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m-IDEALS AND ITS GENERATORS OF TERNARY SEMIGROUPS

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ABSTRACT. The purpose of this paper is to introduce some new classes of m-bi ideals and m-quasi ideals in ternary semigroups and give some characterizations in terms of bi ideals and quasi ideals. Let U_1 be the m-bi ideal of $\mathbb{T}$ and U_2 be the m-bi ideal of U_1 such that $U_2^3 = U_2$, shown that U_2 is a m-bi ideal of $\mathbb{T}$. Let U_1, U_2 and U_3 be the three ternary subsemigroups of $\mathbb{T}$, it has been shown that $U_1U_2U_3$ is a t-bi ideal if at least one of U_1, U_2, U_3 is l-RI or m-LATI or n-LI of $\mathbb{T}$. Also we discuss m-bi ideal generated by $\mathscr{B}$ is $< \mathscr{B} >_m = \mathscr{B} \cup \sum_{finite} \mathscr{B}\mathbb{T}^m\mathscr{B}\mathbb{T}^m\mathscr{B}$. Examples are needed strengthen our results.

1. INTRODUCTION

The notion of ternary semigroup is a natural generalization of ternary group. The notion of ideal play very important role to study the algebraic structures. Lehmer initiated the concept of ternary algebraic systems called triplexes in 1932 [7]. A ternary semigroup is a nonempty set together with a ternary multiplication which is associative. The notion of ternary semigroups was known to banach who is credited with an example of a ternary semigroup which does not reduce to a semigroup. Ternary multiplication compatible with a vector space structure has been studied using both algebraic and functional analytical methods. In [8], Los studied some properties of ternary semigroup and proved that every ternary semigroup can be embedded in a semigroup. In [16], Sioson studied ideal theory in ternary semigroups. The notion of quasi-ideal in semigroup and in ring has been introduced by Steinfeld. In ternary semigroups, quasi-ideals are generalization of right ideals,lateral ideals and left ideals whereas bi-ideals are generalization of quasi-ideals. It is interesting to note that many of the results of semigroup theory can be obtained for ternary semigroups using various ideal concepts. If A, B and C are three subsets of ternary semigroups, $ABC = \left\{ \sum_{finite} a_ib_ic_i : a_i \in A, b_i \in B, c_i \in C \right\}$. Our aim in this paper is threefold.

(1) To study the relationship between bi ideal and m-bi ideal.
(2) To study the relationship between m-quasi ideal and m-bi ideal.
(3) To characterize generators of bi ideal, left ideal and right ideal.

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2. PRELIMINARIES

From now onward, unless stated otherwise, $\mathbb{T}$ will denote a ternary semigroups.

Definition 2.1. [15] A ternary semigroup is a nonempty set $\mathbb{T}$ together with a ternary operation $(a, b, c) \to [abc]$ satisfying the associative law ,$[[abc]uv] = [a[bcu]v] = [ab[cuv]]$, $\forall\ a, b, c, u, v \in \mathbb{T}$.

Definition 2.2. [15] A non empty subset A of $\mathbb{T}$ is said to be

(i) ternary subsemigroup (shortly TSS) if $A^3 = AAA \subseteq A$.
(ii) left ideal (shortly LI) if $\mathbb{TT}A \subseteq A$.
(iii) right ideal (shortly RI) if $A\mathbb{TT} \subseteq A$.
(iv) lateral ideal (shortly LAT) if $\mathbb{T}A\mathbb{T} \subseteq A$.
(v) an ideal if it is a left, right and lateral ideal.

Definition 2.3. [15] For any non empty subset A of $\mathbb{T}$

(i) $A_l = A \cup \mathbb{TT}A$ is the left ideal generated by A.
(ii) $A_r = A \cup A\mathbb{TT}$ is the right ideal generated by A.
(iii) $A_{lat} = A \cup \mathbb{T}A\mathbb{T} \cup \mathbb{TT}A\mathbb{TT}$ is the lateral ideal generated by A.
(iv) $(A) = A \cup \mathbb{TT}A \cup A\mathbb{TT} \cup \mathbb{T}A\mathbb{T} \cup \mathbb{TT}A\mathbb{TT}$ is the ideal generated by A.

Definition 2.4. (i) A $TSS\ B$ of $\mathbb{T}$ is called a bi-ideal if $B\mathbb{T}B\mathbb{T}B \subseteq B$.
(ii) A $TSS\ Q$ of $\mathbb{T}$ is called a quasi-ideal if $Q\mathbb{TT} \cap (\mathbb{T}Q\mathbb{T} \cup \mathbb{TT}Q\mathbb{TT}) \cap \mathbb{TT}Q \subseteq Q$.

Definition 2.5. (i) A subsemigroup L of semigroup S is called an l-left ideal if $S^l L \subseteq L$.
(ii) A subsemigroup N of semigroup S is called an n-right ideal if $NS^n \subseteq N$, where l, n are positive integers.

Remark. *For a semigroup S and l, m are positive integers, $S^m = S \cdot S \cdot S \cdot ... \cdot S(m - times)$. Now $S^2 = S \cdot S \subseteq S$ and $S^3 = S \cdot S \cdot S \subseteq S^2 \subseteq S$. We conclude that $S^l \subseteq S^m$ such that $l \geq m$. Consequently $S^m \subseteq S$.*

3. VARIOUS m-IDEALS

Here m-bi ideals (shortly $m\mathscr{B}$) and m-quasi ideals (shortly mQ) are introduced.

Definition 3.1. The $m\mathscr{B}$ of $\mathbb{T}$ is a TSS of $\mathbb{T}$ such that $\mathscr{B}\mathbb{T}^m\mathscr{B}\mathbb{T}^m\mathscr{B} \subseteq \mathscr{B}$, where $m \in \mathbb{Z}^+$.

Remark. *If $m = 1$, then 1-bi ideal is simply bi ideal of $\mathbb{T}$.*

Theorem 3.1. *For $m \geq 1$, Every bi ideal is a $m\mathscr{B}$ of $\mathbb{T}$.*

Proof. Let $\mathscr{B}$ be the bi ideal of $\mathbb{T}$, then $\mathscr{B}\mathbb{T}\mathscr{B}\mathbb{T}\mathscr{B} \subseteq \mathscr{B}$. Now, $\mathscr{B}\mathbb{T}^1\mathscr{B}\mathbb{T}^1\mathscr{B} = \mathscr{B}\mathbb{T}\mathscr{B}\mathbb{T}\mathscr{B} \subseteq \mathscr{B}$. Also, $\mathscr{B}\mathbb{T}^2\mathscr{B}\mathbb{T}^2\mathscr{B} \subseteq \mathscr{B}\mathbb{T}^1\mathscr{B}\mathbb{T}^1\mathscr{B} \subseteq \mathscr{B}$. Continuing like these, $\mathscr{B}\mathbb{T}^m\mathscr{B}\mathbb{T}^m\mathscr{B} \subseteq \mathscr{B}\mathbb{T}^{m-1}\mathscr{B}\mathbb{T}^{m-1}\mathscr{B} \subseteq \mathscr{B}$. Therefore $\mathscr{B}$ is a $m\mathscr{B}$ of $\mathbb{T}$.

Converse of the Theorem 3.1 is not true which can be illustrated as follows.

Example 3.2. Consider the ternary semigroups

$$\mathbb{T} = \left\{ \left(\begin{smallmatrix} 0 & x_1 & x_2 & x_3 & x_4 & x_5 \\ 0 & 0 & x_6 & x_7 & x_8 & x_9 \\ 0 & 0 & 0 & x_{10} & x_{11} & x_{12} \\ 0 & 0 & 0 & 0 & x_{13} & x_{14} \\ 0 & 0 & 0 & 0 & 0 & x_{15} \\ 0 & 0 & 0 & 0 & 0 & 0 \end{smallmatrix} \right) \middle| x_i^{'s} \in \mathbb{Z}_0^-,\ i = 1, 2, ..., 15 \right\}.$$

Clearly $\mathscr{B} = \left\{ \begin{pmatrix} 0&x&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&y&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&z\\ 0&0&0&0&0&0 \end{pmatrix} \middle| x,\ y,\ z \in \mathbb{Z}_0^-, \right\}$ is a TSS.
Hence $\mathscr{B}$ is a 2-bi ideal but not bi ideal of $\mathbb{T}$.

Theorem 3.2. *The product of any three $m\mathscr{B}^s$ with identity element e is a $m\mathscr{B}^s$ of $\mathbb{T}$.*

Proof. Let Z_1, Z_2 and Z_3 be the three $m\mathscr{B}^s$ of $\mathbb{T}$ with bipotencies m_1, m_2 and m_3 respectively. Then $Z_1\mathbb{T}^{m_1}Z_1\mathbb{T}^{m_1}Z_1 \subseteq Z_1$, $Z_2\mathbb{T}^{m_2}Z_2\mathbb{T}^{m_2}Z_2 \subseteq Z_2$ and $Z_3\mathbb{T}^{m_3}Z_3\mathbb{T}^{m_3}Z_3 \subseteq Z_3$, where $m_i \in \mathbb{Z}^+$, for $i = 1, 2, 3$. Put $t = max\ m_i$ for $i = 1, 2, 3$

$$\begin{aligned}
(Z_1Z_2Z_3)^3 &= (Z_1Z_2Z_3)(Z_1Z_2Z_3)(Z_1Z_2Z_3)\\
&\subseteq (Z_1(\mathbb{T}\mathbb{T}Z_1)(\mathbb{T}\mathbb{T}Z_1))Z_2Z_3\\
&= (Z_1(\mathbb{T}\mathbb{T}eee...eZ_1)(\mathbb{T}\mathbb{T}eee...eZ_1))Z_2Z_3\\
&\subseteq (Z_1(\mathbb{T}\mathbb{T}\mathbb{T}...\mathbb{T}Z_1)(\mathbb{T}\mathbb{T}\mathbb{T}...\mathbb{T}Z_1))Z_2Z_3\\
&= (Z_1\mathbb{T}^mZ_1\mathbb{T}^mZ_1)Z_2Z_3\\
&\subseteq Z_1Z_2Z_3.
\end{aligned}$$

$$\begin{aligned}
(Z_1Z_2Z_3)\mathbb{T}^t(Z_1Z_2Z_3)\mathbb{T}^t(Z_1Z_2Z_3) &= Z_1(Z_2Z_3\mathbb{T}^t)Z_1(Z_2Z_3\mathbb{T}^t)(Z_1Z_2Z_3)\\
&\subseteq Z_1(\mathbb{T}\mathbb{T}\mathbb{T}^t)Z_1(\mathbb{T}\mathbb{T}\mathbb{T}^t)(Z_1Z_2Z_3)\\
&= Z_1\mathbb{T}^{2+t}Z_1\mathbb{T}^{2+t}(Z_1Z_2Z_3)\\
&\subseteq (Z_1\mathbb{T}^{m_1}Z_1\mathbb{T}^{m_1})Z_1Z_2Z_3\\
&= (Z_1\mathbb{T}^{m_1}Z_1\mathbb{T}^{m_1}Z_1)Z_2Z_3\\
&\subseteq Z_1Z_2Z_3
\end{aligned}$$

Thus $Z_1Z_2Z_3$ is an m-bi ideal of $\mathbb{T}$ with bipotency $max\ \ m_i,\ \ i = 1, 2, 3$.

Theorem 3.3. *If Z is a $m\mathscr{B}$ of $\mathbb{T}$ and T_1, T_2 are two TSS^s with e, then ZT_1T_2, T_1ZT_2 and T_1T_2Z are $m\mathscr{B}^s$ of $\mathbb{T}$.*

Proof. Let Z be $m\mathscr{B}$ of $\mathbb{T}$ and T_1, T_2 are two TSS^s with e.

$$\begin{aligned}
(ZT_1T_2)^3 &= (ZT_1T_2)(ZT_1T_2)(ZT_1T_2)\\
&\subseteq (Z(\mathbb{T}\mathbb{T}Z)(\mathbb{T}\mathbb{T}Z))T_1T_2\\
&= (Z(\mathbb{T}\mathbb{T}eee...eZ)(\mathbb{T}\mathbb{T}eee...eZ))T_1T_2\\
&\subseteq (Z(\mathbb{T}\mathbb{T}\mathbb{T}...\mathbb{T}Z)(\mathbb{T}\mathbb{T}\mathbb{T}...\mathbb{T}Z))T_1T_2\\
&= (Z\mathbb{T}^mZ\mathbb{T}^mZ)T_1T_2\\
&\subseteq ZT_1T_2.
\end{aligned}$$

$$\begin{aligned}
(ZT_1T_2)\mathbb{T}^m(ZT_1T_2)\mathbb{T}^m(ZT_1T_2) &= Z(T_1T_2\mathbb{T}^m)Z(T_1T_2\mathbb{T}^m)(ZT_1T_2)\\
&\subseteq Z(\mathbb{T}\mathbb{T}\mathbb{T}^m)Z(\mathbb{T}\mathbb{T}\mathbb{T}^m)(ZT_1T_2)\\
&= Z\mathbb{T}^{2+m}Z\mathbb{T}^{2+m}(ZT_1T_2)\\
&\subseteq (Z\mathbb{T}^mZ\mathbb{T}^m)ZT_1T_2\\
&= (Z\mathbb{T}^mZ\mathbb{T}^mZ)T_1T_2\\
&\subseteq ZT_1T_2.
\end{aligned}$$

Thus, ZT_1T_2 is a $m\mathscr{B}$ of $\mathbb{T}$.
Similarly T_1ZT_2 and T_1T_2Z are $m\mathscr{B}$ of $\mathbb{T}$.

Theorem 3.4. *If Z is a $m\mathscr{B}$ and T_1 is a TSS of $\mathbb{T}$, then $Z \cap T_1$ is a $m\mathscr{B}$ of T_1.*

Proof. Since $Z \cap T_1 \subseteq Z$ and $Z \cap T_1 \subseteq T_1$, $(Z \cap T_1)(Z \cap T_1)(Z \cap T_1) \subseteq ZZZ \subseteq Z$. $(Z \cap T_1)T_1^m(Z \cap T_1)T_1^m(Z \cap T_1) \subseteq (Z \cap T_1)\mathbb{T}^m(Z \cap T_1)\mathbb{T}^m(Z \cap T_1) \subseteq Z\mathbb{T}^m Z\mathbb{T}^m \subseteq Z$. Therefore $Z \cap T_1$ is a $m\mathscr{B}$ of T_1.

Theorem 3.5. *The intersection of family of $m\mathscr{B}$ of $\mathbb{T}$ is a $m\mathscr{B}$ with potencies m_i and potency* $\max\ m_i$ *for* $i = 1, 2, 3$.

Remark. *Sum of two $m\mathscr{B}^s$ of $\mathbb{T}$ is not a $m\mathscr{B}$ of $\mathbb{T}$.*

Example 3.3. Consider the ternary semigroup $\mathbb{T} = \left\{ \begin{pmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{pmatrix} \Big| a_i \in \mathbb{Z}_0^- \right\}$.
Let $X_1 = \left\{ \begin{pmatrix} p & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \Big| p \in \mathbb{Z}_0^- \right\}$ and $X_2 = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & q \end{pmatrix} \Big| q \in \mathbb{Z}_0^- \right\}$ are $m\mathscr{B}$ of $\mathbb{T}$ but $X_1 + X_2$ is not a $m\mathscr{B}$ of $\mathbb{T}$.

Definition 3.4. A $TSS\ Q$ of $\mathbb{T}$ is called a mQ if $Q\mathbb{T}^m \cap \mathbb{T}^m Q\mathbb{T}^m \cap \mathbb{T}^m Q \subseteq Q$.

Remark. *If $m = 1$, then 1-quasi ideal is simply quasi ideal of $\mathbb{T}$.*

Theorem 3.6. *Every mQ is a $m\mathscr{B}$ of $\mathbb{T}$.*

Proof. Let N be mQ of $\mathbb{T}$. Now,

$$\begin{aligned} N\mathbb{T}^m N\mathbb{T}^m N &\subseteq N\mathbb{T}^m\mathbb{T}\mathbb{T}^m\mathbb{T} \\ &= N\mathbb{T}^{m+1}\mathbb{T}^{m+1} \\ &\subseteq N\mathbb{T}^m \end{aligned}$$

Similarly, $N\mathbb{T}^m N\mathbb{T}^m N \subseteq \mathbb{T}^m N\mathbb{T}^m$ and $N\mathbb{T}^m N\mathbb{T}^m N \subseteq \mathbb{T}^m N$.
Thus, $N\mathbb{T}^m N\mathbb{T}^m N \subseteq (N\mathbb{T}^m) \cap (\mathbb{T}^m N\mathbb{T}^m) \cap (\mathbb{T}^m N) \subseteq N$.

Converse of the above Theorem is not true by the following Example.

Example 3.5. Consider the ternary semigroup

$$\mathbb{T} = \left\{ \begin{pmatrix} 0 & x_1 & x_2 & x_3 & x_4 & x_5 \\ 0 & 0 & x_6 & x_7 & x_8 & x_9 \\ 0 & 0 & 0 & x_{10} & x_{11} & x_{12} \\ 0 & 0 & 0 & 0 & x_{13} & x_{14} \\ 0 & 0 & 0 & 0 & 0 & x_{15} \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \Bigg| x_i \in \mathbb{Z}_0^-,\ i = 1, 2, ..., 15 \right\}.$$

Clearly $Q = \left\{ \begin{pmatrix} 0 & x & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & y & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & z \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \Bigg| x, y, z \in \mathbb{Z}_0^- \right\}$ is a TSS.
Hence Q is a 2-bi ideal but not 2-quasi ideal of $\mathbb{T}$.

Theorem 3.7. *Product of three $(l, m, n)Q^s$ N_1, N_2 and N_3 of $\mathbb{T}$ is an $t\mathscr{B}$ of $\mathbb{T}$, where $t \in \mathbb{Z}^+$.*

Proof. Put $Z = N_1N_2N_3 = \left\{ \sum\limits_{finite} l_im_in_i |\ l_i \in N_1, m_i \in N_2, n_i \in N_3, i = 1, 2, ... \right\}$. Now,

$$\begin{aligned} Z^3 &= (N_1N_2N_3)^3 \\ &= (N_1N_2N_3)(N_1N_2N_3)(N_1N_2N_3) \\ &\subseteq N_1(N_2\mathbb{T}N_2\mathbb{T}N_2)N_3 \\ &\subseteq N_1N_2N_3 \\ &= Z. \end{aligned}$$

Put $t = \max\{l_1, l_2, 1_3, m_1, m_2, m_3, n_1, n_2, n_3\}$. Now,

$$\begin{aligned} Z\mathbb{T}^t Z\mathbb{T}^t Z &= (N_1N_2N_3)\mathbb{T}^t(N_1N_2N_3)\mathbb{T}^t(N_1N_2N_3) \\ &\subseteq N_1(N_2\mathbb{T}\mathbb{T}^t\mathbb{T}N_2\mathbb{T}\mathbb{T}^t\mathbb{T}N_2)N_3 \\ &\subseteq N_1(N_2\mathbb{T}^{t+2}N_2\mathbb{T}^{t+2}N_2)N_3 \\ &\subseteq N_1(N_2\mathbb{T}^tN_2\mathbb{T}^tN_2)N_3 \\ &\subseteq N_1N_2N_3 \\ &= Z. \end{aligned}$$

Therefore Z is a $t\mathscr{B}$ of $\mathbb{T}$.

Definition 3.6. (i) A TSS L of $\mathbb{T}$ is called an l-LI if $\mathbb{T}^l L \subseteq L$.
(ii) A TSS M of $\mathbb{T}$ is called an m-LATI if $\mathbb{T}^m M\mathbb{T}^m \subseteq M$.
(iii) A TSS N of $\mathbb{T}$ is called an n-RI if $N\mathbb{T}^n \subseteq N$, where l, m, n are positive integers.

Theorem 3.8. *Every l-LI ,m-LATI and n-RI of $\mathbb{T}$ is an $l\mathscr{B}$, $m\mathscr{B}$,$n\mathscr{B}$ of $\mathbb{T}$ respectively.*

Proof. Let L be the m-LATI of $\mathbb{T}$, then $\mathbb{T}^m L\mathbb{T}^m \subseteq L$.Now, $L\mathbb{T}^m L\mathbb{T}^m L \subseteq L(\mathbb{T}^m L\mathbb{T}^m)$ $L \subseteq LLL \subseteq L$. Therefore L is a $m\mathscr{B}$ of $\mathbb{T}$. Similarly other cases.

Theorem 3.9. *The intersection of l-LI, m-LATI and n-RI is an l-LI, m-LATI and n-RI of $\mathbb{T}$ respectively.*

Theorem 3.10. *Let U_1, U_2, U_3 be an l-LI, m-LATI and n-RI of $\mathbb{T}$ respectively. Then $U_1 \cap U_2 \cap U_3$ is an $t\mathscr{B}$, where $t = \max(l, m, n)$.*

Proof. Clearly, $Z = U_1 \cap U_2 \cap U_3$ is a TSS of $\mathbb{T}$. By Theorem 3.8, U_1, U_2 and U_3 are $l\mathscr{B}$, $m\mathscr{B}$ and $n\mathscr{B}$ respectively. The intersection of U_1, U_2 and U_3 becomes $\max(l, m, n)$-bi ideals by Theorem 3.5.

$$\begin{aligned} Z\mathbb{T}^t Z\mathbb{T}^t Z &\subseteq U_1\mathbb{T}^t U_1\mathbb{T}^t U_1 \\ &\subseteq \mathbb{T}\mathbb{T}^t\mathbb{T}\mathbb{T}^t U_1 \\ &= \mathbb{T}^{t+1}\mathbb{T}^{t+1}U_1 \\ &\subseteq \mathbb{T}^t U_1 \\ &\subseteq U_1. \end{aligned}$$

Therefore $Z\mathbb{T}^t Z\mathbb{T}^t Z \subseteq U_1$.
Similarly, $Z\mathbb{T}^t Z\mathbb{T}^t Z \subseteq U_2$ and $Z\mathbb{T}^t Z\mathbb{T}^t Z \subseteq U_3$.
Hence $Z\mathbb{T}^t Z\mathbb{T}^t Z \subseteq Z$.

Theorem 3.11. *The product of any three tQ^s with e is a tQ^s of $\mathbb{T}$, where $t \in \mathbb{Z}^+$.*

Theorem 3.12. *Let U_1 be an $m\mathscr{B}$ of $\mathbb{T}$ and U_2 be an $m\mathscr{B}$ of U_1 such that $U_2^3 = U_2$. Then U_2 is a $m\mathscr{B}$ of $\mathbb{T}$.*

Proof. Since U_1 is a $m\mathscr{B}$ of $\mathbb{T}$, $U_1\mathbb{T}^mU_1\mathbb{T}^mU_1 \subseteq U_1$ and U_2 is a $m\mathscr{B}$ of U_1, $U_2U_1^mU_2U_1^mU_2 \subseteq U_2$. Now,

$$\begin{aligned}
U_2\mathbb{T}^mU_2\mathbb{T}^mU_2 &= (U_2U_2U_2)\mathbb{T}^mU_2\mathbb{T}^m(U_2U_2U_2) \\
&= U_2U_2(U_2\mathbb{T}^mU_2\mathbb{T}^mU_2)U_2U_2 \\
&\subseteq U_2U_2(U_1\mathbb{T}^mU_1\mathbb{T}^mU_1)U_2U_2 \\
&\subseteq U_2U_2U_1U_2U_2 \\
&= U_2U_2U_1U_2(U_2U_2U_2) \\
&\subseteq U_2(U_2U_1U_2U_1U_2)U_2 \\
&= U_2U_2U_1U_2U_2U_2U_1U_2U_2 \\
&= U_2U_1^3U_2U_1^3U_2 \\
&\cdots \\
&\cdots \\
&= U_2U_1^mU_2U_1^mU_2 \\
&\subseteq U_2
\end{aligned}$$

Thus, U_2 is a $m\mathscr{B}$ of $\mathbb{T}$.

Theorem 3.13. *Let U_1, U_2 and U_3 be the three TSS of $\mathbb{T}$ and $Z = U_1U_2U_3$. Then Z is a $t\mathscr{B}$ if at least one of U_1, U_2, U_3 is l-RI or m-LATI or n-LI of $\mathbb{T}$.*

Proof. Let $Z = U_1U_2U_3$. Suppose U_1 is a l-RI of $\mathbb{T}$. Now,

$$\begin{aligned}
Z\mathbb{T}^tZ\mathbb{T}^tZ &= (U_1U_2U_3)\mathbb{T}^t(U_1U_2U_3)\mathbb{T}^t(U_1U_2U_3) \\
&\subseteq U_1(\mathbb{T}\mathbb{T}\mathbb{T}^t)(\mathbb{T}\mathbb{T}\mathbb{T})\mathbb{T}^t\mathbb{T}U_2U_3 \\
&\subseteq U_1(\mathbb{T}^{t+2}\mathbb{T}^3\mathbb{T}^{t+1})U_2U_3 \\
&\subseteq (U_1\mathbb{T}^t)U_2U_3 \\
&\subseteq U_1U_2U_3 \\
&= Z.
\end{aligned}$$

Suppose that U_2 is a l-RI of $\mathbb{T}$. Now,

$$\begin{aligned}
Z\mathbb{T}^tZ\mathbb{T}^tZ &= (U_1U_2U_3)\mathbb{T}^t(U_1U_2U_3)\mathbb{T}^t(U_1U_2U_3) \\
&\subseteq U_1U_2(\mathbb{T}\mathbb{T}^t\mathbb{T})(\mathbb{T}\mathbb{T}\mathbb{T}^t)\mathbb{T}\mathbb{T}U_3 \\
&\subseteq U_1U_2(\mathbb{T}^t\mathbb{T}^t\mathbb{T})\mathbb{T}U_3 \\
&\subseteq U_1(U_2\mathbb{T}^t)U_3 \\
&\subseteq U_1U_2U_3 \\
&= Z.
\end{aligned}$$

Suppose that U_3 is a l-RI of $\mathbb{T}$. Now,

$$\begin{aligned}
Z\mathbb{T}^tZ\mathbb{T}^tZ &= (U_1U_2U_3)\mathbb{T}^t(U_1U_2U_3)\mathbb{T}^t(U_1U_2U_3) \\
&\subseteq (U_1U_2U_3)(\mathbb{T}^t\mathbb{T}\mathbb{T})(\mathbb{T}\mathbb{T}^t\mathbb{T})\mathbb{T}\mathbb{T} \\
&\subseteq (U_1U_2U_3)(\mathbb{T}^t\mathbb{T}\mathbb{T}^t)\mathbb{T} \\
&\subseteq U_1U_2(U_3\mathbb{T}^t) \\
&\subseteq U_1U_2U_3 \\
&= Z.
\end{aligned}$$

Thus, Z is a $t\mathscr{B}$ of $\mathbb{T}$. Similar proofs for other cases.

Theorem 3.14. *Let Z be a TSS of $\mathbb{T}$. If U_1 is a l-RI, U_2 is a m-LATI and U_3 is a n-LI of $\mathbb{T}$ such that $U_1U_2U_3 \subseteq Z \subseteq U_1 \cap U_2 \cap U_3$, then Z is a $m\mathscr{B}$of $\mathbb{T}$.*

Proof. Suppose that U_1 is a l-RI, U_2 is a m-LATI and U_3 is a n-LI of $\mathbb{T}$ such that $U_1U_2U_3 \subseteq Z \subseteq U_1 \cap U_2 \cap U_3$. Then $Z\mathbb{T}^m Z\mathbb{T}^m Z \subseteq (U_1 \cap U_2 \cap U_3)\mathbb{T}^m(U_1 \cap U_2 \cap U_3)\mathbb{T}^m(U_1 \cap U_2 \cap U_3) \subseteq U_1(\mathbb{T}^m U_2 \mathbb{T}^m)U_3 \subseteq U_1U_2U_3 \subseteq Z$. Thus, Z is a $m\mathscr{B}$ of $\mathbb{T}$.

4. VARIOUS mIDEAL GENERATORS

Theorem 4.1. *Let $\mathscr{B}$ be a nonempty subset of $\mathbb{T}$. Then the $m\mathscr{B}$ generated by $\mathscr{B}$ is $<\mathscr{B}>_m = \mathscr{B} \cup \sum\limits_{finite} \mathscr{B}\mathbb{T}^m\mathscr{B}\mathbb{T}^m\mathscr{B}$.*

Proof. We show that $<\mathscr{B}>_m = \mathscr{B} \cup \sum\limits_{finite} \mathscr{B}\mathbb{T}^m\mathscr{B}\mathbb{T}^m\mathscr{B}$ is the smallest $m\mathscr{B}$ of $\mathbb{T}$ containing $\mathscr{B}$. Let $a, b, c \in <\mathscr{B}>_m$. Then,
$a = i_1$ or $a = \sum\limits_{finite} (s_j a_{j1} a_{j2} ... a_{jm} t_j a_{j1} a_{j2} ... a_{jn} r_j)$,
$b = i_2$ or $b = \sum\limits_{finite} (s_j^{'} a_{j1}^{'} a_{j2}^{'} ... a_{jm}^{'} t_j^{'} a_{j1}^{'} a_{j2}^{'} ... a_{jn}^{'} r_j^{'})$ and
$c = i_3$ or $c = \sum\limits_{finite} (s_j^{''} a_{j1}^{''} a_{j2}^{''} ... a_{jm}^{''} t_j^{''} a_{j1}^{''} a_{j2}^{''} ... a_{jn}^{''} r_j^{''})$.
Where $a_{j1}a_{j2}...a_{jn},\ a_{j1}^{'}a_{j2}^{'}...a_{jn}^{'},\ a_{j1}^{''}a_{j2}^{''}...a_{jn}^{''},\ a_{j1}a_{j2}...a_{jm},\ a_{j1}^{'}a_{j2}^{'}...a_{jm}^{'},$
$a_{j1}^{''}a_{j2}^{''}...a_{jm}^{''} \in \mathbb{T}$ and $s_j, t_j, r_j, s_j^{'}, t_j^{'}, r_j^{'}, s_j^{''}, t_j^{''}, r_j^{''}, i_1, i_2, i_3 \in \mathscr{B}$.
Put $y = s_j a_{j1} a_{j2} ... a_{jm} t_j a_{j1} a_{j2} ... a_{jn} r_j$, $\ y^{'} = s_j^{'} a_{j1}^{'} a_{j2}^{'} ... a_{jm}^{'} t_j^{'} a_{j1}^{'} a_{j2}^{'} ... a_{jn}^{'} r_j^{'}$,
$y^{''} = s_j^{''} a_{j1}^{''} a_{j2}^{''} ... a_{jm}^{''} t_j^{''} a_{j1}^{''} a_{j2}^{''} ... a_{jn}^{''} r_j^{''}$.
Now, $a \cdot b \cdot c = i_1 \cdot i_2 \cdot i_3 \in \mathscr{B} \subseteq <\mathscr{B}>_m$ and $a \cdot b \cdot c = \sum\limits_{finite} y \cdot \sum\limits_{finite} y^{'} \cdot \sum\limits_{finite} y^{''} = \sum\limits_{finite} y \cdot y^{'} \cdot y^{''} \subseteq \sum\limits_{finite} \mathscr{B}\mathbb{T}^m\mathscr{B}\mathbb{T}^m\mathscr{B} \subseteq <\mathscr{B}>_m$. Routine calculation remaining cases, $<\mathscr{B}>_m$ is a TSS of $\mathbb{T}$. Put $Q = \mathscr{B}\mathbb{T}^m\mathscr{B}\mathbb{T}^m\mathscr{B}$. Now, $<\mathscr{B}>_m \mathbb{T}^m <\mathscr{B}>_m \mathbb{T}^m <\mathscr{B}>_m = \Big[\mathscr{B} \cup \sum\limits_{finite} Q\Big]\mathbb{T}^m\Big[\mathscr{B} \cup \sum\limits_{finite} Q\Big]\mathbb{T}^m\Big[\mathscr{B} \cup \sum\limits_{finite} Q\Big] \subseteq \mathscr{B}\mathbb{T}^m\mathscr{B}\mathbb{T}^m\mathscr{B}$. Thus, $<\mathscr{B}>_m \mathbb{T}^m <\mathscr{B}>_m \mathbb{T}^m <\mathscr{B}>_m \subseteq <\mathscr{B}>_m$. Hence $<\mathscr{B}>_m$ is an m-bi ideal containing $\mathscr{B}$. Let Y be any other $m\mathscr{B}$ of $\mathbb{T}$ containing $\mathscr{B}$. Then $\mathscr{B}\mathbb{T}^m\mathscr{B}\mathbb{T}^m\mathscr{B} \subseteq Y\mathbb{T}^m Y\mathbb{T}^m Y \subseteq Y$. Therefore $<\mathscr{B}>_m = \mathscr{B} \cup \sum\limits_{finite} \mathscr{B}\mathbb{T}^m\mathscr{B}\mathbb{T}^m\mathscr{B} \subseteq Y$. Hence, $<\mathscr{B}>_m$ is the smallest $m\mathscr{B}$ of $\mathbb{T}$ containing $\mathscr{B}$.

Corollary 4.2. *Let I_1 be a nonempty subset of $\mathbb{T}$. Then the m-LI generated by I_1 is $<I_1>_l = I_1 \cup \sum\limits_{finite} \mathbb{T}^m I_1$.*

Corollary 4.3. *Let I_2 be a nonempty subset of $\mathbb{T}$. Then the m-RI generated by I_2 is $<I_2>_r = I_2 \cup \sum\limits_{finite} I_2\mathbb{T}^m$.*

Corollary 4.4. *Let I_3 be a nonempty subset of $\mathbb{T}$. Then the m-LATI generated by I_3 is $<I_3>_{lat} = I_3 \cup \sum\limits_{finite} \mathbb{T}^m I_3 \mathbb{T}^m$.*

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