag{31}$$

Hence it holds ($p, q > 1 : \frac{1}{p}+\frac{1}{q} = 1$),

$$\int_{-\infty}^{\infty} |\Delta(x)|^p\, dx = \frac{1}{(2\xi)^p}\int_{-\infty}^{\infty}\left|\int_{-\infty}^{\infty} R(t)\, e^{-\frac{|t|}{\xi}}\, dt\right|^p dx \leq$$

$$\frac{1}{(2\xi)^p}\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} |R(t)|\, e^{-\frac{|t|}{\xi}}\, dt\right)^p dx = \tag{32}$$

$$\frac{1}{(2\xi)^p}\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} |R(t)|\, e^{-\frac{|t|}{2\xi}} e^{-\frac{|t|}{2\xi}}\, dt\right)^p dx \leq$$

$$\frac{1}{(2\xi)^p}\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} |R(t)|^p\, e^{-\frac{|pt|}{2\xi}}\, dt\right)\left(\int_{-\infty}^{\infty} e^{-\frac{|qt|}{2\xi}}\, dt\right)^{\frac{p}{q}} dx =$$

$$\frac{1}{(2\xi)^p}\left(\frac{4\xi}{q}\right)^{\frac{p}{q}}\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} |R(t)|^p\, e^{-\frac{|pt|}{2\xi}}\, dt\right) dx =$$

$$\frac{2^{p-2}\xi^{-1}}{q^{p-1}}\left(\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} |R(t)|^p\, e^{-\frac{|pt|}{2\xi}}\, dt\right) dx\right) =: (*)\,.$$

But, we need to treat

$$|R(t)| \leq \sum_{j=0}^{r} |\alpha_j| j^2 \tag{33}$$

$$\left(\int_0^{|t|} |(f''(x + jsign(t)\theta) + f(x + jsign(t)\theta)) - (f''(x) + f(x))|^p d\theta \right)^{\frac{1}{p}}$$

$$\left(\int_0^{|t|} (|t| - \theta)^q d\theta \right)^{\frac{1}{q}} =$$

$$\sum_{j=0}^{r} |\alpha_j| j^2 \left(\int_0^{|t|} |(f''(x + jsign(t)\theta) + f(x + jsign(t)\theta)) - (f''(x) + f(x))|^p d\theta \right)^{\frac{1}{p}}$$

$$\frac{|t|^{q+\frac{1}{q}}}{(q+1)^{\frac{1}{q}}} =$$

$$\left\{ \sum_{j=0}^{r} \left(\left(|\alpha_j| j^2 \right)^p \right)^{\frac{1}{p}} \right.$$

$$\left. \left(\int_0^{|t|} |(f''(x + jsign(t)\theta) + f(x + jsign(t)\theta)) - (f''(x) + f(x))|^p d\theta \right)^{\frac{1}{p}} \right\}$$

$$\frac{|t|^{q+\frac{1}{q}}}{(q+1)^{\frac{1}{q}}} \leq \tag{34}$$

$(0 < \frac{1}{p} < 1)$

$$\left\{ (r+1)^{\frac{1}{q}} \right.$$

$$\left. \left(\sum_{j=0}^{r} |\alpha_j|^p j^{2p} \int_0^{|t|} |(f''(x + jsign(t)\theta) + f(x + jsign(t)\theta)) - (f''(x) + f(x))|^p d\theta \right)^{\frac{1}{p}} \right\}$$

$$\frac{|t|^{q+\frac{1}{q}}}{(q+1)^{\frac{1}{q}}}.$$

Hence, we find that

$$|R(t)|^p \leq \left(\frac{r+1}{q+1} \right)^{\frac{p}{q}} \left[\sum_{j=0}^{r} |\alpha_j|^p j^{2p} \right.$$

$$\left. \left(\int_0^{|t|} |(f''(x + jsign(t)\theta) + f(x + jsign(t)\theta)) - (f''(x) + f(x))|^p d\theta \right) \right] |t|^{\frac{(q+1)p}{q}}. \tag{35}$$

Therefore, we get

$$(*) \leq \left(\frac{2^{p-2}\xi^{-1}}{q^{p-1}} \left(\frac{r+1}{q+1} \right)^{\frac{p}{q}} \right) \left\{ \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} \left[\sum_{j=0}^{r} |\alpha_j|^p j^{2p} \right. \right. \right.$$

$$\int_0^{|t|} \left|\left(f''\left(x + jsign\left(t\right)\theta\right) + f\left(x + jsign\left(t\right)\theta\right)\right) - \left(f''\left(x\right) + f\left(x\right)\right)\right|^p d\theta\Bigg]$$

$$|t|^{\frac{(q+1)p}{q}} e^{-\frac{|pt|}{2\xi}} dt\Bigg) dx\Bigg\} \tag{36}$$

(set $c_1 = \frac{2^{p-2}\xi^{-1}}{q^{p-1}} \left(\frac{r+1}{q+1}\right)^{\frac{p}{q}}$)

$$= c_1 \left\{ \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} \left[\sum_{j=0}^{r} |\alpha_j|^p j^{2p} \right.\right.\right.$$

$$\left.\left. \int_0^{|t|} \left|\left(f''\left(x + jsign\left(t\right)\theta\right) + f\left(x + jsign\left(t\right)\theta\right)\right) - \left(f''\left(x\right) + f\left(x\right)\right)\right|^p d\theta \right] dx \right)$$

$$\left. |t|^{\frac{(q+1)p}{q}} e^{-\frac{|pt|}{2\xi}} dt \right\}$$

$$= c_1 \left\{ \int_{-\infty}^{\infty} \left[\sum_{j=0}^{r} |\alpha_j|^p j^{2p} \right.\right. \tag{37}$$

$$\left. \int_0^{|t|} \left(\int_{-\infty}^{\infty} \left|\left(f''\left(x + jsign\left(t\right)\theta\right) + f\left(x + jsign\left(t\right)\theta\right)\right) - \left(f''\left(x\right) + f\left(x\right)\right)\right|^p dx \right) d\theta \right]$$

$$\left. |t|^{\frac{(q+1)p}{q}} e^{-\frac{|pt|}{2\xi}} dt \right\} \leq$$

$(\xi > 0)$

$$c_1 \left\{ \int_{-\infty}^{\infty} \left[\sum_{j=0}^{r} |\alpha_j|^p j^{2p} \left(\int_0^{|t|} \omega_1 \left(f'' + f, \xi \frac{j\theta}{\xi} \right)_p^p d\theta \right) \right] \right.$$

$$\left. |t|^{\frac{(q+1)p}{q}} e^{-\frac{|pt|}{2\xi}} dt \right\} \leq$$

$$\omega_1 \left(f'' + f, \xi\right)_p^p c_1 \left\{ \int_{-\infty}^{\infty} \left[\sum_{j=0}^{r} |\alpha_j|^p j^{2p} \left(\int_0^{|t|} \left(1 + \frac{j}{\xi}\theta\right)^p d\theta \right) \right] \right.$$

$$\left. |t|^{(q+1)(p-1)} e^{-\frac{|pt|}{2\xi}} dt \right\} \leq \tag{38}$$

$$c_1 \omega_1 \left(f'' + f, \xi\right)_p^p \left\{ \int_{-\infty}^{\infty} \left[\sum_{j=0}^{r} |\alpha_j|^p j^{2p} 2^{p-1} \left(\int_0^{|t|} \left(1 + \frac{j^p}{\xi^p}\theta^p\right) d\theta \right) \right] \right.$$

$$\left. |t|^{(q+1)(p-1)} e^{-\frac{|pt|}{2\xi}} dt \right\} =$$

$$2^{p-1} c_1 \omega_1 \left(f'' + f, \xi\right)_p^p$$

$$\left\{ \int_{-\infty}^{\infty} \left[\sum_{j=0}^{r} |\alpha_j|^p j^{2p} \left(|t| + \frac{j^p}{\xi^p} \frac{|t|^{p+1}}{p+1} \right) \right] |t|^{(q+1)(p-1)} e^{-\frac{|pt|}{2\xi}} dt \right\} =$$

$$2^{p-1} c_1 \omega_1 \left(f'' + f, \xi\right)_p^p$$

$$\left\{ \sum_{j=0}^{r} |\alpha_j|^p j^{2p} \int_{-\infty}^{\infty} \left(|t| + \frac{j^p}{\xi^p} \frac{|t|^{p+1}}{p+1} \right) |t|^{(q+1)(p-1)} e^{-\frac{|pt|}{2\xi}} dt \right\} = \tag{39}$$

$$2^p c_1 \omega_1 \left(f'' + f, \xi\right)_p^p$$

$$\left\{\sum_{j=0}^{r}|\alpha_j|^p j^{2p}\int_0^{\infty}\left(t+\frac{j^p}{\xi^p(p+1)}t^{p+1}\right)t^{(q+1)(p-1)}e^{-\frac{pt}{2\xi}}dt\right\}=$$

$$2^p c_1\omega_1(f''+f,\xi)_p^p$$

$$\left\{\sum_{j=0}^{r}|\alpha_j|^p j^{2p}\left[\int_0^{\infty}t^{(q+1)(p-1)+1}e^{-\frac{pt}{2\xi}}dt+\frac{j^p}{\xi^p(p+1)}\int_0^{\infty}t^{(q+1)(p-1)+p+1}e^{-\frac{pt}{2\xi}}dt\right]\right\}= \tag{40}$$

$$2^p c_1\omega_1(f''+f,\xi)_p^p\left\{\sum_{j=0}^{r}|\alpha_j|^p j^{2p}\left[\xi^{qp-q+p+1}\int_0^{\infty}\left(\frac{t}{\xi}\right)^{(q+1)(p-1)+1}e^{-\left(\frac{p}{2}\right)\frac{t}{\xi}}d\frac{t}{\xi}\right.\right.$$

$$\left.\left.+\frac{j^p}{\xi^p(p+1)}\xi^{qp-q+2p+1}\int_0^{\infty}\left(\frac{t}{\xi}\right)^{qp-q+2p}e^{-\left(\frac{p}{2}\right)\frac{t}{\xi}}d\frac{t}{\xi}\right]\right\}=$$

(above it is $(q+1)(p-1)+1=qp-q+p>0$, $(q+1)(p-1)+p+1=qp-q+2p>0$)

$$2^p c_1\omega_1(f''+f,\xi)_p^p\xi^{qp-q+p+1}\left[\sum_{j=0}^{r}|\alpha_j|^p j^{2p}\left\{\int_0^{\infty}x^{qp-q+p}e^{-\left(\frac{p}{2}\right)x}dx\right.\right. \tag{41}$$

$$\left.\left.+\frac{j^p}{(p+1)}\int_0^{\infty}x^{qp-q+2p}e^{-\left(\frac{p}{2}\right)x}dx\right]\right\}=$$

(next we use, for $a,b>0$ that it holds $\int_0^{\infty}x^a e^{-bx}dx=b^{-a-1}\Gamma(a+1)$, by Wolfram Alpha computational intelligence)

$$2^p c_1\omega_1(f''+f,\xi)_p^p\xi^{qp-q+p+1}\left[\sum_{j=0}^{r}|\alpha_j|^p j^{2p}\left\{\left(\frac{p}{2}\right)^{-qp+q-p-1}\Gamma(qp-q+p+1)\right.\right.$$

$$\left.\left.+\frac{j^p}{(p+1)}\left(\frac{p}{2}\right)^{-qp+q-2p-1}\Gamma(qp-q+2p+1)\right]\right\}=$$

$$\frac{2^{2(p-1)}}{q^{p-1}}\left(\frac{r+1}{q+1}\right)^{\frac{p}{q}}\omega_1(f''+f,\xi)_p^p\xi^{qp-q+p} \tag{42}$$

$$\left[\sum_{j=0}^{r}|\alpha_j|^p j^{2p}\left\{\left(\frac{p}{2}\right)^{-qp+q-p-1}\left(\Gamma(qp-q+p+1)+\frac{j^p}{(p+1)}\left(\frac{p}{2}\right)^{-p}\Gamma(qp-q+2p+1)\right)\right\}\right].$$

So for $p,q>1:\frac{1}{p}+\frac{1}{q}=1$, $\xi>0$, we have proved:

$$\int_{-\infty}^{\infty}|\Delta(x)|^p dx\leq\frac{2^{2(p-1)}}{q^{p-1}}\left(\frac{r+1}{q+1}\right)^{\frac{p}{q}}\omega_1(f''+f,\xi)_p^p\xi^{qp-q+p}\left(\frac{p}{2}\right)^{-qp+q-p-1} \tag{43}$$

$$\left\{\sum_{j=0}^{r}|\alpha_j|^p j^{2p}\left[\Gamma(qp-q+p+1)+\frac{(2j)^p}{p^p(p+1)}\Gamma(qp-q+2p+1)\right]\right\}.$$

The proof of the theorem is now completed. □

Next comes the L_1 related approximation.

Theorem 2.2. *Let $\xi > 0$, both $f, f'' \in L_1(\mathbb{R}) \cap C(\mathbb{R})$, $x \in \mathbb{R}$. Then*

$$\|\Delta(x)\|_1 \leq \left[\sum_{j=0}^{r} |\alpha_j| j^2 (2+3j)\right] \omega_1 (f''+f,\xi)_1 \xi^2 \to 0, \text{ as } \xi \to 0. \tag{44}$$

Proof. We have that

$$|\Delta(x)| = \frac{1}{2\xi} \left| \int_{-\infty}^{\infty} R(t) e^{-\frac{|t|}{\xi}} dt \right| \leq \frac{1}{2\xi} \int_{-\infty}^{\infty} |R(t)| e^{-\frac{|t|}{\xi}} dt \leq$$

$$\frac{1}{2\xi} \sum_{j=0}^{r} |\alpha_j| j^2 \int_{-\infty}^{\infty} \left(\int_0^{|t|} |(f''(x+j \operatorname{sign}(t)\theta) + f(x+j \operatorname{sign}(t)\theta)) - \right. \tag{45}$$

$$\left. (f''(x)+f(x))| (|t|-\theta) d\theta \right) e^{-\frac{|t|}{\xi}} dt.$$

Thus, we get

$$\|\Delta(x)\|_1 = \int_{-\infty}^{\infty} |\Delta(x)| dx \leq$$

$$\frac{1}{2\xi} \sum_{j=0}^{r} |\alpha_j| j^2 \left(\int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} \left(\int_0^{|t|} |(f''(x+j \operatorname{sign}(t)\theta) + f(x+j \operatorname{sign}(t)\theta)) - \right.\right.\right. \tag{46}$$

$$\left.\left.\left. (f''(x)+f(x))| (|t|-\theta) d\theta \right) e^{-\frac{|t|}{\xi}} dt \right) dx \right) \leq$$

$$\frac{1}{2\xi} \sum_{j=0}^{r} |\alpha_j| j^2 \left(\int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} \left(|t| \int_0^{|t|} |(f''(x+j \operatorname{sign}(t)\theta) + f(x+j \operatorname{sign}(t)\theta)) - \right.\right.\right.$$

$$\left.\left.\left. (f''(x)+f(x))| d\theta \right) e^{-\frac{|t|}{\xi}} dt \right) dx \right) =$$

$$\frac{1}{2\xi} \sum_{j=0}^{r} |\alpha_j| j^2 \left(\int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} \left(|t| \int_0^{|t|} |(f''(x+j \operatorname{sign}(t)\theta) + f(x+j \operatorname{sign}(t)\theta)) - \right.\right.\right.$$

$$\left.\left.\left. (f''(x)+f(x))| d\theta \right) e^{-\frac{|t|}{\xi}} dx \right) dt \right) = \tag{47}$$

$$\frac{1}{2\xi} \sum_{j=0}^{r} |\alpha_j| j^2 \left(\int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} \left(\int_0^{|t|} |(f''(x+j \operatorname{sign}(t)\theta) + f(x+j \operatorname{sign}(t)\theta)) - \right.\right.\right.$$

$$\left. (f''(x)+f(x))| d\theta) dx) |t| e^{-\frac{|t|}{\xi}} dt \right) =$$

$$\frac{1}{2\xi} \sum_{j=0}^{r} |\alpha_j| j^2 \left(\int_{-\infty}^{\infty} \left(\int_0^{|t|} \left(\int_{-\infty}^{\infty} |(f''(x+j \operatorname{sign}(t)\theta) + f(x+j \operatorname{sign}(t)\theta)) - \right.\right.\right.$$

$$\left. (f''(x)+f(x))| dx) d\theta) |t| e^{-\frac{|t|}{\xi}} dt \right) \leq$$

$$\frac{1}{2\xi} \sum_{j=0}^{r} |\alpha_j| j^2 \left(\int_{-\infty}^{\infty} \left(\int_0^{|t|} \omega_1 (f''+f, j\theta)_1 d\theta \right) |t| e^{-\frac{|t|}{\xi}} dt \right) \leq$$

$$\frac{\omega_1(f''+f,\xi)_1}{2\xi} \sum_{j=0}^{r} |\alpha_j| j^2 \left(\int_{-\infty}^{\infty} \left(\int_0^{|t|} \left(1+\frac{j}{\xi}\theta\right) d\theta \right) |t| e^{-\frac{|t|}{\xi}} dt \right) =$$

$$\frac{\omega_1(f''+f,\xi)_1}{2\xi} \sum_{j=0}^{r} |\alpha_j| j^2 \left(\int_{-\infty}^{\infty} \left[|t| + \frac{j}{\xi} \frac{|t|^2}{2} \right] |t| e^{-\frac{|t|}{\xi}} dt \right) = \tag{48}$$

$$\xi^2\omega_1\left(f''+f,\xi\right)_1\sum_{j=0}^{r}\left|\alpha_j\right|j^2\left(\int_0^\infty\left[\frac{t}{\xi}+\frac{j}{2}\left(\frac{t}{\xi}\right)^2\right]\frac{t}{\xi}e^{-\frac{t}{\xi}}d\frac{t}{\xi}\right)=$$

$$\xi^2\omega_1\left(f''+f,\xi\right)_1\sum_{j=0}^{r}\left|\alpha_j\right|j^2\left[\int_0^\infty\left(t+\frac{j}{2}t^2\right)te^{-t}dt\right]=$$

$$\xi^2\omega_1\left(f''+f,\xi\right)_1\sum_{j=0}^{r}\left|\alpha_j\right|j^2\left[\int_0^\infty t^2e^{-t}dt+\frac{j}{2}\int_0^\infty t^3e^{-t}dt\right]=$$

$$\xi^2\omega_1\left(f''+f,\xi\right)_1\sum_{j=0}^{r}\left|\alpha_j\right|j^2\left[2+\frac{j}{2}\cdot 6\right]=$$

$$\xi^2\omega_1\left(f''+f,\xi\right)_1\sum_{j=0}^{r}\left|\alpha_j\right|j^2\left[2+3j\right]. \tag{49}$$

Above we used that

$$\int_0^\infty t^2e^{-t}dt=2 \text{ and } \int_0^\infty t^3e^{-t}dt=6.$$

The theorem is proved. □

We finish with the following

Corollary 2.3. *(to Theorem 2.1) It holds*

$$\left\|P_{r,\xi}(f)-f\right\|_p\leq\left\|f''\right\|_p\left(\sum_{j=0}^{r}\left|\alpha_j\right|\frac{j^2}{\left(j^2\xi^2+1\right)}\right)\xi^2+$$

$$\left(\frac{4(r+1)}{q(q+1)}\right)^{\frac{1}{q}}\left(\frac{2}{p}\right)^{2+\frac{1}{p}}\omega_1\left(f''+f,\xi\right)_p\xi^2 \tag{50}$$

$$\left\{\sum_{j=0}^{r}\left|\alpha_j\right|^p j^{2p}\left[\Gamma(qp-q+p+1)+\frac{(2j)^p}{p^p(p+1)}\Gamma(qp-q+2p+1)\right]\right\}^{\frac{1}{p}}\to 0,$$

as $\xi\to 0$.

Corollary 2.4. *(to Theorem 2.2) It holds*

$$\left\|P_{r,\xi}(f)-f\right\|_1\leq\left\|f''\right\|_1\left(\sum_{j=0}^{r}\left|\alpha_j\right|\frac{j^2}{\left(j^2\xi^2+1\right)}\right)\xi^2+$$

$$\left[\sum_{j=0}^{r}\left|\alpha_j\right|j^2(2+3j)\right]\omega_1\left(f''+f,\xi\right)_1\xi^2\to 0, \text{ as } \xi\to 0. \tag{51}$$

Conclusion: A new type of approximation was introduced for singular integrals. The interesting trigonometric based one.

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