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ROUGH STATISTICAL CONVERGENCE IN NEUTROSOPHIC NORMED SPACES

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ABSTRACT. Neutrosophic normed spaces, one of the recent hot issues in mathematics, are covered in this paper. The Neutrosophic approach is based on the idea that the degree of uncertainty should be taken into consideration and that it is insufficient to categorise problems in daily life as either right or wrong. This paper proposes double sequences' rough statistical convergence in Neutrosophic Normed Spaces. It then specifies the rough statistical limit points and cluster points of a double sequence in these spaces. This paper then looks at some of their fundamental characteristics. The necessity for more research is finally covered.

1. INTRODUCTION

Fast [6] and Steinhaus [17] independently defined statistical convergence in 1951 using the idea of density of positive natural numbers.The idea of statistical convergence in double sequence space was subsequently studied by Mursaleen and Edely [12] in 2003. As a natural generalisation of ordinary convergence, Phu [14] defines the idea of rough convergence in finite dimensional normed spaces. The concepts of rough statistical convergence and rough convergence of double sequences in normed linear spaces have also been studied by Malik and Maity[11].

The first version of the fuzzy set theory was published by Zadeh in 1965 [18]. The concept of fuzzy norms on a linear space was then put out by Cheng and Mordeson [5]. As a generalisation of fuzzy sets, Atanassov [2, 3] proposed the concept of intuitionistic fuzzy sets. Following the definition of intuitionistic fuzzy normed space by Saadati and Park [15], Mursaleen [13] defined statistical convergence of double sequences in an intuitionistic fuzzy normed space. Additionally, Antal et al. [1] have put out the idea of rough statistical convergence in an intuitionistic fuzzy normed space.

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The concept of neutrosophic sets was introduced by Smarandache [16] as a development of the intuitionistic fuzzy set. When the component sum equals 1, the requirement can be satisfied by using neutrosophic set operators. While neutrosophic operators take into account indeterminacy on the same level as truth-membership and falsehood-nonmembership, intuitionistic fuzzy operators neglect indeterminacy, which could lead to different results when used. Statistical Δ^m convergence in neutrosophic normed spaces was recently presented by Jeyaraman and Jenifer[7]. In this paper, we study the rough statistical convergence of double sequences in the neutrosophic normed space.

The present paper can be summarized as follows: In the second part of the present study, some basic definitions and properties to be required for the next section are provided. Section 3 proposes the concepts of rough convergence and rough statistical convergence of double sequences in neutrosophic normed spaces and studies some of their basic properties.

Moreover, it defines the concepts of rough statistical $(\mathfrak{r}-\mathfrak{st})$ limit points and $(\mathfrak{r}-\mathfrak{st})$ cluster points of a double sequence in neutrosophic normed space and investigates some of their basic properties. The final section discusses the need for further research.

2. Preliminaries

This section presents some of the basic definitions to be required in the next sections.

Definition 2.1. [10] Let $(\mathfrak{y}_{\mathfrak{j}\mathfrak{k}})$ be a double sequence (DS) in a normed space $(\mathfrak{NS})$ $(\mathfrak{Y}, \|\cdot\|)$ and $\mathfrak{r} \geq 0$. Then, the (DS) $(\mathfrak{y}_{\mathfrak{j}\mathfrak{k}})$ is said to be rough convergent ($\mathfrak{r}$-convergent) to $\mathfrak{y}_0 \in \mathfrak{Y}$, if for all $\xi > 0$, there exists $N_\xi \in \mathbb{N}$ such that for all $\mathfrak{j}, \mathfrak{k} \geq N_\xi$, $\|\mathfrak{y}_{\mathfrak{j}\mathfrak{k}} - \mathfrak{y}_0\| < \mathfrak{r} + \xi$.
It is denoted by $\mathfrak{y}_{\mathfrak{j}\mathfrak{k}} \xrightarrow{\mathfrak{r}} \mathfrak{x}_0$. The element $\mathfrak{y}_0$ is called a rough limit point ($\mathfrak{r}$-limit point) of the DS $(\mathfrak{y}_{\mathfrak{j}\mathfrak{k}})$.

Definition 2.2. The double natural density of the set $\mathfrak{X} \subseteq \mathbb{N} \times \mathbb{N}$ is defined by $\delta_2(\mathfrak{X}) = \lim\limits_{\mathfrak{m},\mathfrak{n}\to\infty} \dfrac{|\{(\mathfrak{j},\mathfrak{k}) \in \mathfrak{X} : \mathfrak{j} \leq \mathfrak{m} \text{ and } \mathfrak{k} \leq \mathfrak{n}\}|}{\mathfrak{m}\mathfrak{n}}$ where $|\{(\mathfrak{j},\mathfrak{k}) \in \mathfrak{X} : \mathfrak{j} \leq \mathfrak{m} \text{ and } \mathfrak{k} \leq \mathfrak{n}\}|$ denotes the number of elements of $\mathfrak{X}$ not exceeding $\mathfrak{m}$ and $\mathfrak{n}$, respectively. It can be observed that if the set $\mathfrak{X}$ is finite, then $\delta_2(\mathfrak{X}) = 0$.

Definition 2.3. [11] Let $(\mathfrak{y}_{\mathfrak{j}\mathfrak{k}})$ be a DS in a $\mathfrak{NS}(\mathfrak{Y}, \|\cdot\|)$ and $\mathfrak{r} \geq 0$. Then, $(\mathfrak{y}_{\mathfrak{j}\mathfrak{k}})$ is referred to as rough statistically $(\mathfrak{r}-\mathfrak{st})$ convergent to $\mathfrak{y}_0 \in \mathfrak{Y}$, if for all $\xi > 0$,

$$\delta_2(\{(\mathfrak{j},\mathfrak{k}) \in \mathbb{N} \times \mathbb{N} : ||\mathfrak{y}_{\mathfrak{j}\mathfrak{k}} - \mathfrak{y}_0|| \geq \mathfrak{r} + \xi\}) = 0$$

In this case we write $\mathfrak{y}_{\mathfrak{j}\mathfrak{k}} \xrightarrow{\mathfrak{r}-\mathfrak{st}_2} \mathfrak{y}_0$. The element $\mathfrak{y}_0$ is called $(\mathfrak{r}-\mathfrak{st})$ limit point of the DS $(\mathfrak{y}_{\mathfrak{j}\mathfrak{k}})$.

Definition 2.4. [7] The 7-tuple $(\mathfrak{Y}, \varphi, \nu, \psi, *, \diamond, \odot)$ is said to be Neutrosophic Normed Space $(\mathfrak{NNS})$ if $\mathfrak{Y}$ is a linear space, * is a continuous t-norm, $\diamond$ and $\odot$ are continuous t-conorm, φ, ν and ψ are fuzzy sets on $\mathfrak{Y} \times (0, \infty)$ fulfils the following conditions:
For every $a, b \in \mathfrak{Y}$ and $s, t > 0$;

(a) $0 \leq \varphi(a,t) \leq 1; 0 \leq \nu(a,t) \leq 1; 0 \leq \psi(a,t) \leq 1;$
(b) $\varphi(a,t) + \nu(a,t) + \psi(a,t) \leq 3,$
(c) $\varphi(a,t) > 0,$
(d) $\varphi(a,t) = 1$ if and only if $a = 0,$
(e) $\varphi(\alpha a, t) = \varphi\left(a, \frac{t}{|\alpha|}\right)$ for each $\alpha \neq 0,$

(f) $\varphi(a,t) * \varphi(b,s) \leq \varphi(a+b, t+s)$,
(g) $\varphi(a,t) : (0,\infty) \to [0,1]$ is continuous,
(h) $\lim_{t\to\infty} \varphi(a,t) = 1$ and $\lim_{t\to 0} \varphi(a,t) = 0$,
(i) $\nu(a,t) < 1$,
(j) $\nu(a,t) = 0$ if and only if $a = 0$,
(k) $\nu(\alpha a, t) = \nu\left(a, \frac{t}{|\alpha|}\right)$ for each $\alpha \neq 0$,
(l) $\nu(a,t) \diamond \nu(b,s) \geq \nu(a+b, t+s)$,
(m) $\nu(a,t) : (0,\infty) \to [0,1]$ is continuous,
(n) $\lim_{t\to\infty} \nu(a,t) = 0$ and $\lim_{t\to 0} \nu(a,t) = 1$,
(o) $\psi(a,t) < 1$,
(p) $\psi(a,t) = 0$ if and only if $a = 0$,
(q) $\psi(\alpha a, t) = \omega\left(a, \frac{t}{|\alpha|}\right)$ for each $\alpha \neq 0$,
(r) $\psi(a,t) \odot \psi(b,s) \geq \psi(a+b, t+s)$,
(s) $\psi(a,t) : (0,\infty) \to [0,1]$ is continuous,
(t) $\lim_{t\to\infty} \psi(a,t) = 0$ and $\lim_{t\to 0} \psi(a,t) = 1$.

Then (φ, ν, ψ) is known as Neutrosophic Normed Space [$\mathfrak{NNS}$].

Example 2.5. [7] Let $(\mathfrak{Y}, ||\circ||)$ be any $\mathfrak{NS}$. For every $t > 0$ and all $y \in \mathfrak{Y}$, take $\varphi(y,t) = \frac{t}{t+||y||}, \nu(y,t) = \frac{||y||}{t+||y||}$ and $\omega(y,t) = \frac{||y||}{t}$. Also, $u * v = uv, u \diamond v = \min\{u+v, 1\}$ and $u \odot v = \min\{u+v, 1\}$ for all $u, v \in [0,1]$. Then, a 7-tuple $(\mathfrak{Y}, \varphi, \nu, \psi, *, \diamond, \odot)$ is a $\mathfrak{NNS}$ which fulfills the above mentioned conditions.

Definition 2.6. [12] Let $(\mathfrak{Y}, \varphi, \nu, \psi)$ be a $\mathfrak{NNS}, y \in \mathfrak{Y}, \rho \in (0,1)$ and $\upsilon > 0$. Then, the open ball with center $\mathfrak{x}$ and radius ρ is the set $B(\mathfrak{x}, \varepsilon, \upsilon) = \{y \in \mathfrak{Y} : \varphi(\mathfrak{x} - y; \upsilon) > 1 - \rho, \nu(\mathfrak{x} - y; \upsilon) < \rho$ and $\psi(\mathfrak{x} - y; \upsilon) < \rho\}$.

Definition 2.7. [9] Let $(\mathfrak{Y}, \varphi, \nu, \psi)$ be a $\mathfrak{NNS}$. Then, a $DS\,(\mathfrak{y}_{\mathfrak{j}\mathfrak{k}})$ in $\mathfrak{Y}$ is said to convergent to $\mathfrak{y}_0 \in \mathfrak{Y}$ with respect to the $\mathfrak{NNS}(\varphi, \nu, \psi)$, if there exists $N_\xi \in \mathbb{N}$ for all $\upsilon > 0$ and $\xi \in (0,1)$, $\varphi(\mathfrak{y}_{\mathfrak{j}\mathfrak{k}} - \mathfrak{y}_0; \upsilon) > 1 - \varepsilon, \nu(\mathfrak{y}_{\mathfrak{j}\mathfrak{k}} - \mathfrak{y}_0; \upsilon) < \varepsilon$ and $\psi(\mathfrak{y}_{\mathfrak{j}\mathfrak{k}} - \mathfrak{y}_0; \upsilon) < \varepsilon$ for all $\mathfrak{j}, \mathfrak{k} \geq N_\xi$ and denoted by $(\varphi, \nu, \psi) - \lim_{\mathfrak{j},\mathfrak{k}\to\infty} \mathfrak{y}_{\mathfrak{j}\mathfrak{k}} = \mathfrak{y}_0$ or $\mathfrak{y}_{\mathfrak{j}\mathfrak{k}} \xrightarrow{(\varphi,\nu,\psi)} \mathfrak{y}_0$.

Definition 2.8. [9] Let $(\mathfrak{Y}, \varphi, \nu, \psi)$ be a $\mathfrak{NNS}$. Then, a $\mathfrak{DS}(\mathfrak{y}_{\mathfrak{j}\mathfrak{k}})$ is said to be statistically $(\mathfrak{st})$ convergent to $\mathfrak{y}_0 \in \mathfrak{Y}$ with respect to the $\mathfrak{NNS}(\varphi, \nu, \psi)$, for all $\upsilon > 0$ and $\xi \in (0,1)$, if

$$\delta_2\left(\left\{\begin{array}{c} (\mathfrak{j},\mathfrak{k}) \in \mathbb{N}\times\mathbb{N} : \varphi(\mathfrak{y}_{\mathfrak{j}\mathfrak{k}} - \mathfrak{y}_0; \upsilon) \leq 1 - \xi \text{or} \\ \nu(\mathfrak{y}_{\mathfrak{j}\mathfrak{k}} - \mathfrak{y}_0; \upsilon) \geq \xi, \psi(\mathfrak{y}_{\mathfrak{j}\mathfrak{k}} - \mathfrak{y}_0; \upsilon) \geq \xi \end{array}\right\}\right) = 0$$

and denoted by $\mathfrak{st}_2^{(\varphi,\nu,\psi)} - \lim_{\mathfrak{j},\mathfrak{k}\to\infty} \mathfrak{y}_{\mathfrak{j}\mathfrak{k}} = \mathfrak{y}_0$.

3. Rough Statistical Convergence

This section defines the concepts of $\mathfrak{r}$-convergent and $(\mathfrak{r} - \mathfrak{st})$ convergence of DS in the $\mathfrak{NNS}$ and examines some of their basic properties.

Definition 3.1. Let $(\mathfrak{Y}, \varphi, \nu, \psi)$ be a $\mathfrak{NNS}$ and $\mathfrak{r} \geq 0$. Then, a $DS\,(\mathfrak{y}_{\mathfrak{j}\mathfrak{k}})$ in $\mathfrak{Y}$ is said to be $\mathfrak{r}$-convergent to $\mathfrak{y}_0 \in \mathfrak{Y}$, with respect to the norm (φ, ν, ψ), if there exists $N_\xi \in \mathbb{N}$ for all $\upsilon > 0$ and $\xi \in (0,1)$ such that

$$\varphi(\mathfrak{y}_{\mathfrak{j}\mathfrak{k}} - \mathfrak{y}_0; \mathfrak{r} + \upsilon) > 1 - \xi, \nu(\mathfrak{y}_{\mathfrak{j}\mathfrak{k}} - \mathfrak{y}_0; \mathfrak{r} + \upsilon) < \xi \text{ and } \psi(\mathfrak{y}_{\mathfrak{j}\mathfrak{k}} - \mathfrak{y}_0; \mathfrak{r} + \upsilon) < \xi,$$

for all $\mathfrak{j}, \mathfrak{k} \geq N_\xi$.

In this case, we write $\mathfrak{r}_{(\varphi,\nu,\psi)} - \lim_{\mathfrak{j},\mathfrak{k}\to\infty} \mathfrak{y}_{\mathfrak{j}\mathfrak{k}} = \mathfrak{y}_0$ or $\mathfrak{y}_{\mathfrak{j}\mathfrak{k}} \xrightarrow{\mathfrak{r}_{(\varphi,\nu,\psi)}} \mathfrak{y}_0$, where $\mathfrak{y}_0$ is called a $\mathfrak{r}_{(\varphi,\nu,\psi)}$ - limit point of the double sequence $(\mathfrak{y}_{\mathfrak{j}\mathfrak{k}})$.

Note: For $\mathfrak{r} = 0$, the concept of rough convergence in $\mathfrak{NNS}$ becomes the concept of ordinary convergence in $\mathfrak{NNS}$.

The $\mathfrak{r}_{(\varphi,\nu,\psi)}$-limit point of a DS may not be unique. Therefore, the set of all $\mathfrak{r}_{(\varphi,\nu,\psi)}$-limit points for a $DS(\mathfrak{y}_{\mathfrak{j}\mathfrak{k}})$ is as follows:

$$\lim_{\mathfrak{y}_{\mathfrak{j}\mathfrak{k}}}^{\mathfrak{r}} = \{\mathfrak{y}_0 \in \mathfrak{Y} : \mathfrak{y}_{\mathfrak{j}\mathfrak{k}} \xrightarrow{\mathfrak{r}_{(\varphi,\nu,\psi)}} \mathfrak{y}_0\}$$

If $\lim_{\mathfrak{y}_{\mathfrak{j}\mathfrak{k}}}^{\mathfrak{r}} \mathfrak{r} \neq \emptyset$, the DS $(\mathfrak{y}_{\mathfrak{j}\mathfrak{k}})$ is $\mathfrak{r}$-convergent.

Definition 3.2. Let $(\mathfrak{Y}, \varphi, \nu, \psi)$ be a $\mathfrak{NNS}$ and $\mathfrak{r} \geq 0$. Then, a DS $(\mathfrak{y}_{\mathfrak{j}\mathfrak{k}})$ in $\mathfrak{Y}$ is referred to as $(\mathfrak{r} - \mathfrak{st})$ convergent to $\mathfrak{y}_0 \in \mathfrak{Y}$ with respect to the norm (φ, ν, ψ), for all $\upsilon > 0$ and $\xi \in (0, 1)$, if

$$\delta_2\left(\left\{\begin{array}{c} (\mathfrak{j},\mathfrak{k}) \in \mathbb{N}\times\mathbb{N} : \varphi(\mathfrak{y}_{jk} - \mathfrak{y}_0; \mathfrak{r}+\upsilon) \leq 1-\xi \text{ or} \\ \nu(\mathfrak{y}_{jk} - \mathfrak{y}_0; \mathfrak{r}+\upsilon) \geq \xi, \psi(\mathfrak{y}_{jk} - \mathfrak{y}_0; \mathfrak{r}+\upsilon) \geq \xi \end{array}\right\}\right) = 0$$

and denoted by $\mathfrak{r} - \mathfrak{st}_2 - \lim_{\mathfrak{j},\mathfrak{k}\to\infty} \mathfrak{y}_{\mathfrak{j}\mathfrak{k}} = \mathfrak{y}_0$ or $\mathfrak{y}_{\mathfrak{j}\mathfrak{k}} \xrightarrow{\mathfrak{r}-\mathfrak{st}_2} \mathfrak{y}_0$.

Note: If $\mathfrak{r} = 0$, then $(\mathfrak{r} - \mathfrak{st})$ convergence coincide with $\mathfrak{st}$- convergence in $\mathfrak{NNS}$. The $(\mathfrak{r}-\mathfrak{st})$ limit of a (DS) may not be unique. Hence, the set of $(\mathfrak{r}-\mathfrak{st})$ limit points is denoted as follows:

$$\mathfrak{st}_2 - \lim_{\mathfrak{y}_{\mathfrak{j}\mathfrak{k}}}^{\mathfrak{r}} = \left\{\mathfrak{y}_0 \in \mathfrak{Y} : \mathfrak{y}_{\mathfrak{j}\mathfrak{k}} \xrightarrow{\mathfrak{r}-\mathfrak{st}_2} \mathfrak{y}_0\right\}$$

Let $(\mathfrak{y}_{\mathfrak{j}\mathfrak{k}})$ be a unbounded sequence. Then, $\lim_{\mathfrak{y}_{\mathfrak{j}\mathfrak{k}}}^{\mathfrak{r}}$ is an empty set. However, this is not achieved in the case of $(\mathfrak{r} - \mathfrak{st})$ convergence. Hence, $\mathfrak{st}_2 - \lim_{\mathfrak{y}_{\mathfrak{j}\mathfrak{k}}}^{\mathfrak{r}}$ may not be an empty set.

Example 3.3. Let us consider a real $\mathfrak{NS}(\mathfrak{Y}, ||\cdot||)$ and for all $t > 0$ and $y \in \mathfrak{Y}$,

$$\varphi(y,t) = \frac{t}{t+||y||}, \nu(y,t) = \frac{||y||}{t+||y||} \text{ and } \psi(y,u) = \frac{||y||}{t}$$

Then, the $(\mathfrak{Y}, \varphi, \nu, \psi)$ is a $\mathfrak{NNS}$. For all $\mathfrak{j}, \mathfrak{k} \in \mathbb{N}$, define

$$\mathfrak{y}_{\mathfrak{j}\mathfrak{k}} = \begin{cases} (-1)^{\mathfrak{j}+\mathfrak{k}}, & \mathfrak{j} \text{ and } \mathfrak{k} \text{ are non-squares} \\ \mathfrak{j}\mathfrak{k}, & \text{otherwise} \end{cases}$$

Then, $\mathfrak{st}_2 - \lim_{\mathfrak{y}_{\mathfrak{j}\mathfrak{k}}}^{\mathfrak{r}} = \begin{cases} \emptyset, & \mathfrak{r} < 1 \\ [1-\mathfrak{r}, \mathfrak{r}-1], & \mathfrak{r} \geq 1 \end{cases}$ and $\lim_{\mathfrak{y}_{\mathfrak{j}\mathfrak{k}}}^{\mathfrak{r}} = \emptyset$, for all $\mathfrak{r} \geq 0$.

Afterward, the concept of $(\mathfrak{r} - \mathfrak{st})$ bounded sequence in a $\mathfrak{NNS}$ is as follows:

Definition 3.4. Let $(\mathfrak{Y}, \varphi, \nu, \psi)$ be a $\mathfrak{NNS}$ and $\mathfrak{r} \geq 0$. Then, a DS $(\mathfrak{y}_{\mathfrak{j}\mathfrak{k}})$ in $\mathfrak{Y}$ is said to be statistical bounded ($\mathfrak{st}$-bounded) with respect to the norm (φ, ν, ψ), if there exists a real number $\mathfrak{W} > 0$ for all $\upsilon > 0$ and $\varepsilon \in (0, 1)$ such that

$$\delta_2(\{(\mathfrak{j},\mathfrak{k}) \in \mathbb{N}\times\mathbb{N} : \varphi(\mathfrak{y}_{\mathfrak{j}\mathfrak{k}}; \mathfrak{W}) \leq 1-\varepsilon \text{ or } \nu(\mathfrak{y}_{\mathfrak{j}\mathfrak{k}}; \mathfrak{W}) \geq \varepsilon, \psi(\mathfrak{y}_{\mathfrak{j}\mathfrak{k}}; \mathfrak{W}) \geq \varepsilon\}) = 0$$

Definition 3.5. A DS $y' = \left[\mathfrak{y}_{j_p k_p}\right]$ of a DS $y = (\mathfrak{y}_{\mathfrak{j}\mathfrak{k}})$ is called a dense subsequence, if $\delta_2(\{j_p, k_q \in \mathbb{N}\times\mathbb{N} : p, q \in \mathbb{N}\}) = 1$.

Example 3.6. Let us consider the $\mathfrak{NNS}$ in Example (3.3) and, for all $\mathfrak{j}, \mathfrak{k} \in \mathbb{N}$, define

$$\mathfrak{y}_{\mathfrak{j}\mathfrak{k}} = \begin{cases} \mathfrak{j}\mathfrak{k}, & \mathfrak{j} \text{ and } \mathfrak{k} \text{ are squares} \\ 0, & \text{otherwise} \end{cases}$$

Thus, $\mathfrak{st}_2 - \lim\limits_{\mathfrak{y}_{\mathfrak{j}\mathfrak{k}}}^{\mathfrak{r}} = [-\mathfrak{r}, \mathfrak{r}]$. Moreover, for the subsequence $y' = (\mathfrak{y}_{\mathfrak{j}_\mathfrak{t}\mathfrak{k}_\mathfrak{t}})$ of $(\mathfrak{y}_{\mathfrak{j}\mathfrak{k}})$ such that $\mathfrak{j}_\mathfrak{t}$ and $\mathfrak{k}_\mathfrak{t}$ are squares, $\mathfrak{st}_2 - \lim\limits_{y'}^{\mathfrak{r}} = \emptyset$. It can be seen that $\mathfrak{st}_2 - \lim\limits_{\mathfrak{y}_{\mathfrak{j}\mathfrak{k}}}^{\mathfrak{r}} \not\subseteq \mathfrak{st}_2 - \lim\limits_{y'}^{\mathfrak{r}}$. However, this inclusion for the $(\mathfrak{r} - \mathfrak{st})$ convergent sequences and their dense subsequences in the $\mathfrak{NNS}$ is valid. The following theorem explains this state.

Theorem 3.1. *Let $(\mathfrak{Y}, \varphi, \nu, \psi)$ be a $\mathfrak{NNS}$. If $y' = (\mathfrak{y}_{\mathfrak{j}_\mathfrak{p}\mathfrak{k}_\mathfrak{q}})$ is a dense subsequence of $y = (\mathfrak{y}_{\mathfrak{j}_\mathfrak{k}})$, then $\mathfrak{st}_2 - \lim\limits_{y}^{\mathfrak{r}} \subseteq \mathfrak{st}_2 - \lim\limits_{y'}^{\mathfrak{r}}$.*

Proof:
The proof is obvious.

Theorem 3.2. *Let $(\mathfrak{Y}, \varphi, \nu, \psi)$ be a $\mathfrak{NNS}$. A DS $(\mathfrak{y}_{\mathfrak{j}\mathfrak{k}})$ in $\mathfrak{Y}$ is $\mathfrak{st}$- bounded if and only if $\mathfrak{st}_2 - \lim\limits_{\mathfrak{y}_{\mathfrak{j}\mathfrak{k}}}^{\mathfrak{r}} = \emptyset$, for all $\mathfrak{r} > 0$.*

Proof:
Let $(\mathfrak{Y}, \varphi, \nu, \psi)$ be a $\mathfrak{NNS}$.
$(\Rightarrow)$: Let a *DS* $(\mathfrak{y}_{\mathfrak{j}\mathfrak{k}})$ be $\mathfrak{st}-$ bounded in the $\mathfrak{NNS}$. Then, for all $\upsilon > 0, \xi \in (0,1)$ and $\mathfrak{r} > 0$, there exists a real number $\mathfrak{W} > 0$ such that

$$\delta_2(\{(\mathfrak{j}, \mathfrak{k}) \in \mathbb{N} \times \mathbb{N} : \varphi(\mathfrak{y}_{\mathfrak{j}\mathfrak{k}}; \mathfrak{W}) \le 1 - \xi \text{ or } \nu(\mathfrak{y}_{\mathfrak{j}\mathfrak{k}}; \mathfrak{W}) \ge \xi, \psi(\mathfrak{y}_{\mathfrak{j}\mathfrak{k}}; \mathfrak{W}) \ge \xi\}) = 0$$

Let $\mathfrak{K} = \{(\mathfrak{j}, \mathfrak{k}) \in \mathbb{N} \times \mathbb{N} : \varphi(\mathfrak{y}_{\mathfrak{j}\mathfrak{k}}; \mathfrak{W}) \le 1 - \xi \text{ or } \nu(\mathfrak{y}_{\mathfrak{j}\mathfrak{k}}; \mathfrak{W}) \ge \xi, \psi(\mathfrak{y}_{\mathfrak{j}\mathfrak{k}}; \mathfrak{W}) \ge \xi\}$.
For $k \in \mathfrak{K}^c, \varphi(\mathfrak{y}_{\mathfrak{j}\mathfrak{k}}; \mathfrak{W}) > 1 - \xi, \nu(\mathfrak{y}_{\mathfrak{j}\mathfrak{k}}; \mathfrak{W}) < \xi$ and $\psi(\mathfrak{y}_{\mathfrak{j}\mathfrak{k}}; \mathfrak{W}) < \xi$.
Moreover,

$$\begin{aligned} &\varphi(\mathfrak{y}_{\mathfrak{j}\mathfrak{k}}; \mathfrak{r} + \mathfrak{W}) \ge \min\{\varphi(0; \mathfrak{r}), \varphi(\mathfrak{y}_{\mathfrak{j}\mathfrak{k}}; \mathfrak{W})\} = \min\{1, \varphi(\mathfrak{y}_{\mathfrak{j}\mathfrak{k}}; \mathfrak{W})\} > 1 - \xi, \\ &\nu(\mathfrak{y}_{\mathfrak{j}\mathfrak{k}}; \mathfrak{r} + \mathfrak{W}) \le \max\{\nu(0; \mathfrak{r}), \nu(\mathfrak{y}_{\mathfrak{j}\mathfrak{k}}; \mathfrak{W})\} = \max\{0, \nu(\mathfrak{y}_{\mathfrak{j}\mathfrak{k}}; \mathfrak{W})\} < \xi \text{ and} \\ &\psi(\mathfrak{y}_{\mathfrak{j}\mathfrak{k}}; \mathfrak{r} + \mathfrak{W}) \le \max\{\nu(0; \mathfrak{r}), \psi(\mathfrak{y}_{\mathfrak{j}\mathfrak{k}}; \mathfrak{W})\} = \max\{0, \psi(\mathfrak{y}_{\mathfrak{j}\mathfrak{k}}; \mathfrak{W})\} < \xi \end{aligned}$$

Hence, $0 \in \mathfrak{st}_2 - \lim\limits_{\mathfrak{y}_{\mathfrak{j}\mathfrak{k}}}^{\mathfrak{r}}$. Consequently, $\mathfrak{st}_2 - \lim\limits_{\mathfrak{y}_{\mathfrak{j}\mathfrak{k}}}^{\mathfrak{r}} \neq \emptyset$.
$(\Leftarrow)$: Let $\mathfrak{st}_2 - \lim\limits_{\mathfrak{y}_{\mathfrak{j}\mathfrak{k}}}^{\mathfrak{r}} \neq \emptyset$, for all $\mathfrak{r} > 0$. Then, there exists $\mathfrak{y}_0 \in \mathfrak{Y}$ such that $\mathfrak{y}_0 \in \mathfrak{st}_2 - \lim\limits_{\mathfrak{y}_{\mathfrak{j}\mathfrak{k}}}^{\mathfrak{r}}$.
For all $\upsilon > 0$ and $\varepsilon \in (0,1)$,

$$\delta_2\left(\left\{\begin{array}{l} (\mathfrak{j}, \mathfrak{k}) \in \mathbb{N} \times \mathbb{N} : \varphi(\mathfrak{y}_{\mathfrak{j}\mathfrak{k}} - \mathfrak{y}_0; \mathfrak{r} + \upsilon) \le 1 - \xi \text{ or} \\ \nu(\mathfrak{y}_{\mathfrak{j}\mathfrak{k}} - \mathfrak{y}_0; \mathfrak{r} + \upsilon) \ge \xi, \psi(\mathfrak{y}_{\mathfrak{j}\mathfrak{k}} - \mathfrak{y}_0; \mathfrak{r} + \upsilon) \ge \xi \end{array}\right\}\right) = 0$$

Therefore, almost all $\mathfrak{y}_{\mathfrak{j}\mathfrak{k}}$ are contained in some ball with center $\mathfrak{y}_0$ which implies that *DS* $(\mathfrak{y}_{\mathfrak{j}\mathfrak{k}})$ is $\mathfrak{st}-$ bounded in a $\mathfrak{NNS}$. Theorem (3.3) shows that $(\mathfrak{r} - \mathfrak{st})$ convergence of a *DS* in a $\mathfrak{NNS}$ has many arithmetic properties similar to those of ordinary convergence.

Theorem 3.3. *Let $(\mathfrak{y}_{\mathfrak{j}\mathfrak{k}})$ and $(\mathfrak{z}_{\mathfrak{j}\mathfrak{k}})$ be two DSs in a $\mathfrak{NNS}$. Then, for all $\mathfrak{r} \ge 0$, the following holds:*
i. If $\mathfrak{y}_{\mathfrak{j}\mathfrak{k}} \xrightarrow{\mathfrak{r}-\mathfrak{st}_2} \mathfrak{y}_0$ and $\alpha \in \mathfrak{F}$, then $\alpha\mathfrak{y}_{\mathfrak{j}\mathfrak{k}} \xrightarrow{\mathfrak{r}-\mathfrak{st}_2} \alpha\mathfrak{y}_0$.
ii. If $\mathfrak{y}_{\mathfrak{j}\mathfrak{k}} \xrightarrow{\mathfrak{r}-\mathfrak{st}_2} \mathfrak{y}_0$ and $\mathfrak{z}_{\mathfrak{j}\mathfrak{k}} \xrightarrow{\mathfrak{r}-\mathfrak{st}_2} \mathfrak{z}_0$, then $\mathfrak{y}_{\mathfrak{j}\mathfrak{k}} + \mathfrak{z}_{\mathfrak{j}\mathfrak{k}} \xrightarrow{\mathfrak{r}-\mathfrak{st}_2} \mathfrak{y}_0 + \mathfrak{z}_0$.

Proof:
Let $(\mathfrak{y}_{\mathfrak{j}\mathfrak{k}})$ and $(\mathfrak{z}_{\mathfrak{j}\mathfrak{k}})$ be two DSs in a $\mathfrak{N}\mathfrak{N}\mathfrak{S}$ and $\mathfrak{r} \geq 0$.
i. Let $\mathfrak{y}_{\mathfrak{j}\mathfrak{k}} \xrightarrow{\mathfrak{r}-\mathfrak{s}\mathfrak{t}_2} \mathfrak{y}_0$ and $\alpha \in \mathfrak{F}$. Therefore, if for all $\upsilon > 0$ and $\xi \in (0,1)$,

$$\mathfrak{K} = \left\{ \begin{array}{c} (\mathfrak{j},\mathfrak{k}) \in \mathbb{N}\times\mathbb{N} : \varphi\left(\mathfrak{y}_{\mathfrak{j}\mathfrak{k}} - \mathfrak{y}_0; \frac{\mathfrak{r}+\upsilon}{|\alpha|}\right) \leq 1-\xi \text{ or} \\ \nu\left(\mathfrak{y}_{\mathfrak{j}\mathfrak{k}} - \mathfrak{y}_0; \frac{\mathfrak{r}+\upsilon}{|\alpha|}\right) \geq \xi, \psi\left(\mathfrak{y}_{\mathfrak{j}\mathfrak{k}} - \mathfrak{y}_0; \frac{\mathfrak{r}+\upsilon}{|\alpha|}\right) \geq \xi \end{array} \right\}$$

then $\delta_2(\mathfrak{K}) = 0$. Let $(\mathfrak{t},\mathfrak{s}) \in \mathfrak{K}^c$. Then, $\varphi\left(\mathfrak{y}_{\mathfrak{t}\mathfrak{s}} - \mathfrak{y}_0; \frac{\mathfrak{r}+\upsilon}{|\alpha|}\right) > 1-\xi, \nu\left(\mathfrak{y}_{\mathfrak{t}\mathfrak{s}} - \mathfrak{y}_0; \frac{\mathfrak{r}+\upsilon}{|\alpha|}\right) < \xi$ and $\psi\left(\mathfrak{y}_{\mathfrak{t}\mathfrak{s}} - \mathfrak{y}_0; \frac{\mathfrak{r}+\upsilon}{|\alpha|}\right) < \xi$. Hence,

$$\varphi(\alpha\mathfrak{y}_{\mathfrak{t}\mathfrak{s}} - \alpha\mathfrak{y}_0; \mathfrak{r}+\upsilon) = \varphi\left(\mathfrak{y}_{\mathfrak{t}\mathfrak{s}} - \mathfrak{y}_0; \frac{\mathfrak{r}+\upsilon}{|\alpha|}\right) > 1-\xi \tag{3.1}$$

$$\nu(\alpha\mathfrak{y}_{\mathfrak{t}\mathfrak{s}} - \alpha\mathfrak{y}_0; \mathfrak{r}+\upsilon) = \nu\left(\mathfrak{y}_{\mathfrak{t}\mathfrak{s}} - \mathfrak{y}_0; \frac{\mathfrak{r}+\upsilon}{|\alpha|}\right) < \xi \text{ and} \tag{3.2}$$

$$\psi(\alpha\mathfrak{y}_{\mathfrak{t}\mathfrak{s}} - \alpha\mathfrak{y}_0; \mathfrak{r}+\upsilon) = \psi\left(\mathfrak{y}_{\mathfrak{t}\mathfrak{s}} - \mathfrak{y}_0; \frac{\mathfrak{r}+\upsilon}{|\alpha|}\right) < \xi \tag{3.3}$$

Let $\mathfrak{H} = \{(\mathfrak{j},\mathfrak{k}) \in \mathbb{N}\times\mathbb{N} : \varphi(\alpha\mathfrak{y}_{\mathfrak{j}\mathfrak{k}} - \alpha\mathfrak{y}_0; \mathfrak{r}+\upsilon) > 1-\xi, \nu(\alpha\mathfrak{y}_{\mathfrak{j}\mathfrak{k}} - \alpha\mathfrak{y}_0; \mathfrak{r}+\upsilon) < \xi \, and \, \psi(\alpha\mathfrak{y}_{\mathfrak{j}\mathfrak{k}} - \alpha\mathfrak{y}_0; \mathfrak{r}+\upsilon) < \xi\}$.
From the Equations (3.1), (3.2) and (3.3), $(\mathfrak{t},\mathfrak{s}) \in \mathfrak{C}$. Therefore, $\mathfrak{K}^c \subseteq \mathfrak{H}$. Consequently, $\alpha\mathfrak{y}_{\mathfrak{j}\mathfrak{k}} \xrightarrow{\mathfrak{r}-\mathfrak{s}\mathfrak{t}_2} \alpha\mathfrak{y}_0$.
ii. Let $\mathfrak{y}_{\mathfrak{j}\mathfrak{k}} \xrightarrow{\mathfrak{r}-\mathfrak{s}\mathfrak{t}_2} \mathfrak{y}_0$ and $\mathfrak{z}_{\mathfrak{j}\mathfrak{k}} \xrightarrow{\mathfrak{r}-\mathfrak{s}\mathfrak{t}_2} \mathfrak{z}_0$. Therefore, if for all $\upsilon > 0$ and $\xi \in (0,1)$,

$$\mathfrak{A} = \left\{ \begin{array}{c} (\mathfrak{j},\mathfrak{k}) \in \mathbb{N}\times\mathbb{N} : \varphi\left(\mathfrak{y}_{\mathfrak{j}\mathfrak{k}} - \mathfrak{y}_0; \frac{\mathfrak{r}+\upsilon}{2}\right) \leq 1-\xi \text{ or} \\ \nu\left(\mathfrak{y}_{\mathfrak{j}\mathfrak{k}} - \mathfrak{y}_0; \frac{\mathfrak{r}+\upsilon}{2}\right) \geq \xi, \psi\left(\mathfrak{y}_{\mathfrak{j}\mathfrak{k}} - \mathfrak{y}_0; \frac{\mathfrak{r}+\upsilon}{2}\right) \geq \xi \end{array} \right\} \text{and}$$

$$\mathfrak{B} = \left\{ \begin{array}{c} (\mathfrak{j},\mathfrak{k}) \in \mathbb{N}\times\mathbb{N} : \varphi\left(\mathfrak{y}_{\mathfrak{j}\mathfrak{k}} - \mathfrak{y}_0; \mathfrak{r}+\upsilon\right) \leq 1-\xi \text{ or} \\ \nu\left(\mathfrak{y}_{\mathfrak{j}\mathfrak{k}} - \mathfrak{y}_0; \frac{\mathfrak{r}+\upsilon}{2}\right) \geq \xi, \psi\left(\mathfrak{y}_{\mathfrak{j}\mathfrak{k}} - \mathfrak{y}_0; \frac{\mathfrak{r}+\upsilon}{2}\right) \geq \xi \end{array} \right\}$$

Then, $\delta_2(\mathfrak{A}) = 0$ and $\delta_2(\mathfrak{B}) = 0$. Let $(\mathfrak{t},\mathfrak{s}) \in \mathfrak{A}^c \cap \mathfrak{B}^c$. Then,

$$\varphi\left(\mathfrak{y}_{\mathfrak{t}\mathfrak{s}} - \mathfrak{y}_0; \frac{\mathfrak{r}+\upsilon}{2}\right) > 1-\xi, \nu\left(\mathfrak{y}_{\mathfrak{t}\mathfrak{s}} - \mathfrak{y}_0; \frac{\mathfrak{r}+\upsilon}{2}\right) < \xi \text{ and } \psi\left(\mathfrak{y}_{\mathfrak{t}\mathfrak{s}} - \mathfrak{y}_0; \frac{\mathfrak{r}+\upsilon}{2}\right) < \xi \text{and}$$

$$\varphi\left(\mathfrak{z}_{\mathfrak{t}\mathfrak{s}} - \mathfrak{z}_0; \frac{\mathfrak{r}+\upsilon}{2}\right) > 1-\xi, \nu\left(\mathfrak{z}_{\mathfrak{t}\mathfrak{s}} - \mathfrak{z}_0; \frac{\mathfrak{r}+\upsilon}{2}\right) < \xi \text{ and } \psi\left(\mathfrak{z}_{\mathfrak{t}\mathfrak{s}} - \mathfrak{z}_0; \frac{\mathfrak{r}+\upsilon}{2}\right) < \xi.$$

Hence,

$$\begin{aligned} &\varphi(\mathfrak{y}_{\mathfrak{t}\mathfrak{s}} + \mathfrak{z}_{\mathfrak{t}\mathfrak{s}} - (\mathfrak{z}_0 + \mathfrak{y}_0); \mathfrak{r}+\upsilon) \\ &\qquad \geq \min\left\{\varphi\left(\mathfrak{y}_{\mathfrak{t}\mathfrak{s}} - \mathfrak{y}_0; \frac{\mathfrak{r}+\upsilon}{2}\right), \varphi\left(\mathfrak{z}_{\mathfrak{t}\mathfrak{s}} - \mathfrak{z}_0; \frac{\mathfrak{r}+\upsilon}{2}\right)\right\} > 1-\xi \end{aligned} \tag{3.4}$$

$$\begin{aligned} &\nu(\mathfrak{y}_{\mathfrak{t}\mathfrak{s}} + \mathfrak{z}_{\mathfrak{t}\mathfrak{s}} - (\mathfrak{z}_0 + \mathfrak{y}_0); \mathfrak{r}+\upsilon) \\ &\qquad \leq \max\left\{\nu\left(\mathfrak{y}_{\mathfrak{t}\mathfrak{s}} - \mathfrak{y}_0; \frac{\mathfrak{r}+\upsilon}{2}\right), \nu\left(\mathfrak{z}_{\mathfrak{t}\mathfrak{s}} - \mathfrak{z}_0; \frac{\mathfrak{r}+\upsilon}{2}\right)\right\} < \xi \text{ and} \end{aligned} \tag{3.5}$$

$$\begin{aligned} &\psi(\mathfrak{y}_{\mathfrak{t}\mathfrak{s}} + \mathfrak{z}_{\mathfrak{t}\mathfrak{s}} - (\mathfrak{z}_0 + \mathfrak{y}_0); \mathfrak{r}+\upsilon) \\ &\qquad \leq \max\left\{\psi\left(\mathfrak{y}_{\mathfrak{t}\mathfrak{s}} - \mathfrak{y}_0; \frac{\mathfrak{r}+\upsilon}{2}\right), \psi\left(\mathfrak{z}_{\mathfrak{t}\mathfrak{s}} - \mathfrak{z}_0; \frac{\mathfrak{r}+\upsilon}{2}\right)\right\} < \xi \end{aligned} \tag{3.6}$$

Let $\mathfrak{C}=\left\{\begin{array}{c}(\mathfrak{j},\mathfrak{k})\in\mathbb{N}\times\mathbb{N}:\varphi(\mathfrak{y}_{\mathfrak{j}\mathfrak{k}}+\mathfrak{z}_{\mathfrak{j}\mathfrak{k}}-(\mathfrak{y}_0+\mathfrak{z}_0);\mathfrak{r}+\upsilon)>1-\xi,\\ \nu(\mathfrak{y}_{\mathfrak{j}\mathfrak{k}}+\mathfrak{z}_{\mathfrak{j}\mathfrak{k}}-(\mathfrak{y}_0+\mathfrak{z}_0);\mathfrak{r}+\upsilon)<\xi \text{ and}\\ \psi(\mathfrak{y}_{\mathfrak{j}\mathfrak{k}}+\mathfrak{z}_{\mathfrak{j}\mathfrak{k}}-(\mathfrak{y}_0+\mathfrak{z}_0);\mathfrak{r}+\upsilon)<\xi\end{array}\right\}$.
From the Equations (3.4), (3.5) and (3.6), $(\mathfrak{t},\mathfrak{s})\in\mathfrak{C}$.
Therefore, $\mathfrak{C}\subseteq\mathfrak{A}^c\cap\mathfrak{B}^c$. Consequently, $\mathfrak{y}_{\mathfrak{j}\mathfrak{k}}+\mathfrak{z}_{\mathfrak{j}\mathfrak{k}}\xrightarrow{\mathfrak{r}-\mathfrak{st}_2}\mathfrak{y}_0+\mathfrak{z}_0$.
Theorem 3.4 and Theorem 3.5 prove some topological properties of the $(\mathfrak{r}-\mathfrak{st})$ limit set of a *DS* in $\mathfrak{NNS}$.

Theorem 3.4. *Let* $(\mathfrak{y}_{\mathfrak{j}\mathfrak{k}})$ *be a DS in a* $\mathfrak{NNS}(\mathfrak{Y},\varphi,\nu,\psi)$ *and* $\mathfrak{r}\geq 0$. *Then, the set* $\mathfrak{st}_2-\lim\limits_{\mathfrak{y}_{\mathfrak{j}\mathfrak{k}}}^{\mathfrak{r}}$ *is closed.*

Proof:
Let $(\mathfrak{y}_{\mathfrak{j}\mathfrak{k}})$ be a *DS* in a $\mathfrak{NNS}$ and $\mathfrak{r}\geq 0$. If $\mathfrak{st}_2-\lim\limits_{\mathfrak{y}_{\mathfrak{j}\mathfrak{k}}}^{\mathfrak{r}}=\emptyset$, then the theorem is valid. Therefore, let $\mathfrak{st}_2-\lim\limits_{\mathfrak{y}_{\mathfrak{j}\mathfrak{k}}}^{\mathfrak{r}}\neq\emptyset$, for all $\mathfrak{r}\geq 0$ and $\mathfrak{z}_0\in\mathfrak{st}_2-\lim\limits_{\mathfrak{y}_{\mathfrak{j}\mathfrak{k}}}^{\mathfrak{r}}$. Then, $\mathfrak{z}_{\mathfrak{j}\mathfrak{k}}\in\mathfrak{st}_2-\lim\limits_{\mathfrak{y}_{\mathfrak{j}\mathfrak{k}}}^{\mathfrak{r}}$ such that $\mathfrak{z}_{\mathfrak{j}\mathfrak{k}}\xrightarrow{(\varphi,\nu,\psi)}\mathfrak{z}_0$. Then, for all $\upsilon>0$ and $\xi\in(0,1)$, there exists a $\mathfrak{k}_1\in\mathbb{N}$ such that $\varphi\left(\mathfrak{z}_{\mathfrak{j}\mathfrak{k}}-\mathfrak{z}_0;\frac{\upsilon}{2}\right)>1-\xi$, $\nu\left(\mathfrak{z}_{\mathfrak{j}\mathfrak{k}}-\mathfrak{z}_0;\frac{\upsilon}{2}\right)<\xi$ and $\psi\left(\mathfrak{z}_{\mathfrak{j}\mathfrak{k}}-\mathfrak{z}_0;\frac{\upsilon}{2}\right)<\xi$, for all $\mathfrak{j},\mathfrak{k}\geq\mathfrak{k}_1$
Let $\mathfrak{z}_{\mathfrak{m}\mathfrak{n}}\in\mathfrak{st}_2-\lim\limits_{\mathfrak{y}_{\mathfrak{j}\mathfrak{k}}}^{\mathfrak{r}}$ for $\mathfrak{m},\mathfrak{n}>\mathfrak{k}_1$ such that

$$\delta_2\left(\begin{array}{c}(\mathfrak{j},\mathfrak{k})\in\mathbb{N}\times\mathbb{N}:\varphi\left(\mathfrak{y}_{\mathfrak{j}\mathfrak{k}}-\mathfrak{z}_{\mathfrak{m}\mathfrak{n}};\mathfrak{r}+\frac{\upsilon}{2}\right)\leq 1-\xi \text{ or}\\ \nu\left(\mathfrak{y}_{\mathfrak{j}\mathfrak{k}}-\mathfrak{z}_{\mathfrak{m}\mathfrak{n}};\mathfrak{r}+\frac{\upsilon}{2}\right)\geq\xi,\psi\left(\mathfrak{y}_{\mathfrak{j}\mathfrak{k}}-\mathfrak{z}_{\mathfrak{m}\mathfrak{n}};\mathfrak{r}+\frac{\upsilon}{2}\right)\geq\xi\end{array}\right)=0$$

For $(\mathfrak{t},\mathfrak{s})\in\mathfrak{A}=\left\{\begin{array}{c}(\mathfrak{j},\mathfrak{k})\in\mathbb{N}\times\mathbb{N}:\varphi\left(\mathfrak{y}_{\mathfrak{j}\mathfrak{k}}-\mathfrak{z}_{\mathfrak{m}\mathfrak{n}};\mathfrak{r}+\frac{\upsilon}{2}\right)>1-\xi,\\ \nu\left(\mathfrak{y}_{\mathfrak{j}\mathfrak{k}}-\mathfrak{z}_{\mathfrak{m}\mathfrak{n}};\mathfrak{r}+\frac{\upsilon}{2}\right)<\xi \text{ and } \psi\left(\mathfrak{y}_{\mathfrak{j}\mathfrak{k}}-\mathfrak{z}_{\mathfrak{m}\mathfrak{n}};\mathfrak{r}+\frac{\upsilon}{2}\right)<\xi\end{array}\right\}$
$\varphi\left(\mathfrak{y}_{\mathfrak{t}\mathfrak{s}}-\mathfrak{z}_{mn},\mathfrak{r}+\frac{\upsilon}{2}\right)>1-\xi,\nu\left(\mathfrak{y}_{\mathfrak{t}\mathfrak{s}}-\mathfrak{z}_{\mathfrak{m}\mathfrak{n}},\mathfrak{r}+\frac{\upsilon}{2}\right)<\xi$ and $\psi\left(\mathfrak{y}_{\mathfrak{t}\mathfrak{s}}-\mathfrak{z}_{\mathfrak{m}\mathfrak{n}},\mathfrak{r}+\frac{\upsilon}{2}\right)<\xi$
Then,

$$\varphi(\mathfrak{y}_{\mathfrak{t}\mathfrak{s}}-\mathfrak{z}_0,\mathfrak{r}+\upsilon)\geq\min\left\{\varphi\left(\mathfrak{y}_{\mathfrak{j}\mathfrak{k}}-\mathfrak{z}_{\mathfrak{m}\mathfrak{n}},\mathfrak{r}+\frac{\upsilon}{2}\right),\varphi\left(\mathfrak{z}_{\mathfrak{m}\mathfrak{n}}-\mathfrak{z}_0,\frac{\upsilon}{2}\right)\right\}>1-\xi \quad (3.7)$$

$$\nu(\mathfrak{y}_{\mathfrak{t}\mathfrak{s}}-\mathfrak{z}_0,\mathfrak{r}+\upsilon)\leq\max\left\{\nu\left(\mathfrak{y}_{\mathfrak{j}\mathfrak{k}}-\mathfrak{z}_{\mathfrak{m}\mathfrak{n}},\mathfrak{r}+\frac{\upsilon}{2}\right),\nu\left(\mathfrak{z}_{\mathfrak{m}\mathfrak{n}}-\mathfrak{z}_0,\frac{\upsilon}{2}\right)\right\}<\xi \quad (3.8)$$

$$\psi(\mathfrak{y}_{\mathfrak{t}\mathfrak{s}}-\mathfrak{z}_0,\mathfrak{r}+\upsilon)\leq\max\left\{\psi\left(\mathfrak{y}_{\mathfrak{j}\mathfrak{k}}-\mathfrak{z}_{\mathfrak{m}\mathfrak{n}},\mathfrak{r}+\frac{\upsilon}{2}\right),\psi\left(\mathfrak{z}_{\mathfrak{m}\mathfrak{n}}-\mathfrak{z}_0,\frac{\upsilon}{2}\right)\right\}<\xi \quad (3.9)$$

Let

$$\mathfrak{B}=\left\{\begin{array}{c}(\mathfrak{j},\mathfrak{k})\in\mathbb{N}\times\mathbb{N}:\varphi(\mathfrak{y}_{\mathfrak{j}\mathfrak{k}}-\mathfrak{z}_0,\mathfrak{r}+\upsilon)>1-\xi,\\ \nu(\mathfrak{y}_{\mathfrak{j}\mathfrak{k}}-\mathfrak{z}_0,\mathfrak{r}+\upsilon)<\xi \text{ and } \psi(\mathfrak{y}_{\mathfrak{j}\mathfrak{k}}-\mathfrak{z}_0,\mathfrak{r}+\upsilon)<\xi\end{array}\right\}.$$

From the Equations (3.7), (3.8) and (3.9), $(\mathfrak{t},\mathfrak{s})\in\mathfrak{B}$. Since $\mathfrak{A}\subseteq\mathfrak{B}$, then $\delta_2(\mathfrak{A})\leq\delta_2(\mathfrak{B})$. Consequently, $\mathfrak{z}_0\in\mathfrak{st}_2-\lim\limits_{\mathfrak{y}_{\mathfrak{j}\mathfrak{k}}}^{\mathfrak{r}}$.

Theorem 3.5. *Let* $(\mathfrak{y}_{\mathfrak{j}\mathfrak{k}})$ *be a DS in a* $\mathfrak{NNS}(\mathfrak{Y},\varphi,\nu,\psi)$ *and* $\mathfrak{r}\geq 0$. *Then, the set* $\mathfrak{st}_2-\lim\limits_{\mathfrak{y}_{\mathfrak{j}\mathfrak{k}}}^{\mathfrak{r}}$ *is convex.*

Proof:
Let $(\mathfrak{y}_{\mathfrak{j}\mathfrak{k}})$ be a *DS* in a $\mathfrak{NNS}$, $\mathfrak{r}\geq 0$, and $\mathfrak{y}_1,\mathfrak{y}_2\in\mathfrak{st}_2-\lim\limits_{\mathfrak{y}_{\mathfrak{j}\mathfrak{k}}}^{\mathfrak{r}}$. For the convexity of the set $\mathfrak{st}_2-\lim\limits_{\mathfrak{y}_{\mathfrak{j}\mathfrak{k}}}^{\mathfrak{r}}$ we should show that $[(1-t)\mathfrak{y}_1+t\mathfrak{y}_2]\in\mathfrak{st}_2-\lim\limits_{\mathfrak{y}_{\mathfrak{j}\mathfrak{k}}}^{\mathfrak{r}}$, for all $t\in(0,1)$. For all $\upsilon>0$ and $\xi\in(0,1)$, let $\mathfrak{O}_1=\left\{\begin{array}{c}(\mathfrak{j},\mathfrak{k})\in\mathbb{N}\times\mathbb{N}:\varphi\left(\mathfrak{y}_{\mathfrak{j}\mathfrak{k}}-\mathfrak{y}_1;\frac{\mathfrak{r}+\upsilon}{2(1-t)}\right)\leq 1-\xi \text{ or}\\ \nu\left(\mathfrak{y}_{\mathfrak{j}\mathfrak{k}}-\mathfrak{y}_1;\frac{\mathfrak{r}+\upsilon}{2(1-t)}\right)\geq\xi,\psi\left(\mathfrak{y}_{\mathfrak{j}\mathfrak{k}}-\mathfrak{y}_1;\frac{\mathfrak{r}+\upsilon}{2(1-t)}\right)\geq\xi\end{array}\right\}$ and

$$\mathfrak{O}_2 = \left\{ \begin{array}{c} (\mathfrak{j},\mathfrak{k}) \in \mathbb{N}\times\mathbb{N} : \varphi\left(\mathfrak{y}_{\mathfrak{j}\mathfrak{k}} - \mathfrak{y}_2; \frac{\mathfrak{r}+\upsilon}{2t}\right) \leq 1-\xi \text{ or} \\ \nu\left(\mathfrak{y}_{\mathfrak{j}\mathfrak{k}} - \mathfrak{y}_2; \frac{\mathfrak{r}+\upsilon}{2t}\right) \geq \xi, \psi\left(\mathfrak{y}_{\mathfrak{j}\mathfrak{k}} - \mathfrak{y}_2; \frac{\mathfrak{r}+\upsilon}{2t}\right) \geq \xi \end{array} \right\}.$$

By assumption, we have $\delta_2(\mathfrak{O}_1) = 0$ and $\delta_2(\mathfrak{O}_2 = 0$. For $k \in \mathfrak{O}_1^c \cap \mathfrak{O}_2^c$,

$$\begin{aligned}
&\varphi(\mathfrak{y}_{\mathfrak{j}\mathfrak{k}} - [(1-t)\mathfrak{y}_1 + t\mathfrak{y}_2]; \mathfrak{r}+\upsilon) \\
&\quad = \varphi\left((1-t)(\mathfrak{y}_{\mathfrak{j}\mathfrak{k}} - \mathfrak{y}_1) + t(\mathfrak{y}_{\mathfrak{j}\mathfrak{k}} - \mathfrak{y}_2; \mathfrak{r}+\upsilon\right) \\
&\quad \geq \min\left\{\varphi\left((1-t)(\mathfrak{y}_{\mathfrak{j}\mathfrak{k}} - \mathfrak{y}_1); \frac{\mathfrak{r}+\upsilon}{2}\right)\varphi\left(t(\mathfrak{y}_{\mathfrak{j}\mathfrak{k}} - \mathfrak{y}_2; \frac{\mathfrak{r}+\upsilon}{2}\right)\right\} \\
&\quad = \min\left\{\varphi\left(\mathfrak{y}_{\mathfrak{j}\mathfrak{k}} - \mathfrak{y}_1; \frac{\mathfrak{r}+\upsilon}{2(1-t)}\right), \varphi\left(\mathfrak{y}_{\mathfrak{j}\mathfrak{k}} - \mathfrak{y}_2; \frac{\mathfrak{r}+\upsilon}{2t}\right)\right\} \\
&\quad > 1-\xi \\
&\nu(\mathfrak{y}_{\mathfrak{j}\mathfrak{k}} - [(1-t)\mathfrak{y}_1 + t\mathfrak{y}_2]; \mathfrak{r}+\upsilon) \\
&\quad = \nu\left((1-t)(\mathfrak{y}_{\mathfrak{j}\mathfrak{k}} - \mathfrak{y}_1) + t(\mathfrak{y}_{\mathfrak{j}\mathfrak{k}} - \mathfrak{y}_2; \mathfrak{r}+\upsilon\right) \\
&\quad \leq \max\left\{\nu\left((1-t)(\mathfrak{y}_{\mathfrak{j}\mathfrak{k}} - \mathfrak{y}_1); \frac{\mathfrak{r}+\upsilon}{2}\right)\nu\left(t(\mathfrak{y}_{\mathfrak{j}\mathfrak{k}} - \mathfrak{y}_2; \frac{\mathfrak{r}+\upsilon}{2}\right)\right\} \\
&\quad = \max\left\{\nu\left(\mathfrak{y}_{\mathfrak{j}\mathfrak{k}} - \mathfrak{y}_1; \frac{\mathfrak{r}+\upsilon}{2(1-t)}\right), \nu\left(\mathfrak{y}_{\mathfrak{j}\mathfrak{k}} - \mathfrak{y}_2; \frac{\mathfrak{r}+\upsilon}{2t}\right)\right\} \\
&\quad < \xi \text{ and} \\
&\psi(\mathfrak{y}_{\mathfrak{j}\mathfrak{k}} - [(1-t)\mathfrak{y}_1 + t\mathfrak{y}_2]; \mathfrak{r}+\upsilon) \\
&\quad = \psi\left((1-t)(\mathfrak{y}_{\mathfrak{j}\mathfrak{k}} - \mathfrak{y}_1) + t(\mathfrak{y}_{\mathfrak{j}\mathfrak{k}} - \mathfrak{y}_2; \mathfrak{r}+\upsilon\right) \\
&\quad \leq \max\left\{\psi\left((1-t)(\mathfrak{y}_{\mathfrak{j}\mathfrak{k}} - \mathfrak{y}_1); \frac{\mathfrak{r}+\upsilon}{2}\right)\psi\left(t(\mathfrak{y}_{\mathfrak{j}\mathfrak{k}} - \mathfrak{y}_2; \frac{\mathfrak{r}+\upsilon}{2}\right)\right\} \\
&\quad = \max\left\{\psi\left(\mathfrak{y}_{\mathfrak{j}\mathfrak{k}} - \mathfrak{y}_1; \frac{\mathfrak{r}+\upsilon}{2(1-t)}\right), \psi\left(\mathfrak{y}_{\mathfrak{j}\mathfrak{k}} - \mathfrak{y}_2; \frac{\mathfrak{r}+\upsilon}{2t}\right)\right\} \\
&\quad < \xi.
\end{aligned}$$

Thus,

$$\delta_2\left(\left\{ \begin{array}{c} (\mathfrak{j},\mathfrak{k}) \in \mathbb{N}\times\mathbb{N} : \varphi(\mathfrak{y}_{\mathfrak{j}\mathfrak{k}} - [(1-t)\mathfrak{y}_1 + t\mathfrak{y}_2]; \mathfrak{r}+\upsilon) \leq 1-\xi \text{ or} \\ \nu(\mathfrak{y}_k - [(1-t)\mathfrak{y}_1 + t\mathfrak{y}_2]; \mathfrak{r}+\upsilon) \geq 1-\xi, \\ \psi(\mathfrak{y}_k - [(1-t)\mathfrak{y}_1 + t\mathfrak{y}_2]; \mathfrak{r}+\upsilon) \geq 1-\xi \end{array} \right\}\right) = 0.$$

Consequently, $[(1-t)\mathfrak{y}_1 + t\mathfrak{y}_2] \in \mathfrak{st}_2 - \lim\limits_{\mathfrak{y}_{\mathfrak{j}\mathfrak{k}}}^{\mathfrak{r}}$ and so $\mathfrak{st}_2 - \lim\limits_{\mathfrak{y}_{\mathfrak{j}\mathfrak{k}}}^{\mathfrak{r}}$ is a convex set.

Theorem 3.6. *Let $(\mathfrak{y}_{\mathfrak{j}\mathfrak{k}})$ be a DS in a $\mathfrak{N}\mathfrak{N}\mathfrak{S}$ $(\mathfrak{Y}, \varphi, \nu, \psi)$ and $\mathfrak{r} \geq 0$. If there exists a DS$(\mathfrak{z}_{\mathfrak{j}\mathfrak{k}})$ in $\mathfrak{Y}$, $\mathfrak{st}-$ convergent to $\mathfrak{y}_0 \in \mathfrak{Y}$ with respect to the norm (φ, ν, ψ) and, for all $\xi \in (0,1)$ and $\mathfrak{j}, \mathfrak{k} \in N$, $\varphi(\mathfrak{y}_{\mathfrak{j}\mathfrak{k}} - \mathfrak{z}_{\mathfrak{j}\mathfrak{k}}; \mathfrak{r}) > 1-\xi$, $\nu(\mathfrak{y}_{\mathfrak{j}\mathfrak{k}} - \mathfrak{z}_{\mathfrak{j}\mathfrak{k}}; \mathfrak{r}) < \xi$ and $\psi(\mathfrak{y}_{\mathfrak{j}\mathfrak{k}} - \mathfrak{z}_{\mathfrak{j}\mathfrak{k}}; \mathfrak{r}) < \xi$, then $(\mathfrak{y}_{\mathfrak{j}\mathfrak{k}})$ is $(\mathfrak{r} - \mathfrak{st})$ convergent to $\mathfrak{y}_0 \in \mathfrak{Y}$ with respect to the norm (φ, ν, ψ).*

Proof:

Let $(\mathfrak{y}_{\mathfrak{j}\mathfrak{k}})$ be a *DS* in a $\mathfrak{N}\mathfrak{N}\mathfrak{S}, \mathfrak{r} \geq 0, \upsilon > 0$ and there exists a *DS* $(\mathfrak{z}_{\mathfrak{j}\mathfrak{k}})$ in $\mathfrak{Y}$ such that $\mathfrak{z}_{\mathfrak{j}\mathfrak{k}} \xrightarrow{\mathfrak{st}_2^{(\varphi,\nu,\psi)}} \mathfrak{y}_0$ and $\varphi(\mathfrak{y}_{\mathfrak{j}\mathfrak{k}} - \mathfrak{z}_{\mathfrak{j}\mathfrak{k}}; \mathfrak{r}) > 1-\varepsilon, \nu(\mathfrak{y}_{\mathfrak{j}\mathfrak{k}} - \mathfrak{z}_{\mathfrak{j}\mathfrak{k}}; \mathfrak{r}) < \xi$ and $\psi(\mathfrak{y}_{\mathfrak{j}\mathfrak{k}} - \mathfrak{z}_{\mathfrak{j}\mathfrak{k}}; \mathfrak{r}) < \xi$, for all $\mathfrak{j}, \mathfrak{k} \in \mathbb{N}$. For given $\xi \in (0,1)$, let

$$\mathfrak{A} = \left\{ \begin{array}{c} (\mathfrak{j},\mathfrak{k}) \in \mathbb{N}\times\mathbb{N} : \varphi(\mathfrak{z}_{\mathfrak{j}\mathfrak{k}} - \mathfrak{y}_0; \upsilon) \leq 1-\xi \text{ or} \\ \nu(\mathfrak{z}_{\mathfrak{j}\mathfrak{k}} - \mathfrak{y}_0; \upsilon) \geq \xi, \psi(\mathfrak{z}_{\mathfrak{j}\mathfrak{k}} - \mathfrak{y}_0; \upsilon) \geq \xi \end{array} \right\} \text{ and}$$

$$\mathfrak{B} = \left\{ \begin{array}{c} (\mathfrak{j},\mathfrak{k}) \in \mathbb{N}\times\mathbb{N} : \varphi(\mathfrak{y}_{\mathfrak{j}\mathfrak{k}} - \mathfrak{z}_{\mathfrak{j}\mathfrak{k}}; \mathfrak{r}) \leq 1-\xi \text{ or} \\ \nu(\mathfrak{y}_{\mathfrak{j}\mathfrak{k}} - \mathfrak{z}_{\mathfrak{j}\mathfrak{k}}; \mathfrak{r}) \geq \xi, \psi(\mathfrak{y}_{\mathfrak{j}\mathfrak{k}} - \mathfrak{z}_{\mathfrak{j}\mathfrak{k}}; \mathfrak{r}) \geq \xi \end{array} \right\}$$

Clearly, $\delta_2(\mathfrak{A}) = 0$ and $\delta_2(\mathfrak{B}) = 0$. For $(\mathfrak{j},\mathfrak{k}) \in \mathfrak{A}^c \cap \mathfrak{B}^c$,

$$\varphi(\mathfrak{y}_{\mathfrak{j}\mathfrak{k}} - \mathfrak{y}_0; \mathfrak{r}+\upsilon) \geq \min\{\varphi(\mathfrak{y}_{\mathfrak{j}\mathfrak{k}} - \mathfrak{z}_{\mathfrak{j}\mathfrak{k}}; \mathfrak{r}), \varphi(\mathfrak{z}_{\mathfrak{j}\mathfrak{k}} - \mathfrak{y}_0; \upsilon)\} > 1-\xi,$$
$$\nu(\mathfrak{y}_{\mathfrak{j}\mathfrak{k}} - \mathfrak{y}_0; \mathfrak{r}+\upsilon) \leq \max\{\nu(\mathfrak{y}_{\mathfrak{j}\mathfrak{k}} - \mathfrak{z}_{\mathfrak{j}\mathfrak{k}}; \mathfrak{r}), \nu(\mathfrak{z}_{\mathfrak{j}\mathfrak{k}} - \mathfrak{y}_0; \upsilon)\} < \xi \text{ and}$$
$$\psi(\mathfrak{y}_{\mathfrak{j}\mathfrak{k}} - \mathfrak{y}_0; \mathfrak{r}+\upsilon) \leq \max\{\psi(\mathfrak{y}_{\mathfrak{j}\mathfrak{k}} - \mathfrak{z}_{\mathfrak{j}\mathfrak{k}}; \mathfrak{r}), \psi(\mathfrak{z}_{\mathfrak{j}\mathfrak{k}} - \mathfrak{y}_0; \upsilon)\} < \xi.$$

Then, for all $(\mathfrak{j},\mathfrak{k}) \in \mathfrak{A}^c \cap \mathfrak{B}^c$,

$$\varphi(\mathfrak{y}_{\mathfrak{j}\mathfrak{k}} - \mathfrak{y}_0; \mathfrak{r}+\upsilon) > 1-\xi, \nu(\mathfrak{y}_{\mathfrak{j}\mathfrak{k}} - \mathfrak{y}_0; \mathfrak{r}+\upsilon) < \xi \text{ and } \psi(\mathfrak{y}_{\mathfrak{j}\mathfrak{k}} - \mathfrak{y}_0; \mathfrak{r}+\upsilon) < \xi.$$

This implies that

$$\left\{ \begin{array}{c} (\mathfrak{j},\mathfrak{k}) \in \mathbb{N}\times\mathbb{N} : \varphi(\mathfrak{y}_{\mathfrak{j}\mathfrak{k}} - \mathfrak{y}_0; \mathfrak{r}+\upsilon) \leq 1-\xi \text{ or} \\ \nu(\mathfrak{y}_{\mathfrak{j}\mathfrak{k}} - \mathfrak{y}_0; \mathfrak{r}+\upsilon) \geq \xi, \psi(\mathfrak{y}_{\mathfrak{j}\mathfrak{k}} - \mathfrak{y}_0; \mathfrak{r}+\upsilon) \geq \xi \end{array} \right\} \subseteq \mathfrak{A} \cup \mathfrak{B}.$$

Then,

$$\delta_2\left(\left\{ \begin{array}{c} (\mathfrak{j},\mathfrak{k}) \in \mathbb{N}\times\mathbb{N} : \varphi(\mathfrak{y}_{\mathfrak{j}\mathfrak{k}} - \mathfrak{y}_0; \mathfrak{r}+\upsilon) \leq 1-\xi \text{ or} \\ \nu(\mathfrak{y}_{\mathfrak{j}\mathfrak{k}} - \mathfrak{y}_0; \mathfrak{r}+\upsilon) \geq \xi, \psi(\mathfrak{y}_{\mathfrak{j}\mathfrak{k}} - \mathfrak{y}_0; \mathfrak{r}+\upsilon) \geq \xi \end{array} \right\}\right) \leq \delta_2(\mathfrak{A}) + \delta_2(\mathfrak{B}).$$

Hence, $\delta_2\left(\left\{ \begin{array}{c} (\mathfrak{j},\mathfrak{k}) \in \mathbb{N}\times\mathbb{N} : \varphi(\mathfrak{y}_{\mathfrak{j}\mathfrak{k}} - \mathfrak{y}_0; \mathfrak{r}+\upsilon) \leq 1-\xi \text{ or} \\ \nu(\mathfrak{y}_{\mathfrak{j}\mathfrak{k}} - \mathfrak{y}_0; \mathfrak{r}+\upsilon) \geq \xi, \psi(\mathfrak{y}_{\mathfrak{j}\mathfrak{k}} - \mathfrak{y}_0; \mathfrak{r}+\upsilon) \geq \xi \end{array} \right\}\right) = 0.$

Consequently, $\mathfrak{y}_{\mathfrak{j}\mathfrak{k}} \xrightarrow{\mathfrak{r}-\mathfrak{st}_2} \mathfrak{y}_0$.

Theorem 3.7. *Let $y = (\mathfrak{y}_{\mathfrak{j}\mathfrak{k}})$ be a sequence in a $\mathfrak{NNS}$ and $\mathfrak{r} > 0$. Then, there does not exist $\mathfrak{z}, \mathfrak{x} \in \mathfrak{st}_2 - \overset{\mathfrak{r}}{\underset{\mathfrak{y}_{\mathfrak{j}\mathfrak{k}}}{\lim}}$ such that $\varphi(\mathfrak{z}-\mathfrak{x}; \mathfrak{m}\mathfrak{r}) \leq 1-\xi$ or $\nu(\mathfrak{z}-\mathfrak{x}; \mathfrak{m}\mathfrak{r}) \geq \xi, \psi(\mathfrak{z}-\mathfrak{x}; \mathfrak{m}\mathfrak{r}) \geq \xi$ for $\xi \in (0,1)$ and $\mathfrak{m} > 2$.*

Proof:

Let $(\mathfrak{y}_{\mathfrak{j}\mathfrak{k}})$ be a sequence in a $\mathfrak{NNS}$ and $\mathfrak{r} > 0$. Assume that there exists $\mathfrak{z}, \mathfrak{x} \in \mathfrak{st}_2 - \overset{\mathfrak{r}}{\underset{\mathfrak{y}_{\mathfrak{j}\mathfrak{k}}}{\lim}}$ such that for $\mathfrak{m} > 2, \varphi(\mathfrak{z}-\mathfrak{x}; \mathfrak{m}\mathfrak{r}) \leq 1-\xi$ or $\nu(\mathfrak{z}-\mathfrak{x}; \mathfrak{m}\mathfrak{r}) \geq \xi, \psi(\mathfrak{z}-\mathfrak{x}; \mathfrak{m}\mathfrak{r}) \geq \xi$. For given $\xi \in (0,1)$ and $\upsilon > 0$. $\mathfrak{K}_1$ and $\mathfrak{K}_2$ are denoted as follows:

$$\mathfrak{K}_1 = \left\{ \begin{array}{c} (\mathfrak{j},\mathfrak{k}) \in \mathbb{N}\times\mathbb{N} : \varphi\left(\mathfrak{y}_{\mathfrak{j}\mathfrak{k}} - \mathfrak{z}; \mathfrak{r}+\frac{\upsilon}{2}\right) \leq 1-\xi \text{ or} \\ \nu\left(\mathfrak{y}_{\mathfrak{j}\mathfrak{k}} - \mathfrak{z}; \mathfrak{r}+\frac{\upsilon}{2}\right) \geq \xi, \psi\left(\mathfrak{y}_{\mathfrak{j}\mathfrak{k}} - \mathfrak{z}; \mathfrak{r}+\frac{\upsilon}{2}\right) \geq \xi \end{array} \right\}$$

$$\mathfrak{K}_2 = \left\{ \begin{array}{c} (\mathfrak{j},\mathfrak{k}) \in \mathbb{N}\times\mathbb{N} : \varphi\left(\mathfrak{y}_{\mathfrak{j}\mathfrak{k}} - \mathfrak{x}; \mathfrak{r}+\frac{\upsilon}{2}\right) \leq 1-\xi \text{ or} \\ \nu\left(\mathfrak{y}_{\mathfrak{j}\mathfrak{k}} - \mathfrak{x}; \mathfrak{r}+\frac{\upsilon}{2}\right) \geq \xi, \psi\left(\mathfrak{y}_{\mathfrak{j}\mathfrak{k}} - \mathfrak{x}; \mathfrak{r}+\frac{\upsilon}{2}\right) \geq \xi \end{array} \right\}$$

Hence, $\delta(\mathfrak{K}_1) = 0$ and $\delta(\mathfrak{K}_2 = 0$. For $(\mathfrak{j},\mathfrak{k}) \in \mathfrak{K}_1^c \cap \mathfrak{K}_2^c$,

$$\varphi(\mathfrak{z}-\mathfrak{x}; 2\mathfrak{r}+\upsilon) \geq \min\left\{\varphi\left(\mathfrak{y}_{\mathfrak{j}\mathfrak{k}} - \mathfrak{x}; \mathfrak{r}+\frac{\upsilon}{2}\right), \varphi\left(\mathfrak{y}_{\mathfrak{j}\mathfrak{k}} - \mathfrak{z}; \mathfrak{r}+\frac{\upsilon}{2}\right)\right\} > 1-\xi$$
$$\nu(\mathfrak{z}-\mathfrak{x}; 2\mathfrak{r}+\upsilon) \leq \max\left\{\nu\left(\mathfrak{y}_{\mathfrak{j}\mathfrak{k}} - \mathfrak{x}; \mathfrak{r}+\frac{\upsilon}{2}\right), \nu\left(\mathfrak{y}_{\mathfrak{j}\mathfrak{k}} - \mathfrak{z}; \mathfrak{r}+\frac{\upsilon}{2}\right)\right\} < \xi \text{ and}$$
$$\psi(\mathfrak{z}-\mathfrak{x}; 2\mathfrak{r}+\upsilon) \leq \max\left\{\psi\left(\mathfrak{y}_{\mathfrak{j}\mathfrak{k}} - \mathfrak{x}; \mathfrak{r}+\frac{\upsilon}{2}\right), \psi\left(\mathfrak{y}_{\mathfrak{j}\mathfrak{k}} - \mathfrak{z}; \mathfrak{r}+\frac{\upsilon}{2}\right)\right\} < \xi.$$

Hence, $\varphi(\mathfrak{z}-\mathfrak{x}; 2\mathfrak{r}+\upsilon) > 1-\xi$ and $\nu(\mathfrak{z}-\mathfrak{x}; 2\mathfrak{r}+\upsilon) < \xi$ and $\psi(\mathfrak{z}-\mathfrak{x}; 2\mathfrak{r}+\upsilon) < \xi$. Then, $\varphi(\mathfrak{z}-\mathfrak{x}; \mathfrak{m}\mathfrak{r}) > 1-\xi$ or $\nu(\mathfrak{y}-\mathfrak{z}; \mathfrak{m}\mathfrak{r}) < \xi, \psi(\mathfrak{y}-\mathfrak{z}; \mathfrak{m}\mathfrak{r}) < \xi$ for $\mathfrak{m} > 2$ which is

contradiction to the hypothesis. Therefore, there does not exists $\mathfrak{z}, \mathfrak{x} \in \mathfrak{st}_2 - \lim\limits_{\mathfrak{y}_{\mathfrak{j}\mathfrak{k}}}^{\mathfrak{r}}$ such that $\varphi(\mathfrak{z} - \mathfrak{x}; \mathfrak{m}\mathfrak{r}) \leq 1 - \xi$ or $\nu(\mathfrak{z} - \mathfrak{x}; \mathfrak{m}\mathfrak{r}) \geq \xi, \psi(\mathfrak{z} - \mathfrak{x}; \mathfrak{m}\mathfrak{r}) \geq \xi$ for $\xi \in (0,1)$ and $\mathfrak{m} > 2$.
Next, the concept of $(\mathfrak{r} - \mathfrak{st})$ cluster points of a double sequence in a $\mathfrak{NNS}$ is defined, and some related results are proposed.

Definition 3.7. Let $(\mathfrak{Y}, \varphi, \nu, \psi)$ be a $\mathfrak{NNS}, \gamma \in \mathfrak{Y}$ and $\mathfrak{r} \geq 0$. Then, γ is called $(\mathfrak{r} - \mathfrak{st})$ cluster point of the DS $(\mathfrak{y}_{\mathfrak{j}\mathfrak{k}})$ in $\mathfrak{Y}$ with respect to the norm (φ, ν, ψ) if for all $\upsilon > 0$ and $\xi \in (0,1)$

$$\delta_2\left(\left\{\begin{array}{c}(\mathfrak{j}, \mathfrak{k}) \in \mathbb{N} \times \mathbb{N} : \varphi(\mathfrak{y}_{\mathfrak{j}\mathfrak{k}} - \gamma; \mathfrak{r} + \upsilon) > 1 - \xi, \\ \nu(\mathfrak{y}_{\mathfrak{j}\mathfrak{k}} - \gamma; \mathfrak{r} + \upsilon) < \xi \text{ and } \psi(\mathfrak{y}_{\mathfrak{j}\mathfrak{k}} - \gamma; \mathfrak{r} + \upsilon) < \xi\end{array}\right\}\right) > 0.$$

The set of all the $\mathfrak{r} - \mathfrak{st}_2$-cluster points of $y = (\mathfrak{y}_{\mathfrak{j}\mathfrak{k}})$ in a $\mathfrak{NNS}$ is denoted by $\Gamma^{\mathfrak{r}}_{(\varphi,\nu,\psi)_2}(y)$. If $\mathfrak{r} = 0$, then $\Gamma^{\mathfrak{r}}_{(\varphi,\nu,\psi)_2}(y) = \Gamma_{(\varphi,\nu,\psi)_2}(y)$.

Theorem 3.8. *Let $y = (\mathfrak{y}_{\mathfrak{j}\mathfrak{k}})$ be a DS in a $\mathfrak{NNS}$ and $\mathfrak{r} \geq 0$. Then, $\Gamma^{\mathfrak{r}}_{(\varphi,\nu,\psi)_2}(y)$ is a closed set.*

Proof:
Let $(\mathfrak{y}_{\mathfrak{j}\mathfrak{k}})$ be a DS in a $\mathfrak{NNS}$ and $\mathfrak{r} \geq 0$.
i. If $\Gamma^{\mathfrak{r}}_{(\varphi,\nu,\psi)_2}(y) = \emptyset$, then the theorem is valid.
ii. Let $\Gamma^{\mathfrak{r}}_{(\varphi,\nu,\psi)_2}(y) = \emptyset$; and $\mathfrak{z}_0 \in \overline{\Gamma^{\mathfrak{r}}_{(\varphi,\nu,\psi)_2}(y)}$. Then, there is a DS $(\mathfrak{z}_{\mathfrak{j}\mathfrak{k}})$ in $\Gamma^{\mathfrak{r}}_{(\varphi,\nu,\psi)_2}(y)$ such that $(\mathfrak{z}_{\mathfrak{j}\mathfrak{k}}) \xrightarrow{(\varphi,\nu,\psi)} \mathfrak{z}_0$, for all $\mathfrak{j}, \mathfrak{k} \in \mathbb{N}$. It is sufficient to show that $\mathfrak{z}_0 \in \Gamma^{\mathfrak{r}}_{(\varphi,\nu,\psi)_2}(y)$.
As $(\mathfrak{z}_{\mathfrak{j}\mathfrak{k}}) \xrightarrow{(\varphi,\nu,\psi)} \mathfrak{z}_0$, for all $\upsilon > 0$ and $\xi \in (0,1)$, there exists $k_\xi \in \mathbb{N}$ such that $\varphi\left(\mathfrak{z}_{\mathfrak{j}\mathfrak{k}} - \mathfrak{z}_0; \frac{\upsilon}{2}\right) > 1 - \xi, \nu\left(\mathfrak{z}_{\mathfrak{j}\mathfrak{k}} - \mathfrak{z}_0; \frac{\upsilon}{2}\right) < \xi$ and $\psi\left(\mathfrak{z}_{\mathfrak{j}\mathfrak{k}} - \mathfrak{z}_0; \frac{\upsilon}{2}\right) < \xi$, for all $\mathfrak{j}, \mathfrak{k} \geq k_\xi$. Let $\mathfrak{j}_0, \mathfrak{k}_0 \in \mathbb{N}$ such that $\mathfrak{j}_0, \mathfrak{k}_0 \geq k_\xi$.
Then,

$$\varphi\left(\mathfrak{z}_{\mathfrak{j}_0\mathfrak{k}_0} - \mathfrak{z}_0; \frac{\upsilon}{2}\right) > 1 - \xi, \nu\left(\mathfrak{z}_{\mathfrak{j}_0\mathfrak{k}_0} - \mathfrak{z}_0; \frac{\upsilon}{2}\right) < \xi \text{ and } \psi\left(\mathfrak{z}_{\mathfrak{j}_0\mathfrak{k}_0} - \mathfrak{z}_0; \frac{\upsilon}{2}\right) < \xi.$$

Since $\mathfrak{z}_{\mathfrak{j}\mathfrak{k}} \in \Gamma^{\mathfrak{r}}_{(\varphi,\nu,\psi)_2}(y), \mathfrak{z}_{\mathfrak{j}_0\mathfrak{k}_0} \in \Gamma^{\mathfrak{r}}_{(\varphi,\nu,\psi)_2}(y)$.
Thus,

$$\delta_2\left(\left\{\begin{array}{c}(\mathfrak{j}, \mathfrak{k}) \in \mathbb{N} \times \mathbb{N} : \varphi\left(\mathfrak{z}_{\mathfrak{j}\mathfrak{k}} - \mathfrak{z}_{\mathfrak{j}_0\mathfrak{k}_0}; \mathfrak{r} + \frac{\upsilon}{2}\right) > 1 - \xi, \\ \nu\left(\mathfrak{z}_{\mathfrak{j}\mathfrak{k}} - \mathfrak{z}_{\mathfrak{j}_0\mathfrak{k}_0}; \mathfrak{r} + \frac{\upsilon}{2}\right) < \xi \text{ and } \psi\left(\mathfrak{z}_{\mathfrak{j}\mathfrak{k}} - \mathfrak{z}_{\mathfrak{j}_0\mathfrak{k}_0}; \mathfrak{r} + \frac{\upsilon}{2}\right) < \xi\end{array}\right\}\right) > 0.$$

Let

$$\mathfrak{A} = \left\{\begin{array}{c}(\mathfrak{j}, \mathfrak{k}) \in \mathbb{N} \times \mathbb{N} : \varphi\left(\mathfrak{y}_{\mathfrak{j}\mathfrak{k}} - \mathfrak{z}_{\mathfrak{j}_0\mathfrak{k}_0}; \mathfrak{r} + \frac{\upsilon}{2}\right) > 1 - \xi, \\ \nu\left(\mathfrak{y}_{\mathfrak{j}\mathfrak{k}} - \mathfrak{z}_{\mathfrak{j}_0\mathfrak{k}_0}; \mathfrak{r} + \frac{\upsilon}{2}\right) < \xi \text{ and } \psi\left(\mathfrak{y}_{\mathfrak{j}\mathfrak{k}} - \mathfrak{z}_{\mathfrak{j}_0\mathfrak{k}_0}; \mathfrak{r} + \frac{\upsilon}{2}\right) < \xi\end{array}\right\}.$$

Choose $(t, s) \in \mathfrak{A}$. Then,

$$\varphi\left(\mathfrak{y}_{ts} - \mathfrak{z}_{\mathfrak{j}_0\mathfrak{k}_0}; \mathfrak{r} + \frac{\upsilon}{2}\right) > 1 - \xi, \nu\left(\mathfrak{y}_{\mathfrak{j}\mathfrak{k}} - \mathfrak{z}_{\mathfrak{j}_0\mathfrak{k}_0}; \mathfrak{r} + \frac{\upsilon}{2}\right) < \xi \text{ and } \psi\left(\mathfrak{y}_{\mathfrak{j}\mathfrak{k}} - \mathfrak{z}_{\mathfrak{j}_0\mathfrak{k}_0}; \mathfrak{r} + \frac{\upsilon}{2}\right) < \xi.$$

Therefore,

$$\varphi(\mathfrak{y}_{ts} - \mathfrak{z}_0; \mathfrak{r} + \upsilon) \geq \min\left\{\varphi\left(\mathfrak{y}_{ts} - \mathfrak{z}_{\mathfrak{j}_0\mathfrak{k}_0}; \mathfrak{r} + \frac{\upsilon}{2}\right), \varphi\left(\mathfrak{z}_{\mathfrak{j}_0\mathfrak{k}_0} - \mathfrak{z}_0; \frac{\upsilon}{2}\right)\right\} > 1 - \xi \quad (3.10)$$

$$\nu(\mathfrak{y}_{ts} - \mathfrak{z}_0; \mathfrak{r} + \upsilon) \leq \max\left\{\nu\left(\mathfrak{y}_{ts} - \mathfrak{z}_{\mathfrak{j}_0\mathfrak{k}_0}; \mathfrak{r} + \frac{\upsilon}{2}\right), \nu\left(\mathfrak{z}_{\mathfrak{j}_0\mathfrak{k}_0} - \mathfrak{z}_0; \frac{\upsilon}{2}\right)\right\} < \xi \text{ and} \quad (3.11)$$

$$\psi(\mathfrak{y}_{ts} - \mathfrak{z}_0; \mathfrak{r} + \upsilon) \leq \max\left\{\psi\left(\mathfrak{y}_{ts} - \mathfrak{z}_{\mathfrak{j}_0\mathfrak{k}_0}; \mathfrak{r} + \frac{\upsilon}{2}\right), \psi\left(\mathfrak{z}_{\mathfrak{j}_0\mathfrak{k}_0} - \mathfrak{z}_0; \frac{\upsilon}{2}\right)\right\} < \xi \quad (3.12)$$

Let $\mathfrak{B}=\left\{\begin{array}{c}(\mathfrak{j},\mathfrak{k})\in\mathbb{N}\times\mathbb{N}:\varphi(\mathfrak{y}_{\mathfrak{j}\mathfrak{k}}-\mathfrak{z}_0;\mathfrak{r}+\upsilon)>1-\xi,\\ \nu\left(\mathfrak{y}_{\mathfrak{j}\mathfrak{k}}-\mathfrak{z}_0;\mathfrak{r}+\frac{\upsilon}{2}\right)<\xi \text{ and } \psi\left(\mathfrak{y}_{\mathfrak{j}\mathfrak{k}}-\mathfrak{z}_0;\mathfrak{r}+\frac{\upsilon}{2}\right)<\xi\end{array}\right\}$.
From the Equations (3.10), (3.11) and (3.12), $(t,s)\in\mathfrak{B}$.
Thereby, $\mathfrak{A}\subseteq\mathfrak{B}$ and so $\delta_2(\mathfrak{A})\leq\delta_2(\mathfrak{B})$.
Therefore,

$$\delta_2\left(\left\{\begin{array}{c}(\mathfrak{j},\mathfrak{k})\in\mathbb{N}\times\mathbb{N}:\varphi(\mathfrak{y}_{\mathfrak{j}\mathfrak{k}}-\mathfrak{z}_0;\mathfrak{r}+\upsilon)>1-\xi,\\ \nu(\mathfrak{y}_{\mathfrak{j}\mathfrak{k}}-\mathfrak{z}_0;\mathfrak{r}+\upsilon)<\xi,\psi(\mathfrak{y}_{\mathfrak{j}\mathfrak{k}}-\mathfrak{z}_0;\mathfrak{r}+\upsilon)<\xi\end{array}\right\}\right)>0.$$

Consequently, $\mathfrak{z}_0\in\Gamma^{\mathfrak{r}}_{(\varphi,\nu,\psi)_2}(y)$.

Theorem 3.9. *Let $y=(\mathfrak{y}_{\mathfrak{j}\mathfrak{k}})$ be a DS in a $\mathfrak{N}\mathfrak{N}\mathfrak{S}$. Then, for an arbitrary $\gamma\in\Gamma_{(\varphi,\nu,\psi)_2}(y)$ and $\xi\in(0,1),\varphi(\varepsilon-\gamma;\mathfrak{r})>1-\xi,\nu(\varepsilon-\gamma;\mathfrak{r})<\xi$ and $\psi(\varepsilon-\gamma;\mathfrak{r})<\xi$ for all $\varepsilon\in\Gamma^{\mathfrak{r}}_{(\varphi,\nu,\psi)_2}(y)$.*

Proof:
Let $y=(\mathfrak{y}_{\mathfrak{j}\mathfrak{k}})$ be a *DS* in a $\mathfrak{N}\mathfrak{N}\mathfrak{S}$ and $\gamma\in\Gamma_{(\mu,\nu,\psi)_2}(y)$. Then, for all $\upsilon>0$ and $\xi\in(0,1)$,

$$\delta_2\left(\left\{\begin{array}{c}(\mathfrak{j},\mathfrak{k})\in\mathbb{N}\times\mathbb{N}:\varphi(\mathfrak{y}_{\mathfrak{j}\mathfrak{k}}-\gamma;\upsilon)>1-\xi,\\ \nu(\mathfrak{y}_{\mathfrak{j}\mathfrak{k}}-\gamma;\upsilon)<\xi \text{ and } \psi(\mathfrak{y}_{\mathfrak{j}\mathfrak{k}}-\gamma;\upsilon)<\xi\end{array}\right\}\right)>0.$$

Let $\mathfrak{A}=\left\{\begin{array}{c}(\mathfrak{j},\mathfrak{k})\in\mathbb{N}\times\mathbb{N}:\varphi(\mathfrak{y}_{\mathfrak{j}\mathfrak{k}}-\gamma;\upsilon)>1-\xi,\\ \nu(\mathfrak{y}_{\mathfrak{j}\mathfrak{k}}-\gamma;\upsilon)<\xi \text{ and } \psi(\mathfrak{y}_{\mathfrak{j}\mathfrak{k}}-\gamma;\upsilon)<\xi\end{array}\right\}$.
Choose $(t,s)\in\mathfrak{A}$.
Then, $\varphi(\mathfrak{y}_{ts}-\gamma;\upsilon)>1-\xi,\nu(\mathfrak{y}_{ts}-\gamma;\upsilon)<\xi$ and $\psi(\mathfrak{y}_{ts}-\gamma;\upsilon)<\xi$.
Thus,

$$\varphi(\mathfrak{y}_{ts}-\varepsilon;\mathfrak{r}+\upsilon)\geq\min\{\varphi(\mathfrak{y}_{ts}-\gamma;\upsilon),\varphi(\varepsilon-\gamma;\mathfrak{r})\}>1-\xi \text{ and} \tag{3.13}$$
$$\nu(\mathfrak{y}_{ts}-\varepsilon;\mathfrak{r}+\upsilon)\leq\max\{\nu(\mathfrak{y}_{ts}-\gamma;\upsilon),\nu(\varepsilon-\gamma;\mathfrak{r})\}<\xi \tag{3.14}$$
$$\psi(\mathfrak{y}_{ts}-\varepsilon;\mathfrak{r}+\upsilon)\leq\max\{\psi(\mathfrak{y}_{ts}-\gamma;\upsilon),\psi(\varepsilon-\gamma;\mathfrak{r})\}<\xi \tag{3.15}$$

Let $\mathfrak{B}=\left\{\begin{array}{c}(\mathfrak{j},\mathfrak{k})\in\mathbb{N}\times\mathbb{N}:\varphi(\mathfrak{y}_{\mathfrak{j}\mathfrak{k}}-\varepsilon;\mathfrak{r}+\upsilon)>1-\xi,\\ \nu(\mathfrak{y}_{\mathfrak{j}\mathfrak{k}}-\varepsilon;\mathfrak{r}+\upsilon)<\xi \text{ and } \psi(\mathfrak{y}_{\mathfrak{j}\mathfrak{k}}-\varepsilon;\mathfrak{r}+\upsilon)<\xi\end{array}\right\}$.
From the Equations (3.13), (3.14) and (3.15), $(t,s)\in\mathfrak{B}$.
Thereby, $\mathfrak{A}\subseteq\mathfrak{B},\delta_2(\mathfrak{A})\leq\delta_2(\mathfrak{B})$. Therefore,

$$\delta_2\left(\left\{\begin{array}{c}(\mathfrak{j},\mathfrak{k})\in\mathbb{N}\times\mathbb{N}:\varphi(\mathfrak{y}_{\mathfrak{j}\mathfrak{k}}-\varepsilon;\mathfrak{r}+\upsilon)>1-\xi,\\ \nu(\mathfrak{y}_{\mathfrak{j}\mathfrak{k}}-\varepsilon;\mathfrak{r}+\upsilon)<\xi \text{ and } \psi(\mathfrak{y}_{\mathfrak{j}\mathfrak{k}}-\varepsilon;\mathfrak{r}+\upsilon)<\xi\end{array}\right\}\right)>0.$$

Consequently, $\varepsilon\in\Gamma^{\mathfrak{r}}_{(\varphi,\nu,\psi)_2}(y)$.

Theorem 3.10. *Let $y=(\mathfrak{y}_{\mathfrak{j}\mathfrak{k}})$ be a DS in a $\mathfrak{N}\mathfrak{N}\mathfrak{S},\mathfrak{r}>0$ and $c\in\mathfrak{Y}$. Then, $\Gamma^{\mathfrak{r}}_{(\varphi,\nu,\psi)_2}(y)=\bigcup_{c\in\Gamma_{(\varphi,\nu,\psi)_2}(y)}\overline{\mathfrak{B}(c,\xi,\mathfrak{r})}$.*

Proof: Let $y=(\mathfrak{y}_{\mathfrak{j}\mathfrak{k}})$ be a *DS* in a $\mathfrak{N}\mathfrak{N}\mathfrak{S}$ and $\mathfrak{r}>0$. Let $\gamma\in\bigcup_{c\in\Gamma_{(\varphi,\nu,\psi)_2}(y)}\overline{B(c,\xi,\mathfrak{r})}$, then there exists $c\in\Gamma_{(\varphi,\nu,\psi)_2}(y)$ such that for all $\mathfrak{r}>0$ and given $\xi\in(0,1),\varphi(c-\gamma;\mathfrak{r})>1-\xi,\nu(c-\gamma;\mathfrak{r})<\xi$ and $\psi(c-\gamma;\mathfrak{r})<\xi$. Fix $\upsilon>0$. Since $c\in\Gamma_{(\varphi,\nu,\psi)_2}(y)$, there exists a set

$$\mathfrak{K}=\left\{\begin{array}{c}(\mathfrak{j},\mathfrak{k})\in\mathbb{N}\times\mathbb{N}:\varphi(\mathfrak{y}_{\mathfrak{j}\mathfrak{k}}-c;\upsilon)>1-\xi,\\ \nu(\mathfrak{y}_{\mathfrak{j}\mathfrak{k}}-c;\upsilon)<\xi \text{ and } \psi(\mathfrak{y}_{\mathfrak{j}\mathfrak{k}}-c;\upsilon)<\xi\end{array}\right\}$$

such that $\delta_2(\mathfrak{K}) > 0$. For $(\mathfrak{j}, \mathfrak{k}) \in \mathfrak{K}$,

$$\varphi(\mathfrak{y}_{\mathfrak{j}\mathfrak{k}} - \gamma; \mathfrak{r} + \upsilon) \geq \min\{\varphi(\mathfrak{y}_{\mathfrak{j}\mathfrak{k}} - c; \upsilon), \varphi(c - \gamma; \mathfrak{r})\} > 1 - \xi$$
$$\nu(\mathfrak{y}_{\mathfrak{j}\mathfrak{k}} - \gamma; \mathfrak{r} + \upsilon) \leq \max\{\nu(\mathfrak{y}_{\mathfrak{j}\mathfrak{k}} - c; \upsilon), \nu(c - \gamma; \mathfrak{r})\} < \xi \text{ and}$$
$$\psi(\mathfrak{y}_{\mathfrak{j}\mathfrak{k}} - \gamma; \mathfrak{r} + \upsilon) \leq \max\{\psi(\mathfrak{y}_{\mathfrak{j}\mathfrak{k}} - c; \upsilon), \psi(c - \gamma; \mathfrak{r})\} < \xi$$

This implies that

$$\delta_2\left(\left\{\begin{array}{c} (\mathfrak{j}, \mathfrak{k}) \in \mathbb{N} \times \mathbb{N} : \varphi(\mathfrak{y}_{\mathfrak{j}\mathfrak{k}} - \gamma; \mathfrak{r} + \upsilon) > 1 - \xi, \\ \nu(\mathfrak{y}_{\mathfrak{j}\mathfrak{k}} - \gamma; \mathfrak{r} + \upsilon) < \xi \text{ and } \psi(\mathfrak{y}_{\mathfrak{j}\mathfrak{k}} - \gamma; \mathfrak{r} + \upsilon) < \xi \end{array}\right\}\right) > 0.$$

Hence, $\gamma \in \Gamma^{\mathfrak{r}}_{(\varphi,\nu,\psi)_2}(y)$. Therefore, $\bigcup_{c \in \Gamma_{(\varphi,\nu,\psi)_2}(y)} \overline{B(c, \xi, \mathfrak{r})} \subseteq \Gamma^{\mathfrak{r}}_{(\varphi,\nu,\psi)_2}(y)$.
Conversely, let $\gamma \in \Gamma^{\mathfrak{r}}_{(\varphi,\nu,\psi)_2}(y)$ and $\gamma \notin \bigcup_{c \in \Gamma_{(\varphi,\nu,\psi)_2}(y)} \overline{B(c, \xi, \mathfrak{r})}$ and so $\gamma \notin \overline{B(c, \xi, \mathfrak{r})}$, for all $c \in \Gamma_{(\varphi,\nu,\psi)_2}(y)$. Then, $\varphi(\gamma - c; \mathfrak{r}) \leq 1 - \xi$ or $\nu(\gamma - c; \mathfrak{r}) \geq \xi, \psi(\gamma - c; \mathfrak{r}) \geq \xi$ for all $c \in \Gamma_{(\varphi,\nu,\psi)_2}(y)$.
By Theorem (3.9), for $\gamma \in \Gamma_{(\varphi,\nu,\psi)_2}(y), \varphi(\gamma - c; \mathfrak{r}) > 1 - \xi, \nu(\gamma - c; \mathfrak{r}) < \xi$ and $\psi(\gamma - c; \mathfrak{r}) < \xi$ for all $c \in \Gamma^{\mathfrak{r}}_{(\varphi,\nu,\psi)_2}(y)$ which is a contradiction to the assumption.
Therefore, $\gamma \in \bigcup\limits_{c \in \Gamma_{(\varphi,\nu,\psi)_2}(y)} \overline{B(c, \xi, \mathfrak{r})}$.
Hence, $\Gamma^{\mathfrak{r}}_{(\varphi,\nu,\psi)_2}(y) \subseteq \bigcup\limits_{c \in \Gamma_{(\varphi,\nu,\psi)_2}(y)} \overline{B(c, \xi, \mathfrak{r})}$.

Theorem 3.11. *Let $y = (\mathfrak{y}_{\mathfrak{j}\mathfrak{k}})$ be a DS in a $\mathfrak{N}\mathfrak{N}\mathfrak{S}$ and $\mathfrak{r} > 0$. Then, for all $\xi \in (0, 1)$,*
i. If $c \in \Gamma_{(\varphi,\nu,\psi)_2}(y)$, then $\mathfrak{s}t_2 - \lim\limits_{\mathfrak{y}_{\mathfrak{j}\mathfrak{k}}}^{\mathfrak{r}} \subseteq \overline{B(c, \xi, \mathfrak{r})}$.
ii. $\mathfrak{s}t_2 - \lim\limits_{\mathfrak{y}_{\mathfrak{j}\mathfrak{k}}}^{\mathfrak{r}} = \bigcap\limits_{c \in \Gamma_{(\varphi,\nu,\psi)_2}(y)} \overline{B(c, \xi, \mathfrak{r})} = \left\{\xi \in \mathfrak{Y} : \Gamma_{(\varphi,\nu,\psi)_2}(y) \subseteq \overline{B(c, \xi, \mathfrak{r})}\right\}$.

Proof: Let $y = (\mathfrak{y}_{\mathfrak{j}\mathfrak{k}})$ be a DS in a $\mathfrak{N}\mathfrak{N}\mathfrak{S}$.
i. Consider $\xi \in \mathfrak{s}t_2 - \lim\limits_{\mathfrak{y}_{\mathfrak{j}\mathfrak{k}}}^{\mathfrak{r}}$ and $c \in \Gamma_{(\varphi,\nu,\psi)_2}(y)$. For all $\upsilon > 0$ and $\xi \in (0, 1)$, let

$$\mathfrak{A} = \left\{\begin{array}{c} (\mathfrak{j}, \mathfrak{k}) \in \mathbb{N} \times \mathbb{N} : \varphi(\mathfrak{y}_{\mathfrak{j}\mathfrak{k}} - \varepsilon : \mathfrak{r} + \upsilon) > 1 - \xi, \\ \nu(\mathfrak{y}_{\mathfrak{j}\mathfrak{k}} - \varepsilon; \mathfrak{r} + \upsilon) < \xi \text{ and } \psi(\mathfrak{y}_{\mathfrak{j}\mathfrak{k}} - \varepsilon; \mathfrak{r} + \upsilon) < \xi \end{array}\right\}$$

and

$$\mathfrak{B} = \{(\mathfrak{j}, \mathfrak{k}) \in \mathbb{N} \times \mathbb{N} : \varphi(\mathfrak{y}_{\mathfrak{j}\mathfrak{k}} - c; \upsilon) > 1 - \xi, \nu(\mathfrak{y}_{\mathfrak{j}\mathfrak{k}} - c; \upsilon) < \xi \text{ and } \psi(\mathfrak{y}_{\mathfrak{j}\mathfrak{k}} - c; \upsilon) < \xi\}.$$

Thus, $\delta_2(\mathfrak{A}^c) = 0$ and $\delta_2(\mathfrak{B}) \neq 0$. For $(\mathfrak{j}, \mathfrak{k}) \in \mathfrak{A} \cap \mathfrak{B}$,

$$\varphi(\varepsilon - c; \mathfrak{r}) \geq \min\{\varphi(\mathfrak{y}_{\mathfrak{j}\mathfrak{k}} - c; \upsilon), \varphi(\mathfrak{y}_{\mathfrak{j}\mathfrak{k}} - \varepsilon; \mathfrak{r} + \upsilon\} > 1 - \xi$$
$$\nu(\varepsilon - c; \mathfrak{r}) \leq \max\{\nu(\mathfrak{y}_{\mathfrak{j}\mathfrak{k}} - c; \upsilon), \nu(\mathfrak{y}_{\mathfrak{j}\mathfrak{k}} - \varepsilon; \mathfrak{r} + \upsilon)\} < \xi \text{ and}$$
$$\psi(\varepsilon - c; \mathfrak{r}) \leq \max\{\psi(\mathfrak{y}_{\mathfrak{j}\mathfrak{k}} - c; \upsilon), \psi(\mathfrak{y}_{\mathfrak{j}\mathfrak{k}} - \varepsilon; \mathfrak{r} + \upsilon)\} < \xi.$$

Therefore, $\varepsilon \in \overline{B(c, \xi, \mathfrak{r})}$. Hence, $\mathfrak{s}t_2 - \lim\limits_{\mathfrak{y}_{\mathfrak{j}\mathfrak{k}}}^{\mathfrak{r}} \subseteq \overline{B(c, \xi, \mathfrak{r})}$.
ii. From the statement, $\mathfrak{s}t_2 - \lim\limits_{\mathfrak{y}_{\mathfrak{j}\mathfrak{k}}}^{\mathfrak{r}} \subseteq \bigcap\limits_{c \in \Gamma_{(\varphi,\nu,\psi)_2}(y))} \overline{B(c, \xi, \mathfrak{r})}$.
Let $\mathfrak{z} \in \bigcap\limits_{c \in \Gamma_{(\varphi,\nu,\psi)_2}(y)} \overline{B(c, \xi, \mathfrak{r})}$.
Then, $\varphi(\mathfrak{z} - c; \mathfrak{r}) \geq 1 - \xi, \nu(\mathfrak{z} - c; \mathfrak{r}) \leq \xi$ and $\psi(\mathfrak{z} - c; \mathfrak{r}) \leq \xi$ for all $c \in \Gamma_{(\varphi,\nu,\psi)_2}(y)$.

This implies that $\Gamma_{(\varphi,\nu,\psi)_2}(y) \subseteq \overline{B(\mathfrak{z},\xi,\mathfrak{r})}$ and so
$$\bigcap_{c\in\Gamma_{(\varphi,\nu,\psi)_2}(y)} \overline{B(\varepsilon,\xi,\mathfrak{r})} \subseteq \{\varepsilon \in \mathfrak{Y} : \Gamma_{(\varphi,\nu,\psi)_2}(y) \subseteq \overline{B(\varepsilon,\xi,\mathfrak{r})}\}.$$
Further, let $\mathfrak{z} \notin \mathfrak{st}_2 - \lim\limits_{\mathfrak{y}_{\mathfrak{j}\mathfrak{k}}}^{\mathfrak{r}}$. Then, for $\upsilon > 0$,
$$\delta_2\left(\left\{\begin{array}{c} (\mathfrak{j},\mathfrak{k}) \in \mathbb{N}\times\mathbb{N} : \varphi(\mathfrak{y}_{\mathfrak{j}\mathfrak{k}} - \mathfrak{z};\mathfrak{r}+\upsilon) \leq 1-\xi \text{ or} \\ \nu(\mathfrak{y}_{\mathfrak{j}\mathfrak{k}} - \mathfrak{z};\mathfrak{r}+\upsilon) \geq \xi, \psi(\mathfrak{y}_{\mathfrak{j}\mathfrak{k}} - \mathfrak{z};\mathfrak{r}+\upsilon) \geq \xi \end{array}\right\}\right) \neq 0$$
which implies that a statistical cluster point c exists for the sequence y such that
$$\varphi(\mathfrak{z}-c;\mathfrak{r}+\upsilon) \leq 1-\xi \text{ or } \nu(\mathfrak{z}-c;\mathfrak{r}+\upsilon) \geq \xi, \psi(\mathfrak{z}-c;\mathfrak{r}+\upsilon) \geq \xi.$$
Thus, $\Gamma_{(\varphi,\nu,\psi)_2}(y) \nsubseteq \overline{B(y,\xi,\mathfrak{r})}$ and $\mathfrak{z} \notin \left\{\varepsilon \in \mathfrak{Y} : \Gamma_{(\varphi,\nu,\psi)_2}(y) \subseteq \overline{B(\varepsilon,\xi,\mathfrak{r})}\right\}$.
Therefore, $\{\varepsilon \in \mathfrak{Y} : \Gamma_{(\varphi,\nu,\psi)_2}(y) \subseteq \overline{B(\varepsilon,\xi,\mathfrak{r})} \subseteq \mathfrak{st}_2 - \lim\limits_{\mathfrak{y}_{\mathfrak{j}\mathfrak{k}}}^{\mathfrak{r}}$
and so $\bigcap\limits_{c\in\Gamma_{(\varphi,\nu,\psi)}(y)} \overline{B(c,\xi,\mathfrak{r})} \subseteq \mathfrak{st}_2 - \lim\limits_{\mathfrak{y}_{\mathfrak{j}\mathfrak{k}}}^{\mathfrak{r}}$.
Consequently, $\mathfrak{st}_2 - \lim\limits_{\mathfrak{y}_{\mathfrak{j}\mathfrak{k}}}^{\mathfrak{r}} = \bigcap\limits_{c\in\Gamma_{(\varphi,\nu,\psi)_2}(y)} \overline{B(c,\xi,\mathfrak{r})} = \{\xi \in \mathfrak{Y} : \Gamma_{(\varphi,\nu,\psi)_2}(y) \subseteq \overline{B(\varepsilon,\xi,\mathfrak{r})}\}$.

Theorem 3.12. *Let $(\mathfrak{y}_{\mathfrak{j}\mathfrak{k}})$ be a DS in a $\mathfrak{NNS}$. If $(\mathfrak{y}_{\mathfrak{j}\mathfrak{k}})$ is $\mathfrak{st}-$convergent to $\varepsilon \in \mathfrak{Y}$ with respect to the norm (φ,ν,ψ), then for all $\xi \in (0,1)$ and $\mathfrak{r} > 0$, $\mathfrak{st}_2 - \lim\limits_{\mathfrak{y}_{\mathfrak{j}\mathfrak{k}}}^{\mathfrak{r}} = \overline{B(\varepsilon,\xi,\mathfrak{r})}$ is hold.*

Proof:
Let $(\mathfrak{y}_{\mathfrak{j}\mathfrak{k}})$ be a *DS* in a $\mathfrak{NNS}$ and $(\mathfrak{z}_{\mathfrak{j}\mathfrak{k}})$ be $\mathfrak{st}-$convergent to $\varepsilon \in \mathfrak{Y}$ with respect to the norm (φ,ν,ψ) and $\upsilon > 0$. Since $\mathfrak{z}_{\mathfrak{j}\mathfrak{k}} \xrightarrow{\mathfrak{st}_2^{(\varphi,\nu,\psi)}} \varepsilon$, then there exists a set
$$\mathfrak{A} = \{(\mathfrak{j},\mathfrak{k}) \in \mathbb{N}\times\mathbb{N} : \varphi(\mathfrak{y}_{\mathfrak{j}\mathfrak{k}} - \varepsilon;\upsilon) \leq 1-\xi \text{ or } \nu(\mathfrak{y}_{\mathfrak{j}\mathfrak{k}} - \varepsilon;\upsilon) \geq \xi, \psi(\mathfrak{y}_{\mathfrak{j}\mathfrak{k}} - \varepsilon;\upsilon) \geq \xi\}$$
such that $\delta_2(\mathfrak{A}) = 0$.
Let $\mathfrak{z} \in \overline{B(\varepsilon,\xi,\mathfrak{r})} = \{y \in \mathfrak{Y} : \varphi(y-\varepsilon;\mathfrak{r}) \geq 1-\xi, \nu(y-\varepsilon;\mathfrak{r}) \leq \xi \text{ and } \psi(y-\varepsilon;\mathfrak{r}) \leq \xi\}$.
For $(\mathfrak{j},\mathfrak{k}) \in \mathfrak{A}^c$,
$$\varphi(\mathfrak{y}_{\mathfrak{j}\mathfrak{k}} - \mathfrak{z};\mathfrak{r}+\upsilon) \geq \min\{\varphi(\mathfrak{y}_{\mathfrak{j}\mathfrak{k}} - \varepsilon;\upsilon), \varphi(\mathfrak{z}-\varepsilon;\mathfrak{r})\} > 1-\xi$$
$$\nu(\mathfrak{y}_{\mathfrak{j}\mathfrak{k}} - \mathfrak{z};\mathfrak{r}+\upsilon) \leq \max\{\nu(\mathfrak{y}_{\mathfrak{j}\mathfrak{k}} - \varepsilon;\upsilon), \nu(\mathfrak{z}-\varepsilon;\mathfrak{r})\} < \xi \text{ and}$$
$$\psi(\mathfrak{y}_{\mathfrak{j}\mathfrak{k}} - \mathfrak{z};\mathfrak{r}+\upsilon) \leq \max\{\psi(\mathfrak{y}_{\mathfrak{j}\mathfrak{k}} - \varepsilon;\upsilon), \psi(\mathfrak{z}-\varepsilon;\mathfrak{r})\} < \xi$$
This implies that $\mathfrak{z} \in \mathfrak{st}_2 - \lim\limits_{\mathfrak{y}_{\mathfrak{j}\mathfrak{k}}}^{\mathfrak{r}}$, i.e., $\overline{B(\varepsilon,\xi,\mathfrak{r})} \subseteq \mathfrak{st}_2 - \lim\limits_{\mathfrak{y}_{\mathfrak{j}\mathfrak{k}}}^{\mathfrak{r}}$.
On the other hand, $\mathfrak{st}_2 - \lim\limits_{\mathfrak{y}_{\mathfrak{j}\mathfrak{k}}}^{\mathfrak{r}} \subseteq \overline{B(\varepsilon,\xi,\mathfrak{r})}$. Hence, $\mathfrak{st}_2 - \lim\limits_{\mathfrak{y}_{\mathfrak{j}\mathfrak{k}}}^{\mathfrak{r}} = \overline{B(\varepsilon,\xi,\mathfrak{r})}$.

Theorem 3.13. *Let $y = (\mathfrak{y}_{\mathfrak{j}\mathfrak{k}})$ be a DS in a $\mathfrak{NNS}$. If y is $\mathfrak{st}-$ convergent to $\varepsilon \in \mathfrak{Y}$ with respect to the norm (φ,ν,ψ), then $\Gamma^{\mathfrak{r}}_{(\varphi,\nu,\psi)_2}(y) = \mathfrak{st}_2 - \lim\limits_{\mathfrak{y}_{\mathfrak{j}\mathfrak{k}}}^{\mathfrak{r}}$ for some $\mathfrak{r} > 0$.*

Proof:
Let $y = (\mathfrak{y}_{\mathfrak{j}\mathfrak{k}})$ be a *DS* in a $\mathfrak{NNS}$ and y be $\mathfrak{st}-$convergent to $\varepsilon \in \mathfrak{Y}$ with respect to the norm (φ,ν,ψ). Then, $\Gamma^{\mathfrak{r}}_{(\varphi,\nu,\psi)_2}(y) = \{\varepsilon\}$. By Theorem (3.10), for some $\mathfrak{r} > 0$ and $\xi \in (0,1)$, $\Gamma^{\mathfrak{r}}_{(\varphi,\nu,\psi)_2}(y) = \overline{B(\varepsilon,\xi,\mathfrak{r})}$.

Moreover, by Theorem (3.12), $\overline{B(\varepsilon, \xi, \mathfrak{r})} = \mathfrak{st}_2 - \lim\limits_{\mathfrak{y}_{\mathfrak{j}\mathfrak{k}}}^{\mathfrak{r}}.$

Hence, $\Gamma^{\mathfrak{r}}_{(\varphi,\nu,\psi)_2}(y) = \mathfrak{st}_2 - \lim\limits_{\mathfrak{y}_{\mathfrak{j}\mathfrak{k}}}^{\mathfrak{r}}.$

4. Conclusion

This paper studies the concept of $(\mathfrak{r} - \mathfrak{st})$ convergence, a generalization of rough convergence, and statistical convergence in $\mathfrak{NNS}$. Then, it defines the concepts of $(\mathfrak{r} - \mathfrak{st})$ limit and $(\mathfrak{r} - \mathfrak{st})$ cluster points' sets and investigates some of their basic properties. In future studies, researchers can study the concepts proposed herein for triple sequences. Moreover, they can define the concept of rough ideal convergence of a double sequence in a $\mathfrak{NNS}$ and examines its basic properties.

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