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A STUDY OF ORDERED QUASI-HYPERIDEALS AND ORDERED BI-HYPERIDEALS IN REGULAR ORDERED SEMIHYPERGROUPS

ABUL BASAR* AND MOHAMMAD YAHYA ABBASI

ABSTRACT. In this paper, we introduce the concept of ordered quasi-hyperideals of regular ordered semihypergroups, and study the basic results on ordered quasi-hyperideals of ordered semihypergroups. We also investigate regular ordered semihypergroups in terms of its ordered quasi-hyperideals, ordered right hyperideals and ordered left hyperideals. We prove that: (i) A partially ordered semihypergroup S is regular if and only if for every ordered bi-hyperideal B, every ordered hyperideal I and every ordered quasi-hyperideal Q, we have $B \cap I \cap Q \subseteq (B \circ I \circ Q]$, and (ii) A partially ordered semihypergroup S is regular if and only if for every ordered quasi-hyperideal Q, every ordered left hyperideal L and every ordered right-hyperideal R, we have $R \cap Q \cap L \subseteq (R \circ Q \circ L]$.

1. INTRODUCTION AND BASIC DEFINITIONS

Steinfeld [29], [30], [31] gave the concept of quasi-ideal in rings and in semigroups. Thereafter, this notion has been the subject of study of mathematicians of all over the world and lots of results have been derived by applying the concept of ideals, quasi-ideals and bi-ideals to semigroups, ordered semigroups, ternary semigroups, regular rings, near rings, and various other algebraic structures as can be seen from the vast literature available on the subject matter [2], [11], [17], [21], [33], [22], [23], [24], [25], [26], [28], [19], [20], [38], [39]. Our objective in this paper is to investigate some results based on ordered left(right) hyperideals, ordered quasi-hyperideals and ordered bi-hyperideals in ordered semihypergroups. Basar [1] introduced relative weakly hyperideals and relative prime bi-hyperideals in ordered hypersemigroups and in involution ordered hypersemigroups.

The notion of algebraic hyperstructures was introduced by Marty [16]. Algebraic hyperstructures are a standard generalization of classical algebraic structures. In a classical algebraic structures, the composition of two elements is an element while in an algebraic hyperstructuress, the composition of two elements is a set. An interesting book has been written on the subject based on hypergroups by Corsini [37].

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*Corresponding author.

On the other hand, algebraic hyperstructure theory has many applications to other fields: geometry, graphs, and hypergraphs, binary relations, lattices, groups, fuzzy sets and rough sets, automata, cryptography, codes, median algebras, relation algebras, C-algebras, artificial intelligence, probabilities and many more fields. The book on application of hyperstructures to various fields has been written by Corsini and Leoreanu [35] . Prenowitz and Jantosciak [40] showed its applications in geometry. Davvaz and Leoreanu-Fotea [8] wrote a monograph starting with some basic results concerning ring theory and algebraic hyperstructures which represent the most general algebraic context in which the real world problems can be modelled. Several kinds of hyperrings are introduced and analyzed in this book. For application in Chemistry and Physics, one can refer to [10], [12], [13], [15]. It describes various kinds of hyperstructures: ehyperstructures and transposition hypergroups.

The concept of Hv-structures constitute a generalization of the well-known algebraic hyperstructures (hypergroup, hyperring, hypermodule). For ordered semigroups, we refer to [3], [4], [5], [27], [32]. Heideri and Davvaz [14] studied ordered hyperstructures. For semihypergroups, we refer to [6], [7], [9]. .Hila and Davvaz[18] studied quasi-hyperideals of ordered semihypergroups. Corsini [36], [37] also studied hypergroup theory.

Let S be a nonempty set. Furthermore, let A and B be two nonempty subsets of S, then the hyperproduct of A and B is defined as below:

$$A \circ B = \bigcup_{a \in A, b \in B} a \circ b \ , \ x \circ A = \{x\} \circ A \ , \ A \circ x = A \circ \{x\}.$$

Also, $A \circ B = \bigcup \{a \circ b \mid a \in A, b \in B\}$. A mapping $\circ : S \times S \to P^{\star}(S)$, where $P^{\star}(S)$ is the family of all nonempty subsets of S, is called hyperoperation on S. The couple $(S, \circ)$ is called a hypergroupoid. A hypergroupoid $(S, \circ)$ is called a semihypergroup if for every $x, y, z \in S$, we have the following:

$$x \circ (y \circ z) = (x \circ y) \circ z,$$

i.e.,

$$\bigcup_{u \in y \circ z} x \circ u = \bigcup_{v \in x \circ y} v \circ z.$$

If A and B are nonempty subsets of S, then we say that $A \leq B$ if for every $a \in A$ there exists $b \in B$ such that $a \leq b$. Clearly, every ordered semigroup is an ordered semihypergroup. A nonempty subset A of an ordered semihypergroup $(S, \circ, \leq)$ is called a subsemihypergroup of S if $A \circ A \subseteq A$. An ordered semihypergroup $(S, \circ, \leq)$ is a semihypergroup $(S, \circ)$ together with a partial order $\leq$ that is compatible with the hyperoperation, that is, for any $x, y, z \in S$, we have the following:

$$x \leq y \Rightarrow z \circ x \leq z \circ y,$$

and

$$x \circ z \leq y \circ z.$$

Here, $z \circ x \leq z \circ y$ implies that for any $a \in z \circ x$, there exists $b \in z \circ y$ such that $a \leq b$. The case

$$x \circ z \leq y \circ z$$

is defined similarly.

Throughout this paper, S will denote an ordered semihypergroup unless otherwise specified. An ordered semihypergroup S is called regular if for each $s \in S$, there exists $a \in S$ such that $s \leq s \circ a \circ s$. Equivalent definitions of regular ordered semihypergroup is as follows:

(1): $A \subseteq (A \circ S \circ A]$ for each $A \subseteq S$.
(2): $s \in (s \circ S \circ s]$ for each $s \in S$.

Let $(S, \leq)$ be an ordered semihypergroup, and N be a sub-semihypergroup of S, then $(N, \leq)$ is an ordered semihypergroup. Let A be a nonempty subset of N. Then in a similar fashion, we easily have the following:

$$(A]_N = \{n \in N : n \leq a \text{ for some a} \in \text{A}\}$$

and $A \cup a = A \cup \{a\}$. We also write $(A]_N$ by simply $(A]$ if $N = S$. A nonempty subset I of an ordered semihypergroup S is called an ordered right-hyperideal (left-hyperideal) of S if $I \circ S \subseteq I(S \circ I \subseteq I)$, and for any $x \in I$, $(x] \subseteq I$. We recall that I is called an ordered hyperideal of S if it is both a left and a right ordered hyperideal of S. Also, for any $s \in S$, we have $(S \circ s]$ is an ordered left hyperideal of S, and $(s \circ S]$ is an ordered right hyperideal of S.

A nonempty subset Q of S is called an ordered quasi-hyperideal of S if

(1): $(Q \circ S] \cap (S \circ Q] \subseteq Q$, and
(2): $(Q] \subseteq Q$.

A subsemihypergroup B of an ordered semihypergroup S is called an ordered bi-hyperideal of S if $B \circ S \circ B \subseteq B$, and for any $x \in B$, $(x] \subseteq B$.

Let X be a nonempty subset of S. Then the least right(left) ordered hyperideal of S containing X is given by

$$R(X) = (X \cup X \circ S](L(X) = (S \circ X \cup X]).$$

If $X = \{s\}$, $s \in S$, we write $R(s)$ and $L(s)$, respectively by $R(s)$ and $L(s)$ and, $R(s) = (s \cup s \circ S]$ and $L(s) = (S \circ s \cup s]$ and the hyperideal generated by $s \in S$ is given by $I(s) = (s \cup S \circ s \cup s \circ S \cup S \cup S \circ s \circ S]$. Also, the least quasi-hyperideal of S containing X is denoted by $Q(X)$. Moreover, we introduce the following notations:

(1): $N_Q = \{Q : Q \neq \phi \text{ where Q} \subseteq \text{S and (Q]} \subseteq \text{Q}\}$,
(2): R_I is a set of ordered right hyperideal of S,
(3): L_I is a set of ordered left hyperideal of S, and
(3): I_T is a two-sided hyperideal of S.

Now for any two elements $Q_1, Q_2 \in N_Q$, we define an operation $\star$ on N_Q as follows:

$$Q_1 \star Q_2 = (Q_1 \circ Q_2].$$

Furthermore, let N be a sub-semihypergroup of S. Then easily we observe the following:

(1): $A \subseteq (A]_N \subseteq (A] = ((A]]$ for $A \subseteq N$.
(2): For $A \subseteq N$ and $B \subseteq N$, we have $(A \cup B] = (A] \cup (B]$.
(3): For $A \subseteq N$ and $B \subseteq N$, we have $(A \cap B] \subseteq (A] \cap (B]$.
(4): For $a, b \in N$ with $a \leq b$, we have $(a \circ N] \subseteq (b \circ N]$ and $(N \circ a] \subseteq (N \circ b]$.
(5): $(A] \circ (B] \subseteq (A \circ B]$.
(6): For every left(right, two-sided) ordered hyperideal L of S, $(L] = L$.
(7): If A and B are two ordered hyperideals of S, then $(A \circ B]$ and $A \cup B$ are also ordered hyperideals of S.

(8): For any $s \in S$, $(S \circ s \circ S]$ is an ordered hyperideal of S.

2. Ordered Semihypergroups, Ordered Quasi-hyperideals and Ordered Bi-hyperideals

In this section, we study some properties of semihypergroups based on ordered right(left) hyperideals, ordered hyperideals, ordered quasi-hyperideals and ordered bi-hyperideals. We start with the following:

Lemma 2.1. *Let S be an ordered semihypergroup. Then:*

(1): $(N_Q, \star, \subseteq)$ *is an ordered semihypergroup.*
(2): $(L_I, \star, \subseteq), (R_I, \star, \subseteq)$ *and* $(I_T, \circ, \subseteq)$ *are sub-semihypergroups of* $(N_Q, \star, \subseteq)$.

Proof. Suppose that $P, Q, R \in N_Q$. Since $P \circ Q \subseteq (P \circ Q]$, we obtain $((P \circ Q) \circ R] \subseteq ((P \circ Q] \circ R]$. Next, we have $(P \star Q) \star R = (P \circ Q \circ R]$ by using the following:

$$(P \star Q) \star R = (P \circ Q] \star R = ((P \circ Q] \circ R] \subseteq ((P \circ Q) \circ R] = (P \circ Q \circ R].$$

In a similar fashion, we can show that $P \star (Q \star R) = (P \circ Q \circ R]$. Therefore $(P \star Q) \star R = P \star (Q \star R)$. Hence $(N_Q, \star)$ is a semihypergroup. Suppose that $P \subseteq Q$. Then $P \star R = (P \circ Q] \subseteq (Q \circ R] = Q \star R$ and $R \star P = (R \circ P] \subseteq (R \circ Q] = R \star Q$. Hence $(N_Q, \star, \subseteq)$ is an ordered semihypergroup.

(ii) We have L_I, R_I and I_T are nonempty subsets of N_Q. Suppose that $L_1, L_2 \in L_I$. Then obviously, we have $(L_1 \star L_2] = ((L_1 \circ L_2]] = (L_1 \circ L_2]$. Moreover, using the following:

$$\begin{aligned} S \circ (L_1 \star L_2) &= S \circ (L_1 \circ L_2] \\ &\subseteq (S \circ (L_1 \circ L_2]] \\ &\subseteq ((S \circ L_1) \circ L_2] \\ &\subseteq (L_1 \circ L_2] \\ &= L_1 \star L_2, \end{aligned}$$

we infer that $L_1 \star L_2$ is a left hyperideal of S, that is, $L_1 \star L_2 \subseteq L_1$. Thus $(L_1, \star, \subseteq)$ is a sub-semihypergroup of $(N_Q, \star, \subseteq)$.

Dually, we can prove that $(R_I, \star, \subseteq)$ is a sub-semihypergroup of $(N_Q, \star, \subseteq)$. Since $I_T = L_I \cap R_I$, it follows that $(I_T, \star, \subseteq)$ is a sub-semihypergroup of $(N_Q, \star, \subseteq)$. □

Let $Q_I = \{Q : Q \text{ is an ordered quasi} - \text{hyperideal of S}\}$. Then, obviously, we have $L_I \cup R_I \subseteq Q_I \subseteq N_Q$. This implies that every one-sided hyperideal of an ordered semihypergroup is a quasi-hyperideal of S. Thus the class of ordered quasi-hyperideal of S is a generalization of the class of one-sided ordered hyperideals of S.

Lemma 2.2. *Each ordered quasi-hyperideal Q of an ordered semihypergroup S is a sub-semihypergroup of S.*

Proof. The proof is straightforward. Furthermore, we have the following:

$$Q \circ Q \subseteq (Q \circ S] \cap (S \circ Q] \subseteq Q.$$

□

Lemma 2.3. *For every ordered right hyperideal R and an ordered left hyperideal L of an ordered semihypergroup S, $R \cap L$ is an ordered quasi-hyerideal of S.*

Proof. As $R \circ L \subseteq S \circ L \subseteq L$, and $R \circ L \subseteq R \circ S \subseteq R$, we obtain $R \circ L \subseteq R \cap L$, so $R \cap L \neq \phi$. Now the fact that $R \cap L$ is an ordered quasi-hyperideal of S follows from the below:

(1): $(R \cap L] \subseteq (R] \cap (L] \subseteq R \cap L$,
(2): $((R \cap L) \circ S] \cap (S \circ (R \cap L)] \subseteq (R \circ S] \cap (S \circ L] \subseteq (R] \cap (L] \subseteq R \cap L$.

□

Lemma 2.4. *Let Q be an ordered quasi-hyperideal of S, then we obtain $Q = L(Q) \cap R(Q) = (S \circ Q \cup Q] \cap (Q \cup Q \circ S]$.*

Proof. Obviously, we have the following relation:

$$Q \subseteq (S \circ Q \cup Q] \cap (Q \cup Q \circ S].$$

Conversely, suppose that $a \in (S \circ Q \cup Q] \cap (Q \cup Q \circ S]$. Then $a \leq b$, or $a \leq x \circ u$ and $a \leq v \circ y$ for some $b, u, v \in Q$; $x, y \in S$. As Q is an ordered quasi-hyperideal of S, the former case implies $a \in (Q] \subseteq Q$ and the later case implies $a \in (S \circ Q] \cap (Q \circ S] \subseteq Q$. Therefore, $(S \circ Q \cup Q] \cap (Q \cup Q \circ S] = Q$. □

We recall here that if X is a nonempty subset of an ordered semihypergroup S, then we write the least quasi-hyperideal of S containing X by $Q(X)$. If $X = \{a\}$, we write $Q(\{a\})$ by $Q(a)$.

Theorem 2.5. *Suppose that S is an ordered semihypergroup. Then we have the following:*

(1): *For every $s \in S$, $Q(s) = L(s) \cap R(s) = (S \circ s \cup s] \cap (s \cup s \circ S]$.*
(2): *Let $\phi \neq X \subseteq S, Q(X) = L(X) \cap R(X) = (S \circ X \cup X] \cap (X \cup X \circ S]$.*

Proof. Suppose that $s \in S$. Using Lemma 2.3, we have $L(s) \cap R(s)$ is a quasi-hyperideal of S containing s. Therefore, we have $Q(s) \subseteq L(s) \cap R(s)$. By Lemma 2.4, we obtain the following:

$$\begin{aligned} L(s) \cap R(s) &= (S \circ s \cup s] \cap (s \cup s \circ S] \\ &\subseteq (S \circ Q(s) \cup Q(s)] \cap (Q(s) \cup Q(s) \circ S] \\ &= Q(s). \end{aligned}$$

Hence $Q(s) = L(s) \cap R(s)$.
(ii) Its proof can be given as (i). □

The notion of an ordered bi-hyeridеal is a generalization of the notion of an ordered quasi-hyperideal. In a similar fashion, the class of ordered quasi-hyperideals of ordered semihypergroups is a particular case of the class of ordered bi-hyperideal of ordered semihypergroups.

Theorem 2.6. *Suppose that I is a two-sided ordered hyperideal of an ordered semihypergroup S and Q is an ordered quasi-hyperideal of I, then Q is an ordered bi-hyperideal of S.*

Proof. Since Q is an ordered quasi-hyperideal of I, and $Q \subseteq I$, we obtain the following:

$$\begin{aligned} Q \circ Q &\subseteq Q \circ S \circ I \\ &= Q \circ (S \circ I) \\ &\subseteq Q \circ I \\ &\subseteq (Q \circ I] \\ &\subseteq (S \circ I] \\ &\subseteq (I] \\ &\subseteq I, \end{aligned}$$

and

$$\begin{aligned} Q \circ S \circ Q &\subseteq I \circ Q \circ S \\ &= (I \circ S) \circ Q \\ &\subseteq I \circ Q \\ &\subseteq (I \circ Q] \\ &\subseteq (I \circ S] \\ &\subseteq (I] \\ &\subseteq I, \end{aligned}$$

and $q \in (Q]$ implies that there exists $q_1 \in Q \subseteq I$ such that

$$q \leq q_1 \Rightarrow q \in (I] = I.$$

and

$$q \in (Q] \Rightarrow q \in I \cap (Q] = (Q]_I \subseteq Q.$$

Therefore, we have the following:

$$\begin{aligned} Q \circ S \circ Q &\subseteq (I \cap (I \circ Q]) \cap (I \cap (Q \circ I]) \\ &= (I \circ Q]_I \cap (Q \circ I]_I \\ &\subseteq Q, \end{aligned}$$

and $(Q] \subseteq Q$. Hence, applying these assertions together with Lemma 2.2, we have proved that Q is an ordered bi-hyperideal of S. □

3. Regular Ordered Semihypergroups and Ordered Quasi-hyperideals

In this section, we use the concept of ordered hyperideals, ordered quasi-hyperideals, ordered bi-hyperideals to obtain various equivalent conditions and investigate regular ordered semihypergroups and idempotent semihypergroups.

Lemma 3.1. *Let S be an ordered semihypergroup. Then the ordered sub-semihypergroup of $(N_Q, \star)$ generated by $(L_I, \star)$ and $(R_I, \star)$ is in the following form:*

$$< L_I \cup R_I >= L_I \cup R_I \cup (R_I \star L_I).$$

Proof. We can easily observe that

$$< L_I \cup R_I >= \{Y_1 \star Y_2 \star \cdots Y_{n-1} \star Y_n \mid Y_j \in L_I \text{ or } Y_j \in R_I \text{ , } j = 1, \cdots n, n \in Z^+\}.$$

Suppose that $Y_j, Y_{j+1} \in L_1 \cup R_I$. Then the conditions arise are as follows:

(1): $Y_j, Y_{j+1} \in L_I$. In this condition, by Lemma 2.1, we obtain $Y_j \star Y_{j+1} \in L_I$.
(2): $Y_j, Y_{j+1} \in R_I$. In this condition, $Y_j \star Y_{j+1} \in R_I$ by the same Lemma 2.1.

(3): $Y_j \in L_I, Y_{j+1} \in R_I$. In this condition, $Y_j \star Y_{j+1} = (Y_j \star Y_{j+1}]$ is an ordered hyperideal of S, so $Y_j \star Y_{j+1} \in I_T = L_I \cap R_I$.

(4): $Y_j \in R_I, Y_{j+1} \in L_I$. In this condition, $Y_j \star Y_{j+1} \in R_I \star L_I$ in $(N_Q, \star)$. Therefore, for any $Y_1, \cdots, Y_n \in L_I \cup R_I$, where $n \in Z^+$, using (1)-(4), there arise the three conditions as follows:

(5): If $Y_1 \in L_I$, then $Y_1 \star Y_2 \star \cdots Y_{n-1} \star Y_n \in L_I$,
(6): If $Y_n \in R_I$, then $Y_1 \star Y_2 \star \cdots Y_{n-1} \star Y_n \in R_I$,
(7): If $Y_1 \in R_I$ and $Y_n \in L_I$, where $n \geq 2$, then $Y_1 \star Y_2 \star \cdots Y_{n-1} \star Y_n \in R_I \star L_I$.

Hence, holds the Lemma. □

Theorem 3.2. *Let S be an ordered semihypergroup. Then the following assertions on S are equivalent:*

(i): *S is a regular ordered semihypergroup.*
(ii): *For every ordered left hyperideal L and every ordered right hyperideal R, we have*
$$(R \circ L] = R \cap L;$$
(iii): *For every ordered right hyperideal R and ordered left hyperideal L of S;*
(I) $(R \circ R] = R$.
(II) $(L \circ L] = L$.
(III) $(R \circ L]$ is an ordered quasi-hyperideal of S.
(iv): *$(L_I, \star)$ and $(R_I, \star)$ are ordered idempotent semihypergroups and $(Q_I, \star)$ is the sub-semihypergroup of $(N_Q, \star)$ generated by $(L_I, \star)$ and $(R_I, \star)$;*
(v): *$(Q_I, \star)$ is a regular ordered sub-semihypergroup of the semihypergroup $(N_Q, \star)$;*
(vi): *Every ordered quasi-hyperideal Q of S is given by $Q = (Q \circ S \circ Q]$;*
(vii): *$(Q_I, \star, \subseteq)$ is a regular ordered sub-semihypergroup of the ordered semihypergroup of $(N_Q, \star, \subseteq)$.*

Proof. $(i) \Rightarrow (ii)$ Suppose that R and L are ordered right and left hyperideals of S, respectvely; then we have the following:
$$(R \circ L] \subseteq R \cap L.$$
Let S be regular, we need to show only that $R \cap L \subseteq (R \circ L]$. Suppose that $a \in R \cap L$. Since S is regular, we obtain $a \leq a \circ x \circ a$ for some $x \in S$, and so $a \in R$ and $x \circ a \subseteq L$. Therefore, $a \circ x \circ a \subseteq R \circ L$. So, $a \in (R \circ L]$. Thus $R \cap L \subseteq (R \circ L]$.
$(ii) \Rightarrow (iii)$ $(R \circ L]$ is an ordered quasi-hyperideal of S follows directly from Lemma2.3 and the condition (ii). As the ordered two-sided hyperideal of S generated by $R = (R \cup S \circ R]$, so the condition (ii) implies that
$$R = R \cap (R \cup S \circ R] = (R \circ (R \cup S \circ R]],$$
therefore, we have the following:
$$(R \circ R] \subseteq (R \circ (R \cup S \circ R]] = R.$$
Conversely, suppose that $a \in (R \circ (R \cup S \circ R]]$. Then, $a \leq r \circ b$ for $r \in R$ and $b \in (R \cup S \circ R]$. From $b \in (R \cup S \circ R]$, we have the following: $b \leq c$, where $c = r^{'} \in R$ or $c = s \circ r^{''}$ for some $s \in S$ and $r^{''} \in R$.
Therefore $a \leq r \circ c = r \circ r^{'} \subseteq R \circ R$ or $a \leq r \circ c = r \circ (s \circ r^{''}) = (r \circ s) \circ r^{''} \subseteq R \circ R$. Thus $a \in (R \circ R]$. Therefore $R \subseteq (R \circ R]$ so that $(R \circ R] = R$. In a similar fashion, we can prove that $(L \circ L] = L$, dually.

$(iii) \Rightarrow (iv)$. The conditions (I), (II) in (iii) and the Lemma 3.1 imply that $(L_I, \star)$ and $(R_I, \star)$ is an idempotent semihypergroup, respectively. Applying (iii)(III), we obtain $R_I \star L_I \subseteq Q_I$. Therefore, $< L_I \cup R_I > \subseteq Q_I$ in $(N_Q, \star)$.

Conversely, suppose that $Q \in Q_I$. Then $(Q \cup S \circ Q]$ is the ordered left hyperideal of S generated by Q. The condition (iii) (II) implies that

$$\begin{aligned} Q &\subseteq (Q \cup S \circ Q] \\ &= ((Q \cup S \circ Q)] \circ (Q \cup S \circ Q)]] \\ &\subseteq (Q \circ Q \cup S \circ Q \circ Q \cup Q \circ S \circ Q \cup (S \circ Q) \circ (S \circ Q)] \\ &\subseteq (S \circ Q]. \end{aligned}$$

We can dually prove that $Q \subseteq (Q \circ S]$. Therefore, using thse facts and Lemma 2.4, it follows that

$$Q \subseteq (S \circ Q] \cap (Q \circ S] \subseteq (S \circ Q \cup Q] \cap (Q \cup Q \circ S] = Q.$$

Therefore, for $Q \in Q_I$, we have the following:

$$Q = (S \circ Q] \cap (Q \circ S].$$

and the conditions (iii)(III) implies that

$$(R \circ L] = (S \circ (R \circ L]] \cap ((R \circ L] \circ S].$$

Moreover, by the assertion (iii)(II), we have $S = (S \circ S]$ and

$$\begin{aligned} (S \circ Q] &= ((S \circ Q]^2] \\ &= ((S \circ Q] \circ (S \circ Q]] \\ &= ((S \circ Q] \circ ((S \circ S] \circ Q]] \\ &\subseteq (S \circ Q \circ S \circ S \circ Q] \\ &\subseteq (S \circ (Q \circ S] \circ (S \circ Q]] \\ &\subseteq (S \circ ((Q \circ S] \circ (S \circ Q]]] \\ &\subseteq (S \circ (Q \circ S \circ S \circ Q)] \\ &\subseteq (S \circ Q]. \end{aligned}$$

Therefore

$$(S \circ Q] = (S \circ ((Q \circ S] \circ (S \circ Q]]].$$

Dually, we can prove that

$$(Q \circ S] = (((Q \circ S] \circ (S \circ Q]] \circ S].$$

From these facts, we obtain the following:

$$\begin{aligned} Q &= (Q \circ S] \cap (S \circ Q] \\ &= (((Q \circ S] \circ (S \circ Q]] \circ a \circ S] \cap (S \circ ((Q \circ S] \circ (S \circ Q]]] \\ &= ((Q \circ S] \circ (S \circ Q]] \\ &= (Q \circ S] \star (S \star Q] \in R_I \star L_I \\ &\subseteq < L_I \cup R_I > \end{aligned}$$

by Lemma 3.1. Therefore, $Q_I \subseteq < L_I \cup R_I >$. Hence, $Q_I = < L_I \cup R_I >$ in $(N_Q, \star)$.
$(iv) \Rightarrow (iii)$ It is a consequence of Lemma 3.1.
$(iii) \Rightarrow (v)$. By $(iii) \Rightarrow (iv)$, we have the desired result. Suppose that Q_1, Q_2 are two ordered quasi-hyperideals of S. Then $(S \circ (Q_1 \circ Q_2] \cup (Q_1 \circ Q_2]]$ is the least ordered left

hyperideal of S containing $(Q_1 \circ Q_2]$. Then the condition (iii)(II) gives

$$\begin{array}{rcl} (Q_1 \circ Q_2] & \subseteq & (S \circ (Q_1 \circ Q_2] \cup (Q_1 \circ Q_2]] \\ & = & ((S \circ (Q_1 \circ Q_2] \cup (Q_1 \circ Q_2]]^2] \\ & \subseteq & (S \circ (Q_1 \circ Q_2]] \\ & = & ((S \circ S] \circ (Q_1 \circ Q_2]] \\ & \subseteq & (S \circ (S \circ (Q_1 \circ Q_2]]]. \end{array}$$

Dually, we can show that

$$\begin{array}{rcl} (Q_1 \circ Q_2] & \subseteq & ((Q_1 \circ Q_2] \cup (Q_1 \circ Q_2] \circ S] \\ & \subseteq & (((Q_1 \circ Q_2] \circ S] \circ S]. \end{array}$$

These facts imply that

$$\begin{array}{rcl} (Q_1 \circ Q_2] & \subseteq & (S \circ (Q_1 \circ Q_2] \cup (Q_1 \circ Q_2]] \cap ((Q_1 \circ Q_2] \cup (Q_1 \circ Q_2] \circ S] \\ & \subseteq & (S \circ (S \circ (Q_1 \circ Q_2]]] \cap (((Q_1 \circ Q_2] \circ S] \circ S] \\ & = & (((Q_1 \circ Q_2] \circ S] \circ (S \circ (Q_1 \circ Q_2]]] \\ & \subseteq & ((Q_1 \circ (Q_2 \circ S \circ S) \circ Q_2] \\ & \subseteq & (Q_1 \circ Q_2]. \end{array}$$

By Theorem 2.5 (ii), we have the following:

$$(Q_1 \circ Q_2] = (S \circ (Q_1 \circ Q_2] \cup (Q_1 \circ Q_2]] \cap ((Q_1 \circ Q_2] \cup (Q_1 \circ Q_2] \circ S]$$

is an ordered quasi-hyperideal of S. Therefore, $Q_1 \star Q_2 \in Q_I$. Hence $(Q_I, \star)$ is a sub-semihypergroup of $(N_Q, \star)$. For every $Q \in Q_I$, we obtain the following:

$$\begin{array}{rcl} Q & = & ((Q \circ S] \circ (S \circ Q]] \\ & \subseteq & (Q \circ S \circ S \circ Q] \\ & \subseteq & (Q \circ a \circ S \circ Q] \\ & \subseteq & Q, \end{array}$$

and so

$$Q = (Q \circ S \circ Q] = Q \star S \star Q,$$

where $S \in Q_I$. Thus $(Q_I, \star)$ is a regular sub-semihypergroup of $(N_Q, \star)$.
$(v) \Rightarrow (vi)$. Suppose that Q is an ordered quasi-hyperideal of S. Applying the condition (iv), there is an ordered quasi-hyperideal of Q_1 of S so that by Lemma 2.4, we have the following:

$$\begin{array}{rcl} Q & = & Q \star Q_1 \star Q \\ & = & (Q \circ Q_1 \star Q] \\ & \subseteq & (Q \circ S \star Q] \\ & \subseteq & (S \circ Q] \cap (Q \circ S] \\ & \subseteq & (S \circ Q \cup Q] \cap (Q \cup Q \circ S] \\ & = & Q. \end{array}$$

Therefore, we have $Q = (Q \circ S \circ Q]$.
$(vi) \Rightarrow (vii)$ Straightforward.
$(vii) \Rightarrow (i)$ For every $s \in S$, using Theorem 2.5, we have $R(s) \cap L(s)$ is an ordered quasi-hyperideal of S containing s. By (vii), there exists $Q \in Q_S$ so that we have the

following:

$$\begin{aligned} s \in R(s) \cap L(s) &\subseteq (R(s) \cap L(s)) \star Q \star (R(s) \cap L(s)) \\ &= ((R(s) \cap L(s) \circ Q \circ (R(s) \cap L(s)] \\ &\subseteq (R(s) \circ S \circ L(s)] \\ &= ((s \cup s \circ S] \circ S \circ (S \circ s \cup s]] \\ &\subseteq (s \circ S \circ s]. \end{aligned}$$

Hence S is a regular ordered semihypergroup. □

Lemma 3.3. *Every two-sided ordered hyperideal I of a regular ordered semihypergroup S is a regular sub-semihypergroup of S.*

Proof. Suppose that $i \in I$. As S is regular, there exists $s \in S$ so that, we have the following:

$$i \leq i \circ s \circ i \leq i \circ s \circ i \circ s \circ i = i \circ (s \circ i \circ s) \circ i.$$

As $s \circ i \circ s \in S \circ I \circ S \subseteq I$, we observe that $i \in (i \circ I \circ i]_I$.

□

Theorem 3.4. *Suppose that S is a regular ordered semihypergroup. Then the following statements are true:*

- **(i):** *Every ordered quasi-hyperideal of S can be expressed as follows:*
  $Q = R \cap L = (R \circ L]$, where R and L are respectively the ordered right and left hyperideals of S generated by Q.
- **(ii):** *Let Q be an ordered quasi-hyperideal of S, then $(Q \circ Q] = (Q \circ Q \circ Q]$.*
- **(iii):** *Every ordered bi-hyperideal of S is an ordered quasi-hyperideal of S.*
- **(iv):** *Every ordered bi-hyperideal of any ordered two sided-hyperideal of S is a quasi-hyperideal of S.*
- **(v):** *For every $L_1, L_2 \in L_I$ and $R_1, R_2 \in R_I$, we obtain the following:*
  $L_1 \cap L_2 \subseteq (L_1 \circ L_2]$ and $R_1 \cap R_2 \subseteq (R_1 \circ R_2]$.

Proof. Because S is a regular ordered semihypergroup, by Lemma 2.4 and Theorem 3.2, the statement (i) is done.
Since $(Q \circ Q \circ Q] \subseteq (Q \circ Q]$ is always true, we need to show that $(Q \circ Q] \subseteq (Q \circ Q \circ Q]$. We have $(Q \circ Q]$ is also an ordered quasi-hyperideal of S by Theorem 3.2. Moreover, we have the following:

$$\begin{aligned} (Q \circ Q] &= (Q \circ Q \circ S \circ Q \circ Q] \\ &= (Q \circ (Q \circ (Q \circ S \circ Q) \circ Q] \\ &\subseteq (Q \circ Q \circ Q]. \end{aligned}$$

Suppose that Q_I is an ordered bi-hyperideal of S. Then $(S \circ Q_1]$ is an ordered left hyperideal and $(Q_1 \circ S]$ is an ordered right hyperideal of S. Applying Theorem 3.2, we obtain the following:

$$\begin{aligned} (S \circ Q_1] \cap (Q_1 \circ S] &= ((Q_1 \circ S] \circ (S \circ Q_1]] \\ &\subseteq (Q_1 \circ S \circ Q_1] \\ &\subseteq (Q_1] \\ &\subseteq Q_1, \end{aligned}$$

Therefore, Q_I is an ordered quasi-hyperideal of S.
Suppose that I is a two-sided ordered hyperideal of S and B is an ordered bi-hyperideal of

I. By the relation (iii) and Lemma 3.3, B is an ordered quasi-hyperideal of I. Therefore, using Theorem 2.6, B is an ordered bi-hyperideal of S. Also, from the relation (iii) again, we obtain B is an ordered quasi-hyperideal of S.
Lastly, suppose that $L_1, L_2 \in L_I$. Because S is regular and $L_1 \cap L_2$ is an ordered quasi-hyperideal of S, using Theorem 3.2, we obtain the following:

$$\begin{aligned} L_1 \cap L_2 &= ((L_1 \cap L_2) \circ S \circ (L_1 \cap L_2)] \\ &\subseteq (L_1 \circ (S \circ L_2)] \\ &\subseteq (L_1 \circ L_2]. \end{aligned}$$

Dually, we can show that $R_1 \cap R_2 \subseteq (R_1 \circ R_2]$ for all $R_1, R_2 \in R_I$. □

Theorem 3.5. *A partiall ordered semihypergroup S is regular if and only if for every ordered bi-hyperideal B, every ordered hyperideal I and every ordered quasi-hyperideal Q, we have the following:*

$$B \cap I \cap Q \subseteq (B \circ I \circ Q].$$

Proof. Let S be regular. Then for any $a \in B \cap I \cap Q$ there exists $s \in S$ such that

$$\begin{aligned} a \leq a \circ s \circ a &\leq (a \circ s \circ a) \circ s \circ (a \circ s \circ a) \\ &= (a \circ s \circ a) \circ (s \circ a \circ s) \circ a \\ &\in (B \circ B) \circ (S \circ I \circ S) \circ Q \\ &\subseteq B \circ I \circ Q. \end{aligned}$$

Hence $a \in (B \circ I \circ Q]$.

Conversely, let

$$B \cap I \cap Q \subseteq (B \circ I \circ Q]$$

for every ordered bi-hyperideal B, every ordered hyperideal I and every ordered quasi-hyperideal Q of S. Suppose that $s \in S$. Let $B(s)$ and $Q(s)$ be the ordered bi-hyperideal and ordered quasi-hyperideal of S generated by s, respectively. So we have the following:

$$\begin{aligned} s \in B(s) \cap I(s) \cap Q(s) &\subseteq (B(s) \circ I(s) \circ Q(s)] \\ &\subseteq ((s \cup s \circ S \circ s] \circ S \circ (s \cup (S \circ s \cap s \circ S)]] \\ &\subseteq ((s \cup s \circ S \circ s] \circ S \circ (s \cup S \circ]] \\ &\subseteq (s \circ S \circ s]. \end{aligned}$$

Hence S is regular. □

Next consider R in place of Q in the above Theorem 3.5 to obtain the following.

Corollary 3.6. *An ordered semihypergroup S is regular if and only if for every ordered bi-hyperideal B, every ordered hyperideal I and every right hyperideal R of S,*

$$B \cap I \cap R \subseteq (B \circ I \circ R].$$

Theorem 3.7. *A partially ordered semihypergroup S is regular if and only if for every ordered quasi-hyperideal Q, every ordered left hyperideal L and every ordered right hyperideal R, we have the following:*

$$R \cap Q \cap L \subseteq (R \circ Q \circ L].$$

Proof. Let S be regular, then for any $a \in R \cap Q \cap L$, there exists $s \in S$ such that

$$\begin{aligned} a &\leq a \circ s \circ a \\ &\leq (a \circ s \circ a) \circ s \circ (a \circ s \circ a) \\ &= (a \circ s) \circ a \circ (s \circ a \circ s \circ a) \\ &\subseteq (R \circ S) \circ Q \circ (S \circ L \circ S \circ L) \\ &\subseteq R \circ Q \circ L. \end{aligned}$$

Hence $a \in (R \circ Q \circ L]$.

Conversely, let

$$R \cap Q \cap L \subseteq (R \circ Q \circ L].$$

for every ordered right hyperideal R, every ordered quasi-hyperideal Q and every ordered left hyperideal L of S. Suppose that $s \in S$. So we have the following:

$$\begin{aligned} s \in R(s) \cap Q(s) \cap L(s) &\subseteq (R(s) \circ Q(s) \circ L(s)] \\ &\subseteq (R(s) \circ S \circ L(s)] \\ &\subseteq (R(s) \circ L(s)] \\ &\subseteq ((s \cup s \circ S] \circ (s \cup S \circ s]] \\ &\subseteq ((s \circ s \cup s \circ S \circ s]]. \end{aligned}$$

So, $s \leq s \circ s$ or $s \leq s \circ x \circ s$ for some $x \in S$. If $s \leq s \circ s$, then we have the following:

$$\begin{aligned} s &\leq s \circ s \\ &\leq (s \circ s) \circ (s \circ s) \\ &= s \circ (s \circ s)^2 \circ s \\ &\in s \circ S \circ s. \end{aligned}$$

If $s \leq s \circ x \circ s$ for some $x \in S$. Then $s \in s \circ S \circ s$. So, finally, we obtain $s \in (s \circ S \circ s]$. Hence S is regular. □

Corollary 3.8. *If we consider an ordered left hyperideal L (or an ordered right hyperideal R) in place of the ordered quasi-hyperideal Q in the above Theorem 3.7, we obtain the following:*

$$L \cap R \subseteq (R \circ L].$$

4. CONCLUSION

We study mathematical objects in set theoretical forms. This property is inbuilt in the algebraic hyperstructures and so is in semihypergroups. This feature makes hyperstructures all the more applicable to model real life problems. In this paper, we have characterized regular ordered semihypergroups in terms of ordered quasi-hyperideals and ordered bi-hyperideals. For future scope of these and other results, researchers can see to study or generalize it in ordered Γ-semihypergroups, and in other algebraic structures using quasi-ideals, bi-ideals, bi-quasi-ideals, m-bi-ideals, relative quasi-ideals and relative bi-ideals.

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ABUL BASAR, DEPARTMENT OF NATURAL AND APPLIED SCIENCES, GLOCAL UNIVERSITY, MIRZAPUR, SAHARANPUR, U. P., INDIA
Email address: basar.jmi@gmail.com

MOHAMMAD YAHYA ABBASI
DEPARTMENT OF MATHEMATICS, JAMIA MILLIA ISLAMIA, NEW DELHI, INDIA
Email address: mabbasi@jmi.ac.in