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QUOTIENT QUASI-ORDERED RESIDUATED SYSTEMS INDUCED BY QUASI-VALUATION MAPS

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ABSTRACT. The concept of quasi-ordered residuated systems was introduced in 2018 by Bonzio and Chajda as a generalization both of commutative residuated lattices and hoop-algebras. Then this author investigated the substructures of ideals and filters in these algebraic structures. As a continuation of these research, in this article we design the concept of quotient quasi-ordered residuated systems induced by a quasi-valuation on it. Additionally, we prove some important properties of the thus constructed quotient structure.

1. INTRODUCTION

Song, Roh and Jun, in [15], introduced the notion of quasi-valuation maps based on a subalgebra and an ideal in BCK/BCI-algebras, and then they investigated several their properties. They provided relations between a quasi-valuation map based on a subalgebra and a quasi-valuation map based on an ideal. Using the notion of a quasi-valuation map based on an ideal, they constructed appropriate (pseudo) metric spaces. In [1], Aaly Kologani et al. introduced the notion of quasi-valuation maps on hoops based on subalgebras and filters and related properties of them are investigated. The idea of designing (quasi- or pseudo-) valuation maps was also applied to some other algebraic structures (for example: [9, 10, 15]). Song, Bordbar and Jun in [16], have described the quotient structure on BCK/BCI - algebras generated by a pseudo-valuation on them. Designing the quotient structure on some other algebraic structures is also shown in the papers [4, 5, 12].

Quasi-ordered residuated systems are quasi-ordered commutative residuated integral monoids ([2]). In the last few years, the theory of quasi-ordered residuated systems and related structures was enriched with more results both about the interior of these structures and about some of their substructures such as ideals and filters ([11, 13]). This class of

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algebraic structures is a generalization of both the class of commutative residuated lattices ([6, 7, 8]) and the class of hoop-algebras ([3]).

In this paper we design (Theorem 3.4) a congruence R_v on $A/\equiv_{\preccurlyeq}$ based on a quasi-valuation map $v : A/\equiv_{\preccurlyeq}\longrightarrow \mathbb{R}$ on a quasi-ordered residuated system $\mathfrak{A}$, where $F_v = [1]_{R_v}$ holds (Theorem 3.7) with the property that the quotient A/R_v is a quasi-ordered system again (Theorem 3.5). In addition, it was shown (Theorem 3.9) that if for quasi-valuation maps v and w on a quasi-ordered residual system $\mathfrak{A}$ the following holds $[1]_{R_v} = [1]_{R_w}$, then R_v and R_w coincide.

2. Preliminaries

In this section, the necessary notions and notations and some of their interrelationships are listed in order to enable a reader to comfortably follow the presentation in this report. It should be pointed out here that the notations for logical conjunction, logical implication and other logical functions have a literal meaning. For example, if a formula is not closed by some quantifier, it is understood that it is under universal quantification.

2.1. **Concept of quasi-ordered residuated systems.** In article [2], S. Bonzio and I. Chajda introduced and analyzed the concept of residual relational systems.

Definition 2.1 ([2], Definition 2.1)**.** A *residuated relational system* is a structure $\mathfrak{A} = \langle A, \cdot, \rightarrow, 1, R\rangle$, where $\langle A, \cdot, \rightarrow, 1\rangle$ is an algebra of type $\langle 2, 2, 0\rangle$ and R is a binary relation on A and satisfying the following properties:
(1) $(A, \cdot, 1)$ is a commutative monoid;
(2) $(\forall x \in A)((x, 1) \in R)$;
(3) $(\forall x, y, z \in A)((x \cdot y, z) \in R \Longleftrightarrow (x, y \rightarrow z) \in R)$.
We will refer to the operation $\cdot$ as multiplication, to $\rightarrow$ as its residuum and to condition (3) as residuation.

Recall that a *quasi-order relation* $'\preccurlyeq'$ on a set A is a binary relation which is reflexive and transitive.

Definition 2.2 ([2])**.** A *quasi-ordered residuated system* is a residuated relational system $\mathfrak{A} = \langle A, \cdot, \rightarrow, 1, \preccurlyeq\rangle$, where $\preccurlyeq$ is a quasi-order relation in the monoid $(A, \cdot)$

The following proposition shows the basic properties of quasi-ordered residuated systems.

Proposition 2.1 ([2], Proposition 3.1)**.** *Let $\mathfrak{A}$ be a quasi-ordered residuated system. Then*
(4) *The operation '$\cdot$' preserves the pre-order in both positions;*

$$(\forall x, y, z \in A)(x \preccurlyeq y \implies (x \cdot z \preccurlyeq y \cdot z \,\wedge\, z \cdot x \preccurlyeq z \cdot y));$$

(5) $(\forall x, y, z \in A)(x \preccurlyeq y \implies (y \rightarrow z \preccurlyeq x \rightarrow z \,\wedge\, z \rightarrow x \preccurlyeq z \rightarrow y))$*;*
(6) $(\forall y, z \in A)(x \cdot (y \rightarrow z) \preccurlyeq y \rightarrow x \cdot z)$*;*
(7) $(\forall x, y, z \in A)(x \cdot y \rightarrow z \preccurlyeq x \rightarrow (y \rightarrow z))$*;*
(8) $(\forall x, y, z \in A)(x \rightarrow (y \rightarrow z) \preccurlyeq x \cdot y \rightarrow z)$*;*
(9) $(\forall x, y, z \in A)(x \rightarrow (y \rightarrow z) \preccurlyeq y \rightarrow (x \rightarrow z))$*;*
(10) $(\forall x, yz \in A)((x \rightarrow y) \cdot (y \rightarrow z) \preccurlyeq x \rightarrow z)$*;*
(11) $(\forall x, y \in A)((x \cdot y \preccurlyeq x) \,\wedge\, (x \cdot y \preccurlyeq y))$*;*
(12) $(\forall x, y, z \in A)(x \rightarrow y \preccurlyeq (y \rightarrow z) \rightarrow (x \rightarrow z))$*;*
(13) $(\forall x, y, z \in A)(y \rightarrow z \preccurlyeq (x \rightarrow y) \rightarrow (x \rightarrow z))$.

It is generally known that a quasi-order relation $\preccurlyeq$ on a set A generates a equivalence relation $\equiv_{\preccurlyeq} := \preccurlyeq \cap \preccurlyeq^{-1}$ on A. Due to properties (4) and (5), this equality relation is compatible with the operations in A. Thus, $\equiv_{\preccurlyeq}$ is a congruence on A.

In the light of the previous note, it is easy to see that the following applies:
(7) and (8) give:
(H3) $(\forall x, y, z \in A)(x \cdot y \to z \equiv_{\preccurlyeq} x \to (y \to z))$.
Due to the universality of formula (9), we have:

$$(\forall x, y, z \in A)(x \to (y \to z) \equiv_{\preccurlyeq} y \to (x \to z)).$$

Example 2.3. By a hoop ([3]) we mean an algebra $(H, \cdot, \to, 1)$ in which $(H, \cdot, 1)$ is a commutative semigroup with the identity and the following assertions are valid:
(H1) $(\forall x \in H)(x \to x = 1)$,
(H2) $(\forall x, y \in H)(x \cdot (x \to y) = y \cdot (y \to x))$ and
(H3) $(\forall x, y, z \in A)(x \cdot y \to z = x \to (y \to z))$.
In this algebra, order is determined as follows:

$$(\forall x, y \in A)(x \leqslant y \iff x \to y = 1).$$

It is easy to see that $(H, \leqslant)$ is a poset. It is easy to see that every hoop is a (quasi-)ordered residuated system and vice versa does not have to be.

Since, in the general case, the formula

$$(\forall x, y \in A)(x \cdot (x \to y) \equiv_{\preccurlyeq} y \cdot (y \to x))$$

does not have to be valid in a quasi-ordered residuated system, we conclude that this last mentioned system is a generalization of the hoop-algebra.

Example 2.4. For a commutative monoid A, let $\mathfrak{P}(A)$ denote the powerset of A ordered by set inclusion and '$\cdot$' the usual multiplication of subsets of A. Then $\langle \mathfrak{P}(A), \cdot, \to, A, \subseteq \rangle$ is a quasi-ordered residuated system in which the residuum are given by

$$(\forall X, Y \in \mathfrak{P}(A))(Y \to X := \{z \in A : Yz \subseteq X\}).$$

Example 2.5. Let $A = \{1, 2, 3, 4\}$ and operations '$\cdot$' and '$\to$' defined on A as follows:

$\cdot$	1	a	b	c	d
1	1	a	b	c	d
a	a	a	a	a	a
b	b	a	b	b	b
c	c	a	b	c	b
d	d	a	b	b	d

and

$\to$	1	a	b	c	d
1	1	a	b	c	d
a	1	1	1	1	1
b	1	a	1	1	1
c	1	a	d	1	d
d	1	a	c	c	1

Then $\mathfrak{A} = \langle A, \cdot, \to, 1 \rangle$ is a quasi-ordered residuated systems where the relation '$\preccurlyeq$' is defined as follows $\preccurlyeq := \{(1,1), (a,1), (a,b), (a,c), (a,d), (b,b), (b.c), (b,d), (b,1), (c,c), (c,1), (d,d), (d,1)\}$.

2.2. **Concept of filters.**

Definition 2.6 ([11], Definition 3.1)**.** For a subset F of a quasi-ordered residuated system $\mathfrak{A}$ we say that it is a *filter* of $\mathfrak{A}$ if it satisfies conditions
(F2) $(\forall u, v \in A)((u \in F \wedge u \preccurlyeq v) \implies v \in F)$, and
(F3) $(\forall u, v \in A)((u \in F \wedge u \to v \in F) \implies v \in F)$.

Let it note that the empty subset of A satisfies the conditions (F2) and (F3). Therefore, $\emptyset$ is a filter in $\mathfrak{A}$. It is shown ([11], Proposition 3.4 and Proposition 3.2), that if a non-empty

subset F of a quasi-ordered system $\mathfrak{A}$ satisfies the condition (F2), then it also satisfies the following conditions
(F0) $1 \in F$ and
(F1) $(\forall u, v \in A)((u \cdot v \in F \implies (u \in F \wedge v \in F))$.
Also, it can be seen without difficulty that $(F3) \implies (F2)$ is valid. Indeed, if (F3) holds, then the formula $u \in F \wedge u \preccurlyeq v$, can be transformed into the formula $u \in F \wedge u \to v \equiv_{\preccurlyeq} 1 \in F$ by (F0) so from here, according to (F3) it can be demonstrate the validity of implications (F2). However, the reverse does not have to be valid.

If $\mathfrak{F}(A)$ is the family of all filters in a QRS $\mathfrak{A}$, then $\mathfrak{F}(A)$ is a complete lattice ([11], Theorem 3.1).

Example 2.7. Let $\mathfrak{A}$ be a quasu-ordered residuated system as in Example 2.5. Then $F_1 := \{1\}$, $F_2 := \{c, 1\}$, $F_3 := \{1, d\}$ and $F_4 := \{1, c, d\}$ and $F_5 := \{1, b, c, d\}$ are filters of $\mathfrak{A}$.

2.3. **Concept of ideals.** In the article [13], the concepts of pre-ideal and ideal in quasi-ordered residuated systems were analyzed. Before that, the conditions were analyzed
(J1) $(\forall y, v \in A)((u \in J \vee v \in J) \implies u \cdot v \in J)$,
(J2) $(\forall u, v \in A)((u \preccurlyeq v \wedge v \in J) \implies u \in J)$, and
(J3) $(\forall u, v \in A)((u \to v \notin J \wedge v \in J) \implies u \in J)$.
Furthermore, in that paper it was proved that (J2) $\implies$ (J1) holds and that (J3) $\implies$ (J2) also holds for the proper subset J. With respect to the above, we have:

Definition 2.8. Let $\mathfrak{A}$ be a quasi-ordered residuated system. For a subset J of the set A we say that it is an pre-ideal in $\mathfrak{A}$ if the condition (J2) is valid. For a subset J of the set A we say that it is an ideal in $\mathfrak{A}$ if $J = A$ or the condition (J3) is valid.

It can easily be seen that if J is a proper (pre-)ideal of $\mathfrak{A}$, then it holds
(J0) $1 \notin J$.

2.4. **Quasi-valuation on QRS.** The following definition gives the concept of quasi-valuation maps on a quasi-ordered residuated system.

Definition 2.9. ([14], Definition 3.1) Let $\mathfrak{A} =: \langle A, \cdot, \to, 1\rangle$ a quasi-ordered residuated system. A real valued function $v : A/\equiv_{\preccurlyeq} \longrightarrow \mathbb{R}$ is called quasi-valuation on $\mathfrak{A}$ if holds
(V0) $v(1) = 0$ and
(V1) $(\forall x, y \in A)(v(y) \geqslant v(x) + v(x \to y))$.
If a quasi-valuation map $v : A/\equiv_{\preccurlyeq} \longrightarrow \mathbb{R}$ satisfies:
(V2) $(\forall x \in A)(\neg(x \equiv_{\preccurlyeq} 1) \implies v(x) \neq 0)$,
then we say that v is a valuation map on $\mathfrak{A}$.

In the following proposition, some of the fundamental properties of the mapping $v : A/\equiv_{\preccurlyeq} \longrightarrow \mathbb{R}$ designed in this way are given .

Proposition 2.2 ([14], Proposition 3.2)**.** *For any quasi-valuation map v on a quasi-ordered residuated system $\mathfrak{A}$, we have the following assertions:*
(14) $(\forall x, y \in A)(x \preccurlyeq y \implies v(x) \leqslant v(y))$.
(15) $(\forall x \in A)(v(x) \leqslant 0)$.
(16) $(\forall x, y \in A)(2v(x \cdot y) \leqslant v(x) + v(y))$.
(17) $(\forall x, y \in A)(v(x \to y) \leqslant v(y) - v(x))$.

On the other hand, we have

Proposition 2.3 ([14], Proposition 3.3). *For any quasi-valuation map v on a quasi-ordered residuated system $\mathfrak{A}$, we have the following assertions:*
(20) $(\forall x, y \in A)(v(x \to y) \geqslant v(x) + v(y))$.
(21) $(\forall x, y \in A)(v(x \cdot y) \geqslant v(x) + v(y))$.

The following two theorems connect a quasi-valuation $v : A/\equiv_{\preccurlyeq} \longrightarrow \mathbb{R}$ on a quasi-ordered residuated system $\mathfrak{A}$ and the concept of filters in $\mathfrak{A}$.

Theorem 2.4 ([14], Theorem 3.5). *If $v : A/\equiv_{\preccurlyeq} \longrightarrow \mathbb{R}$ is a quasi-valuation map on a quasi-ordered redisuated system $\mathfrak{A}$, then the set*

$$F_v := \{x \in A : v(x) = 0\}$$

is a filter of $\mathfrak{A}$.

Theorem 2.5 ([14], Theorem 3.6). *Let G be a non-empty filter in a quasi-ordered residuated system $\mathfrak{A} = \langle A, \cdot, 1, \to \rangle$. For any negative real number k, let v_G be a real valued function on $A/\equiv_{\preccurlyeq}$ defined by $v_G(x) := 0$ if $x \in G$ and $v_G(x) := k$ if $x \in A \setminus G$. Then v_G is a quasi-valuation on $\mathfrak{A}$ and $F_{v_G} = G$ holds.*

Example 2.10. Let $\mathfrak{A}$ be a quasi-ordered residuated system as in Example 2.5. Then the set $F := \{1, b\}$ is a filter of $\mathfrak{A}$. If $v : A/\equiv_{\preccurlyeq} \longrightarrow \mathbb{R}$ is defined by $v(1) = v(b) = 0$ and $v(a) = v(c) = v(d) = -7$, then v is a quasi-valuation on $\mathfrak{A}$ according to the Theorem 2.5.

Theorem 2.6 ([14], Theorem 3.8). *If $v : A/\equiv_{\preccurlyeq} \longrightarrow \mathbb{R}$ is a quasi-valuation map on a quasi-ordered redisuated system $\mathfrak{A}$, then the set*

$$J_v := \{x \in A : v(x) < 0\}$$

is an ideal of $\mathfrak{A}$.

Example 2.11. Let $A = H$ as in article [1], Example 3.3 and let $v : A/\equiv_{\preccurlyeq} \longrightarrow \mathbb{R}$ is determined as in Example 3.9 in the same paper. Then v is a quasi-valuation map on $\mathfrak{A}$. Then $J_v = \{0, a, b\}$ is an ideal and $F_v = \{1\}$ is a filter in $\mathfrak{A}$ because $v(1) = 0$.

In what follows, we will design a pseudo-metric space on a quasi-ordered residuated system generated by a pseudo-valuation on it. By a pseudo-metric on a quasi-ordered residuated system $\mathfrak{A}$, we mean a real-valued function $d : A/\equiv_{\preccurlyeq} \times A/\equiv_{\preccurlyeq} \longrightarrow \mathbb{R}$ satisfying the following properties: $d(x, y) \geqslant 0$, $d(x, x) = 0$, $\mathrm{d}(x, y) = d(y, x)$ and $d(x, z) \leqslant d(x, y) + d(y, z)$ for every $x, y, z \in A$. A pseudo-metric d_v on $\mathfrak{A}$ is said to be a metric on $\mathfrak{A}$ and a pseudo metric space (A, d_v) is said to be a metric space if additionally the following holds $(\forall x, y \in A)(d_v(x, y) = 0 \implies x \equiv_v y)$.

Theorem 2.7 ([14], Theorem 3.11). *Let $v : A/\equiv_{\preccurlyeq} \longrightarrow \mathbb{R}$ be a quasi-valuation on a quasi-ordered residuated system $\mathfrak{A}$. Then*

$$d_v : A/\equiv_{\preccurlyeq} \times A/\equiv_{\preccurlyeq} \ni (x, y) \longmapsto d_v(x, y) := -(v(x \to y) + v(y \to x)) \in \mathbb{R}$$

is a pseudo-metric on $\mathfrak{A}$ and so (A, d_v) is a pseudo-metric space.

3. The main results

This section is the main part of this paper. In the first subsection, several important properties of the induced pseudo-metric d_v by quasi-valuation map v in a quasi-ordered residuated system $\mathfrak{A}$ are shown. In the second subsection, we proceed by designing the congruence relation induced by the pseudo-metric d_v.

3.1. **Some additional properties of induced pseudo-metric.** We begin this subsection with an important result:

Proposition 3.1. *If* $v : A/\equiv_{\preccurlyeq} \longrightarrow \mathbb{R}$ *is a valuation map on a quasi-ordered residuated system* $\mathfrak{A}$*, then the pseudo-metric space* (A, d_v) *induced by* v *satisfies the following assertion:*

(22) $(\forall x, y \in A)(d_v(x, y) = 0 \implies x \equiv_{\preccurlyeq} y)$.

Proof. Let v be a valuation map of a quasi-ordered residuated system $\mathfrak{A}$. Then v is a quasi-valuation map on $\mathfrak{A}$. Thus, by Theorem 2.4, d_v is a pseudo-metric. Let $x, y \in A$ be such that $d_v(x, y) = 0$. Then $v(x \to y) + v(y \to x) = 0$. Since v is a quasi-valuation map on $\mathfrak{A}$, for any $u \in A$ holds $v(x) \leqslant 0$ by (15). So, $v(x \to y) \leqslant 0$, $v(y \to x) \leqslant 0$ and $v(x \to y) = -v(y \to x)$. Thus $0 \geqslant v(x \to y) = -v(y \to x) \geqslant 0$. Hence $v(x \to y) = 0 = v(y \to x)$. From here it follows $x \to y \equiv_{\preccurlyeq} 1$ and $y \to x \equiv_{\preccurlyeq} 1$ according to the contraposition of (V2). Therefore $x \equiv_{\preccurlyeq} y$. □

The following proposition considers the condition when a quasi-valuation map on a quasi-ordered residuated system $\mathfrak{A}$ will be a valuation map on $\mathfrak{A}$.

Proposition 3.2. *Let the pseudo-metric space* (A, d_v) *induced by a quasi-valuation map* $v : A/\equiv_{\preccurlyeq} \longrightarrow \mathbb{R}$ *in a quasi-ordered residuated system* $\mathfrak{A}$ *satisfies the condition (22). Then* v *is a valuation map in* $\mathfrak{A}$*.*

Proof. Let the quasi-metric space (A, d_v) induced by a quasi-valuation map $v : A/\equiv_{\preccurlyeq} \longrightarrow \mathbb{R}$ in a quasi-ordered residuated system $\mathfrak{A}$ satisfies the condition (22). Let us prove that v satisfies the condition (V2). Let $x \in A$ be such that $v(x) = 0$. Then, from $v(1) + v(x) \leqslant v(1 \to x) \leqslant v(1) - v(x)$ and $v(1) + v(x) \leqslant v(x \to 1) \leqslant v(x) - v(1)$ it follows

$$0 = -2(v(1) + v(x)) \geqslant -(v(1 \to x) + v(x \to 1) = d_v(1, x) \geqslant 0.$$

So, $d_v(1, x) = 0$. As (A, d_v) satisfies the condition (22), we get $x \equiv_{\preccurlyeq} 1$. As (A, d_v) satisfies the condition (22), we get $x \equiv_{\preccurlyeq} 1$. Thus, we have obtained a contradiction of the formula (V2). This proves that v is a valuation in $\mathfrak{A}$. □

The following proposition gives some properties of induced pseudo-metric d_v by a quasi-valuation v on a quasi-ordered residuated system $\mathfrak{A}$.

Proposition 3.3. *Let* $v : A/\equiv_{\preccurlyeq} \longrightarrow \mathbb{R}$ *is a quasi-valuation map on a quasi-ordered residuated system* $\mathfrak{A}$*. Then the pseudo-metric space* (A, d_v) *induced by a quasi-valuation map* v *satisfies the following assertions:*

(23) $(\forall x, y, z \in A)(d_v(x, y) \geqslant d_v(z \to x, z \to y))$,

(24) $(\forall x, y, z \in A)(d_v(x, y) \geqslant d_v(x \to z, y \to z))$,

(25) $(\forall x, y, z \in A)(d_v(x, y) \geqslant d_v(z \cdot x, z \cdot y))$.

Proof. Let $v : A/\equiv_{\preccurlyeq} \longrightarrow \mathbb{R}$ is a quasi-valuation map on a quasi-ordered residuated system $\mathfrak{A}$.

For arbitrary elements $x, y, z \in A$,

$$x \to y \preccurlyeq (z \to x) \to (z \to y) \quad \text{and} \quad y \to x \preccurlyeq (z \to y) \to (z \to z)$$

are valid according to (13). Then

$$v(x \to y) \leqslant v((z \to x) \to (z \to y)) \quad \text{and} \quad v(y \to x) \leqslant v((z \to y) \to (z \to z))$$

also holds by (14). Hence

$$\begin{aligned} d_v(x,y) &= -(v(x \to y) + v(y \to x)) \\ &\geqslant -(v((z \to x) \to (z \to y)) + v((z \to y) \to (z \to z)) \\ &\geqslant d_v(z \to x, z \to y). \end{aligned}$$

This proves the validity of formula (23).

Similarly, we can prove the condition (24) starting from the formula (12).

For arbitrary elements $x, y, z \in A$, the following $x \to y \preccurlyeq x \to y$ is valid due to the reflexivity of the relation $\preccurlyeq$. Then $(x \to y) \cdot x \preccurlyeq y$ by (3). Thus $(x \to y) \cdot x \cdot z \preccurlyeq y \cdot z$ by (4). From here we get $x \to y \preccurlyeq x \cdot z \to y \cdot z$ according to (3). Replacing the variables x and y in the previous inequality, we get $y \to x \preccurlyeq y \cdot z \to x \cdot z$. Now, according to (14), we have $v(x \to y) \leqslant v(x \cdot z \to y \cdot z)$ and $v(y \to x) \leqslant v(y \cdot z \to x \cdot z)$. Hence

$$\begin{aligned} d_v(x,y) &= -(v(x \to y) + v(y \to x)) \\ &\geqslant -(v(x \cdot z \to y \cdot z) + v(y \cdot z \to x \cdot z)) \\ &= d_v(z \cdot x, z \cdot y). \end{aligned}$$

This proves the inequality (25). □

3.2. **A construction of a congruence induced by a quasi-valuation.**

In [13], the notion of congruence on a quasi-ordered residuated system was introduced as follows:

Definition 3.1. An equivalence relation θ on a quasi-ordered residuated system $\mathfrak{A} = \langle A, \cdot, \to, \preccurlyeq, 1 \rangle$ is a congruence on $\mathfrak{A}$ if the the following holds

$(\forall x, y, z \in A)((x, y) \in \theta \implies (x \cdot z, y \cdot z) \in \theta)$ and

$(\forall x, y, z \in A)((x, y) \in \theta \implies ((x \to z, y \to z) \in \theta \wedge (z \to x, z \to y) \in \theta))$.

The following theorem designs an equivalence relation on a quasi-ordered residuated system $\mathfrak{A}$ using a quasi-valuation map in $\mathfrak{A}$.

Theorem 3.4. *Let $v : A/\equiv_{\preccurlyeq} \longrightarrow \mathbb{R}$ be a quasi-valuation map on a quasi-ordered residuated system $\mathfrak{A}$. Then the relation R_v on A, defined by*

$$(\forall x, y \in A)((x, y) \in R_v \iff d_v(x, y) = 0),$$

is an equivalence relation on $\mathfrak{A}$ compatible with the operations in $\mathfrak{A}$.

Proof. It is clear that R_v is reflexive and symmetric relation on A. Suppose $(x, y) \in R_v$ and $(y, z) \in R_v$. Then $d_v(x, y) = 0$ and $d_v(y, z) = 0$. On the other hand, for arbitrary variables $x, y, z \in A$ $(x \to y) \cdot (y \to z) \preccurlyeq x \to z$ holds according to (12). Hence, due to the validity of the implication (14), $v((x \to y) \cdot (y \to z)) \preccurlyeq v(x \to z)$ follows. From here, due to (21), we get

$$v(x \to y) + v(y \to z) \preccurlyeq v((x \to y) \cdot (y \to z)) \preccurlyeq v(x \to z).$$

By replacing the variables x and z in the previous formula, we also get

$$v(z \to y) + v(y \to x) \preccurlyeq v(z \to x).$$

Hence

$$\begin{aligned} 0 &= -d_v(x,y) - d_v(y,z) \\ &= v(x \to y) + v(y \to x) + v(y \to z) + v(z \to y) \\ &= (v(x \to y) + v(y \to z)) + (v(z \to y) + v(y \to x)) \\ &\leqslant v(x \to z) + v(z \to x) = -d_v(x,z) \leqslant 0. \end{aligned}$$

Then $d_v(x,z) = 0$ and so, $(x,z) \in R_v$. Therefore, R_v is a transitive relation on $\mathfrak{A}$. This shows that R_v is an equivalence relation on A.

Let us prove that R_v is compatible with the operation $\rightarrow$. For arbitrary elements $x, t, z \in A$ such that $d_v(x,y) = 0$, we have

$$x \rightarrow y \preccurlyeq (y \rightarrow z) \rightarrow (x \rightarrow z) \text{ and } y \rightarrow x \preccurlyeq (x \rightarrow z) \rightarrow (y \rightarrow z),$$

according to (12). From here, in accordance with (14), we get

$$v(x \rightarrow y) \preccurlyeq v((y \rightarrow z) \rightarrow (x \rightarrow z)) \text{ and } v(y \rightarrow x) \preccurlyeq v((x \rightarrow z) \rightarrow (y \rightarrow z)).$$

Hence

$$\begin{aligned} 0 = -d_v(x,y) &= v(x \rightarrow y) + v(y \rightarrow x) \\ &\leqslant v((y \rightarrow z) \rightarrow (x \rightarrow z)) + v((x \rightarrow z) \rightarrow (y \rightarrow z)) = -d_v(x \rightarrow z, y \rightarrow z) \\ &\leqslant v(x \rightarrow z) - v(y \rightarrow z) + v(y \rightarrow z) - v(x \rightarrow z) \quad \text{according to (17)} \\ &= 0. \end{aligned}$$

The second required result $d(z \rightarrow x, z \rightarrow y) = 0$ can be obtained in an analogous way starting with formula (13).

It remains to show that the relation R_v is compatible with the multiplication operation in $\mathfrak{A}$. Let $x, y, z \in A$ be arbitrary elements such that $d_v(x,y) = 0$. If we start from the valid formula $y \cdot z \preccurlyeq y \cdot z$, we get $y \preccurlyeq z \rightarrow y \cdot z$ according to (3). From here, according to (13), we get $x \rightarrow y \preccurlyeq x \rightarrow (z \rightarrow y \cdot z)$ and from here, according to (8), we have $x \rightarrow y \preccurlyeq x \cdot z \rightarrow y \cdot z$. Now, according to (14), we have $v(x \rightarrow y) \leqslant v(x \cdot z \rightarrow y \cdot z)$. Therefore, we can now calculate

$$\begin{aligned} 0 = -d_v(x,y) &= v(x \rightarrow y) + v(y \rightarrow x) \leqslant v(x \cdot z \rightarrow y \cdot z) + v(y \cdot z \rightarrow x \cdot z) \\ &= -d_v(x \cdot z, y \cdot z) \leqslant 0. \end{aligned}$$

Therefore, $d_v(x \cdot z, y \cdot z) = 0$, which means that $(x \cdot z, y \cdot z) \in R_v$ holds. This proves the compatibility of the relation R_v with the multiplication operation in $\mathfrak{A}$. $\square$

The importance of a congruence relation R_v on a quasi-ordered residuated system $\mathfrak{A}$ is justified by the fact that the quotient A/R_v turns naturally into an ordered set. It is commonly known that if $(A, \preccurlyeq)$ is a quasi-ordered set and R_v is an equivalence relation on A, then the relation $\leqq$, defined by

$$(\forall x, y \in A)([x]_{R_v} \leqq [y]_{R_v} \iff x \preccurlyeq y)$$

is an order relation on A/R_v. Let us define operations '$\odot$ and '$\rightrightarrows$' as

$$(\forall x, y \in A)([x]_{R_v} \odot [y]_{R_v} = [x \cdot y]_{R_v}) \text{ and}$$
$$(\forall x, y \in)([x]_{R_v} \rightrightarrows [y]_{R_v} = [x \rightarrow y]_{R_v}).$$

Theorem 3.5. *Let $\mathfrak{A}$ be a quasi-ordered relational system and let $v : A/\equiv_{\preccurlyeq} \longrightarrow \mathbb{R}$ be a quasi-valuation map on $\mathfrak{A}$. Then*

$$\langle A/R_v, \odot, \rightrightarrows, [1]_{R_v}, \leqq \rangle$$

is a (quasi-)ordered residuated system.

Proof. This is a special case of Theorem 5.3 in the article [13]. $\square$

In what follows, we need the following lemma:

Lemma 3.6. *Let $v : A/\equiv_{\preccurlyeq} \longrightarrow \mathbb{R}$ be a quasi-valuation map on a quasi-ordered residuated system $\mathfrak{A}$. Then holds*

$$(\forall x \in A)(v(x \to 1) = 0) \;\; and \;\; (\forall x \in A)(v(x) = v(1 \to x)).$$

Proof. Let $x \in A$ be an arbitrary element. Then $x \preccurlyeq 1$ implies $1 \preccurlyeq x \to 1 \preccurlyeq 1$. Thus $v(1) \leqslant v(x \to 1) \leqslant v(1)$ by (14). On the other hand, from $x \preccurlyeq 1 \to x$ it follows $v(x) \leqslant v(1 \to x) \leqslant v(x) - v(1) = v(x)$ according to (14) and (17). □

Theorem 3.7. *Let $v : A/\equiv_{\preccurlyeq} \longrightarrow \mathbb{R}$ be a quasi-valuation map on a quasi-ordered residuated system $\mathfrak{A}$. Then $F_v = [1]_{R_v}$ and $J_v = \bigcup_{d_v(x,1)>0}[x]_{R_v}$.*

Proof.
$$\begin{aligned} F_v &= \{x \in A : v(x) = 0\} = \{x \in A : v(1 \to x) + v(x \to 1) = 0\} \\ &= \{A : d_v(1, x) = 0\} = \{x \in A : (x, 1) \in R_v\} = [1]_{R_v}. \end{aligned}$$

Also, we have

$$\begin{aligned} J_v &= \{x \in A : v(x) < 0\} = \{x \in A : v(1 \to x) + v(x \to 1) < 0\} \\ &= \{x \in A : -d_v(1, x) < 0\} = \{x \in A : d_v(1, x) > 0\} \\ &= \textstyle\bigcup_{d_v(x,1)>0}[x]_{R_v}. \end{aligned}$$ □

Let $C(F_v)$ be a relation on $\mathfrak{A}$ defined by

$$(\forall x, y \in A)((x, y) \in C(F_v) \iff (x \to y \in F_v \wedge y \to x \in F_v)).$$

On the other hand, we have:

Theorem 3.8. *Let $v : A/\equiv_{\preccurlyeq} \longrightarrow \mathbb{R}$ be a quasi-valuation map on a quasi-ordered residuated system $\mathfrak{A}$. Then $C(F_v) = R_v$.*

Proof. Let $x, y \in A$. Then

$$\begin{aligned} (x, y) \in C(F_v) &\iff (x \to y \in F_v \wedge y \to x \in F_v) \\ &\iff v(x \to y) = v(y \to x) = 0 \\ &\iff v(x \to y) + v(y \to x) = 0 \\ &\iff d_v(x, y) = 0 \\ &\iff (x, y) \in R_v. \end{aligned}$$ □

Theorem 3.9. *Let v and w be quasi-valuation maps on a quasi-ordered residuated system $\mathfrak{A}$ with $v \neq w$. If $[1]_{R_v} = [1]_{R_w}$, then R_v and R_w coincide and so $A/R_v = A/R_w$.*

Proof. Let $x, y \in A$ such that $(x, y) \in R_v$. Then $d_v(x, y) = 0$ and $v(x \to y) + v(y \to x) - 0$. Thus $v(x \to y) = -v(y \to x) \geqslant 0$ by (15). This is possible only if $v(x \to y) = 0$ and, therefore, $v(y \to x) = 0$. This means that $x \to y \in [1]_{R_v}$ and $y \to x \in [1]_{R_v}$. On the other hand, since $[1]_{R_v} = [1]_{R_w}$ by assumption, we have $x \to y \in [1]_{R_w}$ and $y \to x \in [1]_{R_w}$. So,$(x, y) \in R_w$. The reverse implication $R_w \subseteq R_v$ it can be proved analogously to the previous one. Thus $R_v = R_w$. □

4. Conclusions

This report is a continuation of papers on our research of quasi-ordered residuated systems. More precisely, this paper is a continuation in the literal sense of the paper [14]. In articles [11, 13] the concepts of filters and ideals in such algebraic structures are analyzed. Article [14] is dedicated to designing the concept of quasi-valuation map in a quasi-ordered residuated system $\mathfrak{A}$ and analyzing its properties. In this paper, a congruence on a quasi-ordered residuated system $\mathfrak{A}$, generated by a quasi-valuation in $\mathfrak{A}$, is designed. In addition, it was shown (Theorem 3.9) that if for quasi-valuation maps v and w on a quasi-ordered residual system $\mathfrak{A}$ the following holds $[1]_{R_v} = [1]_{R_w}$, then R_v and R_w coincide.

The author is convinced that the results announced in this report raise academic knowledge about quasi-ordered residuated systems and that they can be one of the bases for further research into these algebraic structures.

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