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SOME MORE RESULTS ON $\epsilon - LP-$SASAKIAN MANIFOLDS ADMITTING η-RICCI SOLITONS

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ABSTRACT. The object of the present paper is to characterize $\epsilon - LP-$Sasakian manifolds with a quarter-symmetric metric connection admitting $\eta-$Ricci solitons. Finally, the existence of $\eta-$Ricci soliton in an $\epsilon - LP-$Sasakian manifold has been proved by a concrete example.

1. INTRODUCTION

The study of manifolds with indefinite metrics is of high interest in physics and relativity theory. In 1993, the concept of ϵ-Sasakian manifolds was introduced by Bejancu and Duggal [2]. Later, it was shown by Xufeng and Xiaoli [18] that these manifolds are real hypersurfaces of indefinite Kahlerian manifolds. In 2012, Prasad and Srivastava [14] have studied ϵ-Lorentzian para-Sasakian manifolds and shown its existence by an example. On the other hand, the concept of ϵ-Kenmotsu manifold was introduced by U. C. De and A. Sarkar [6] who showed that the existence of new structure on an indefinite metrics influences the curvatures. Recently, the manifolds with indefinite metrics have been studied by various authors in several ways to a different extent such as ([10], [12], [17]) and many others.

As a generalization of Ricci solitons, the notion of η-Ricci solitons was introduced by Cho and Kimura [3]. They have studied Ricci solitons of real hypersurfaces in a non-flat complex space form and defined η-Ricci solitons, which satisfies the equation

$$\pounds_V g + 2S + 2\lambda g + 2\mu\eta \otimes \eta = 0, \tag{1.1}$$

where S is the Ricci tensor associated to g, η is a 1-form and λ, μ are real numbers. In particular, if $\mu = 0$, then the notion of η-Ricci soliton (g, V, λ, μ) reduces to the notion of Ricci soliton (g, V, λ). Recenty, $\eta-$Ricci solitons have been studied by various authors such as ([4], [9], [11], [13], [15], [19]) and many others.

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2. PRELIMINARIES

A differentiable manifold M of dimension n is called an ϵ-Lorentzian para-Sasakian (briefly, $\epsilon - LP-$Sasakian), if it admits a $(1,1)$-tensor field ϕ, a contravariant vector field ξ, a 1-form η and a Lorentzian like metric g which satisfy

$$\phi^2 X = X + \eta(X)\xi,\ \eta(\xi) = -1, \tag{2.1}$$

$$g(\xi,\xi) = -\epsilon,\ \eta(X) = \epsilon g(X,\xi),\ \phi\xi = 0,\ \eta(\phi X) = 0, \tag{2.2}$$

$$g(\phi X, \phi Y) = g(X,Y) + \epsilon\eta(X)\eta(Y), \tag{2.3}$$

$$(\nabla_X \phi)Y = g(X,Y)\xi + \epsilon\eta(Y)X + 2\epsilon\eta(X)\eta(Y)\xi, \tag{2.4}$$

$$\nabla_X \xi = \epsilon\phi X \tag{2.5}$$

for all $X, Y \in \chi(M)$, where $\chi(M)$ is the Lie algebra of vector fields on the manifold M, ϵ is -1 or 1 according to the vector field ξ being spacelike or timelike and ∇ denotes the Levi-Civita connection with respect to g.
If we put

$$\Phi(X,Y) = g(\phi X, Y) \tag{2.6}$$

for all vector fields X and Y on M, then $\Phi(X,Y)$ is a symmetric $(0,2)$ tensor field. Also since the 1-form η is closed in an $\epsilon - LP-$Sasakian manifold, so we have [14]

$$(\nabla_X \eta)(Y) = \Phi(X,Y),\ \Phi(X,\xi) = 0 \tag{2.7}$$

for all $X, Y \in \chi(M)$.
Moreover, the curvature tensor R, the Ricci tensor S and the Ricci operator Q in an $\epsilon - LP-$Sasakian manifold with the Levi-Civita connection satisfy the following equations [14]:

$$R(X,Y)\xi = \eta(Y)X - \eta(X)Y, \tag{2.8}$$

$$R(\xi,X)Y = \epsilon g(X,Y)\xi - \eta(Y)X, \tag{2.9}$$

$$R(\xi,X)\xi = -R(X,\xi)\xi = X + \eta(X)\xi, \tag{2.10}$$

$$\eta(R(X,Y)Z) = \epsilon[g(Y,Z)\eta(X) - g(X,Z)\eta(Y)], \tag{2.11}$$

$$S(X,\xi) = (n-1)\eta(X), Q\xi = \epsilon(n-1)\xi, \tag{2.12}$$

where $X, Y, Z \in \chi(M)$ and $g(QX,Y) = S(X,Y)$.
We note that if $\epsilon = 1$ and the structure vector field ξ is timelike, then an $\epsilon - LP-$Sasakian manifold is usual $LP-$Sasakian manifold.

Definition 2.1. An $\epsilon - LP-$Sasakian manifold is said to be an η-Einstein manifold if its non-vanishing Ricci tensor S of type $(0,2)$ satisfies [20]

$$S(Y,Z) = \alpha g(Y,Z) + \beta\eta(Y)\eta(Z) \tag{2.13}$$

where α and β are the smooth functions on M. If $\beta = 0$, then M is said to be an Einstein manifold.

3. CURVATURE TENSOR IN AN $\epsilon - LP-$SASAKIAN MANIFOLD WITH A QUARTER-SYMMETRIC METRIC CONNECTION

A linear connection $\bar\nabla$ in a Riemannian manifold M is said to be a quarter-symmetric connection [7] if the torsion tensor T of the connection $\bar\nabla$ is of the form

$$T(X,Y) = \bar\nabla_X Y - \bar\nabla_Y X - [X,Y] = \eta(Y)\phi X - \eta(X)\phi Y, \tag{3.1}$$

where η is a 1-form and ϕ is a $(1,1)$ tensor field. If moreover, a quarter-symmetric connection $\bar{\nabla}$ satisfies the condition

$$(\bar{\nabla}_X g)(Y,Z) = 0 \tag{3.2}$$

for all $X,Y,Z \in \chi(M)$, then the connection $\bar{\nabla}$ is said to be a quarter-symmetric metric, otherwise it is said to be a quarter-symmetric non-metric.

A quarter-symmetric metric connection $\bar{\nabla}$ in an $\epsilon - LP-$Sasakian manifold is given by

$$\bar{\nabla}_X Y = \nabla_X Y + \eta(Y)\phi X - \epsilon g(\phi X, Y)\xi. \tag{3.3}$$

A quarter symmetric metric connection have been studied by various authors such as Ahmad et al. [1], De and Mondal [5], Singh and Pandey [16] and many others.

If $\bar{R}$ and $\bar{S}$, respectively, are the curvature tensor and the Ricci tensor of a quarter-symmetric metric connection $\bar{\nabla}$ in an $\epsilon - LP-$Sasakian manifold M, then we have [9]

$$\bar{R}(X,Y)Z = R(X,Y)Z + (2-\epsilon)[g(\phi X, Z)\phi Y - g(\phi Y, Z)\phi X] \tag{3.4}$$
$$+\epsilon\eta(Z)[\eta(Y)X - \eta(X)Y] + [\eta(X)g(Y,Z) - \eta(Y)g(X,Z)]\xi,$$
$$\bar{R}(X,Y)\xi = (1-\epsilon)[\eta(Y)X - \eta(X)Y], \tag{3.5}$$
$$\bar{R}(\xi,X)Y = -(1-\epsilon)[g(X,Y)\xi + \eta(Y)X], \tag{3.6}$$
$$\bar{R}(\xi,X)\xi = (1-\epsilon)[X + \eta(X)\xi], \tag{3.7}$$
$$\bar{S}(Y,Z) = S(Y,Z) + (1-\epsilon)g(Y,Z) + (n\epsilon - 1)\eta(Y)\eta(Z) - (2-\epsilon)g(\phi Y, Z)\psi, \tag{3.8}$$
$$\bar{S}(X,\xi) = (1-\epsilon)(n-1)\eta(X),\ \bar{Q}\xi = -(1-\epsilon)(n-1)\xi \tag{3.9}$$

for all $X,Y,Z \in \chi(M)$.

Lemma 3.1. *Let M be an n-dimensional $\epsilon - LP-$Sasakian manifold with a quarter-symmetric metric connection. Then we have*

$$(\bar{\nabla}_X\phi)Y = (1-\epsilon)(g(X,Y)\xi - \eta(Y)X - 2\eta(X)\eta(Y)\xi), \tag{3.10}$$
$$\bar{\nabla}_X\xi = -(1-\epsilon)\phi X, \tag{3.11}$$
$$(\bar{\nabla}_X\eta)Y = (1-\epsilon)g(\phi X, Y), \tag{3.12}$$
$$(\bar{\mathcal{L}}_\xi g)(X,Y) = 2\epsilon g(\phi X, Y) \tag{3.13}$$

for all $X,Y \in \chi(M)$.

Proof. By the covariant differentiation of ϕY with respect to X, we have

$$\bar{\nabla}_X\phi Y = (\bar{\nabla}_X\phi)Y + \phi(\bar{\nabla}_X Y)$$

which in view of (3.3) takes the form

$$(\bar{\nabla}_X\phi)Y = (\nabla_X\phi)Y - \epsilon g(\phi X, \phi Y)\xi - \eta(Y)\phi^2 X. \tag{3.14}$$

By making use of (2.1), (2.3) and (2.4) in the last equation, (3.10) follows.
Next, we replace $Y = \xi$ in (3.3) and using (2.2) we find

$$\bar{\nabla}_X\xi = \nabla_X\xi + \eta(\xi)\phi X. \tag{3.15}$$

By using (2.1) and (2.5) in (3.15), we get (3.11).
In order to prove (3.12), we differentiate $\eta(Y)$ covariantly with respect to X and using (3.2), we find

$$(\bar{\nabla}_X\eta)(Y) = \epsilon g(Y, \bar{\nabla}_X\xi) \tag{3.16}$$

which in view of (3.11) gives (3.12).
In view of (3.1), the expression $(\bar{\mathcal{L}}_\xi g)(X,Y) = \bar{\mathcal{L}}_\xi g(X,Y) - g(\bar{\mathcal{L}}_\xi X, Y) - g(X, \bar{\mathcal{L}}_\xi Y)$ takes the form

$$(\bar{\mathcal{L}}_\xi g)(X,Y) = \bar{\nabla}_\xi g(X,Y) - g[\bar{\nabla}_\xi X - \bar{\nabla}_X\xi - \phi X, Y] + g[X, \bar{\nabla}_\xi Y - \bar{\nabla}_Y\xi - \phi Y]$$

in which using (3.11), we obtain (3.13). □

In [9], A. Haseeb and R. Prasad have studied $\eta-$Ricci solitons in an $\epsilon - LP-$Sasakian manifold with a quarter-symmetric metric connection and proved the following:

Lemma 3.2. *If (g, ξ, λ, μ) is an $\eta-$Ricci soliton in an $\epsilon - LP-$Sasakian manifold with a quarter-symmetric metric connection, then*

$$\bar{S}(Y, Z) = -\epsilon g(\phi Y, Z) - \lambda g(Y, Z) - \mu\eta(Y)\eta(Z), \tag{3.17}$$

$$\lambda - \epsilon\mu = (1 - \epsilon)(n - 1) \tag{3.18}$$

for all $Y, Z \in \chi(M)$.

4. $\eta-$Ricci solitons in $\epsilon - LP-$Sasakian manifolds with a quarter-symmetric metric connection admitting Codazzi type of Ricci tensor

In this section we consider $\eta-$Ricci solitons in $\epsilon - LP-$Sasakian manifolds with a quarter-symmetric metric connection admitting Codazzi type of Ricci tensor. A. Gray [8] introduced the notion of cyclic parallel Ricci tensor and Codazzi type of Ricci tensor.

Definition 4.1. An $\epsilon-LP-$Sasakian manifold with a quarter-symmetric metric connection is said to have Codazzi type of Ricci tensor if its Ricci tensor $\bar{S}$ of type $(0, 2)$ is non-zero and satisfies the following condition

$$(\bar{\nabla}_X\bar{S})(Y, Z) = (\bar{\nabla}_Y\bar{S})(Z, X) \tag{4.1}$$

for all $X, Y, Z \in \chi(M)$.

Taking covariant derivative of (3.17) and making use of (2.2), (3.10) and (3.12), we find

$$(\bar{\nabla}_X\bar{S})(Y, Z) = (1 - \epsilon)[\epsilon g(X, Y)\eta(Z) - \eta(Y)g(X, Z) - 2\epsilon\eta(X)\eta(Y)\eta(Z)] \tag{4.2}$$
$$-(1 - \epsilon)\mu[g(\phi X, Y)\eta(Z) + g(\phi X, Z)\eta(Y)].$$

If the Ricci tensor $\bar{S}$ is of Codazzi type, then

$$(\bar{\nabla}_X\bar{S})(Y, Z) = (\bar{\nabla}_Y\bar{S})(Z, X). \tag{4.3}$$

Using (4.2) in (4.3), we have

$$g(X, Y)\eta(Z) - g(Y, Z)\eta(X) + \mu[g(\phi X, Y)\eta(Z) - g(\phi Y, Z)\eta(X)] = 0,\ (1 - \epsilon) \neq 0,$$

which by taking $Z = \xi$ and using (2.2) reduces to

$$g(X, Y) + \epsilon\eta(X)\eta(Y) + \mu g(\phi X, Y) = 0. \tag{4.4}$$

By putting $Y = \phi Y$ in (4.4), we find

$$g(X, \phi Y) + \mu g(\phi X, \phi Y) = 0. \tag{4.5}$$

Replacing X by ϕX in (4.5), we have

$$g(\phi X, \phi Y) + \mu g(X, \phi Y) = 0. \tag{4.6}$$

Now adding (4.5) and (4.6), we obtain

$$(1 + \mu)[g(X, \phi Y) + g(\phi X, \phi Y)] = 0 \tag{4.7}$$

from which it follows that $\mu = -1$. From the relation (3.18), we get $\lambda = n(1 - \epsilon) - 1$. Thus we have the following:

Theorem 4.1. *Let (g, ξ, λ, μ) be an $\eta-$Ricci soliton in an n-dimensional $\epsilon-LP-$Sasakian manifold with a quarter-symmetric metric connection and if the manifold has Ricci tensor of Codazzi type, then $\mu = -1$ and $\lambda = n(1 - \epsilon) - 1$.*

5. $\eta-$RICCI SOLITONS IN $\epsilon - LP-$SASAKIAN MANIFOLDS WITH A QUARTER-SYMMETRIC METRIC CONNECTION ADMITTING CYCLIC PARALLEL RICCI TENSOR

Definition 5.1. An $\epsilon-LP-$Sasakian manifold with a quarter-symmetric metric connection is said to have cyclic parallel Ricci tensor if its Ricci tensor $\bar{S}$ of type $(0,2)$ is non-zero and satisfies the following condition [8]

$$(\bar{\nabla}_X\bar{S})(Y,Z)+(\bar{\nabla}_Y\bar{S})(Z,X)+(\bar{\nabla}_Z\bar{S})(X,Y)=0 \tag{5.1}$$

for all $X,Y,Z\in\chi(M)$.

Let (g,ξ,λ,μ) be an $\eta-$Ricci soliton in an $\epsilon-LP-$Sasakian manifold with a quarter-symmetric metric connection and the manifold M has cyclic parallel Ricci tensor, then (5.1) holds. Taking covariant derivative of (3.17) and making use of (2.2), (3.10) and (3.12), we find

$$\begin{aligned}(\bar{\nabla}_X\bar{S})(Y,Z)&=(1-\epsilon)[\epsilon g(X,Y)\eta(Z)-\eta(Y)g(X,Z)-2\epsilon\eta(X)\eta(Y)\eta(Z)]\\&-(1-\epsilon)\mu[g(\phi X,Y)\eta(Z)+g(\phi X,Z)\eta(Y)].\end{aligned} \tag{5.2}$$

Similarly, we have

$$\begin{aligned}(\bar{\nabla}_Y\bar{S})(Z,X)&=(1-\epsilon)[\epsilon g(Y,Z)\eta(X)-\eta(Z)g(Y,X)-2\epsilon\eta(X)\eta(Y)\eta(Z)]\\&-(1-\epsilon)\mu[g(\phi Y,Z)\eta(X)+g(\phi Y,X)\eta(Z)],\end{aligned} \tag{5.3}$$

and

$$\begin{aligned}(\bar{\nabla}_Z\bar{S})(X,Y)&=(1-\epsilon)[\epsilon g(Z,X)\eta(Y)-\eta(X)g(Z,Y)-2\epsilon\eta(X)\eta(Y)\eta(Z)]\\&-(1-\epsilon)\mu[g(\phi Z,X)\eta(Y)+g(\phi Z,Y)\eta(X)].\end{aligned} \tag{5.4}$$

By using (5.2)-(5.4) in (5.1), we obtain

$$\begin{aligned}&-2[g(X,Y)\eta(Z)+g(Y,Z)\eta(X)+g(Z,X)\eta(Y)]+6\eta(X)\eta(Y)\eta(Z)\\&-2\mu[g(\phi X,Y)\eta(Z)+g(\phi Y,Z)\eta(X)+g(\phi Z,X)\eta(Y)]=0,\qquad(1-\epsilon)\neq 0\end{aligned}$$

which by taking $Z=\xi$ reduces to

$$g(\phi X,\phi Y)+\mu g(\phi X,Y)=0. \tag{5.5}$$

Replacing Y by ϕY in (5.5), we have

$$g(\phi X,Y)+\mu g(\phi X,\phi Y)=0. \tag{5.6}$$

By adding (5.5) and (5.6), we obtain

$$(1+\mu)[g(\phi X,Y)+g(\phi X,\phi Y)]=0 \tag{5.7}$$

from which it follows that $\mu=-1$. From the relation (3.18), we get $\lambda=n(1-\epsilon)-1$. Thus we have the following:

Theorem 5.1. *Let (g,ξ,λ,μ) be an $\eta-$Ricci soliton in an n-dimensional $\epsilon-LP-$Sasakian manifold with a quarter-symmetric metric connection and if the manifold has cyclic parallel Ricci tensor, then $\mu=-1$ and $\lambda=n(1-\epsilon)-1$.*

Definition 5.2. An $\epsilon-LP-$Sasakian manifold with a quarter-symmetric metric connection is said to have cyclic $\eta-$recurrent Ricci tensor, if

$$\begin{aligned}&(\bar{\nabla}_X\bar{S})(Y,Z)+(\bar{\nabla}_Y\bar{S})(Z,X)+(\bar{\nabla}_Z\bar{S})(X,Y)\\&=\eta(X)\bar{S}(Y,Z)+\eta(Y)\bar{S}(Z,X)+\eta(Z)\bar{S}(X,Y)\end{aligned} \tag{5.8}$$

for all $X,Y,Z\in\chi(M)$.

Suppose that the manifold M with a quarter-symmetric metric connection has cyclic $\eta-$recurrent Ricci tensor, then (5.8) holds. By using (3.17) and (5.2)-(5.4) in (5.8), we get

$$(\lambda-2(1-\epsilon))(g(X,Y)\eta(Z)+g(Y,Z)\eta(X)+g(Z,X)\eta(Y))+(6(1-\epsilon)+3\mu)\eta(X)\eta(Y)\eta(Z)$$
$$+(\epsilon-2(1-\epsilon)\mu)(g(\phi X,Y)\eta(Z)+g(\phi Y,Z)\eta(X)+g(\phi Z,X)\eta(Y))=0$$

which by putting $Y=Z=\xi$ gives $\mu=\epsilon\lambda$. Thus we have the following:

Theorem 5.2. *Let (g,ξ,λ,μ) be an $\eta-$Ricci soliton in an n-dimensional $\epsilon-LP-$Sasakian with a quarter-symmetric metric connection and if the manifold has cyclic $\eta-$recurrent, then $\mu=\epsilon\lambda$.*

6. ϕ-Ricci symmetric $\eta-$Ricci solitons in $\epsilon-LP-$Sasakian manifolds with a quarter-symmetric metric connection

Definition 6.1. An $\epsilon-LP-$Sasakian manifold with a quarter-symmetric metric connection is said to be ϕ-Ricci symmetric if the Ricci operator $\bar{Q}$ satisfies

$$\phi^2((\bar{\nabla}_X\bar{Q})(Y))=0 \tag{6.1}$$

for any $X,Y\in\chi(M)$.

Let (g,ξ,λ,μ) be an $\eta-$Ricci soliton in an n-dimensional $\epsilon-LP-$Sasakian manifold with a quarter-symmetric metric connection and the manifold M is ϕ-Ricci symmetric. Then (6.1) holds, which in view of (2.1) yields

$$(\bar{\nabla}_X\bar{Q})Y+\eta((\bar{\nabla}_X\bar{Q})Y)\xi=0. \tag{6.2}$$

Taking the inner product of (6.2) with Z and using (2.2), we find

$$g((\bar{\nabla}_X\bar{Q})Y,Z)+\epsilon\eta((\bar{\nabla}_X\bar{Q})Y)\eta(Z)=0$$

which can be written as

$$g(\bar{\nabla}_X\bar{Q}Y,Z)-\bar{S}(\bar{\nabla}_XY,Z)+\epsilon\eta((\bar{\nabla}_X\bar{Q})Y)\eta(Z)=0. \tag{6.3}$$

Now putting $Y=\xi$ in (6.3) and using (3.9) and (3.11), we get

$$(1-\epsilon)^2(n-1)g(\phi X,Z)+(1-\epsilon)\bar{S}(\phi X,Z)+\epsilon\eta((\bar{\nabla}_X\bar{Q})\xi)\eta(Z)=0.$$

Replacing Z by ϕZ in the last equation and using (2.2), we get

$$(1-\epsilon)(n-1)g(\phi X,\phi Z)+\bar{S}(\phi X,\phi Z)=0, \quad (1-\epsilon)\neq 0. \tag{6.4}$$

By virtue of (3.17), (6.4) takes the form

$$[\lambda-(1-\epsilon)(n-1)]g(\phi X,\phi Z)+\epsilon(X,\phi Z)=0. \tag{6.5}$$

Replacing X by ϕX in (6.5), we have

$$[\lambda-(1-\epsilon)(n-1)]g(X,\phi Z)+\epsilon(\phi X,\phi Z)=0. \tag{6.6}$$

By adding (6.5) and (6.6), we get

$$[\lambda-(1-\epsilon)(n-1)+\epsilon](g(X,\phi Z)+\epsilon(\phi X,\phi Z))=0 \tag{6.7}$$

from which it follows that $\lambda=(1-\epsilon)(n-1)-\epsilon$. From the relation (3.18), we get $\mu=-1$. Thus we have the following theorem:

Theorem 6.1. *Let (g,ξ,λ,μ) be an $\eta-$Ricci soliton in an n-dimensional $\epsilon-LP-$Sasakian manifold with a quarter-symmetric metric connection and if the manifold is ϕ-Ricci symmetric, then $\lambda=(1-\epsilon)(n-1)-\epsilon$ and $\mu=-1$.*

Now from the relations (2.3), (3.18) and (6.5), we obtain

$$\bar{S}(X,Z) = -(n-1)(1-\epsilon)g(X,Z). \tag{6.8}$$

Thus we have

Corollary 6.2. *An n-dimensional ϕ-Ricci symmetric $\epsilon - LP-$Sasakian manifold with a quarter-symmetric metric connection admitting an η-Ricci soliton (g,ξ,λ,μ) is an Einstein manifold of the form* (6.8).

7. The $\eta-$parallel $\phi-$tensor in $\epsilon - LP-$Sasakian manifolds with a quarter-symmetric metric connection

In this section we study the $\eta-$parallel $\phi-$tensor in $\epsilon - LP-$Sasakian manifolds with a quarter-symmetric metric connection. If the $(1,1)-$tensor ϕ is $\eta-$parallel, then we have

$$g((\bar{\nabla}_X\phi)Y,Z) = 0 \tag{7.1}$$

for any $X,Y,Z \in \chi(M)$.
By using (3.10) in (7.1), we have

$$(1-\epsilon)g(g(X,Y)\xi - \eta(Y)X - 2\eta(X)\eta(Y)\xi, Z) = 0.$$

By using (2.2) in the last equation, we find

$$(1-\epsilon)(\epsilon g(X,Y)\eta(Z) - \eta(Y)g(X,Z) - 2\epsilon\eta(X)\eta(Y)\eta(Z)) = 0. \tag{7.2}$$

Putting $Z = \xi$ in (7.2), we get

$$(1-\epsilon)(g(X,Y) - \eta(X)\eta(Y)) = 0 \tag{7.3}$$

from which it follows that either
(i) $\epsilon = 1$(i.e., the vector field ξ is timelike), or (ii) $g(X,Y) = \eta(X)\eta(Y)$.
Putting $X = \bar{Q}X$ in (ii), we obtain

$$\bar{S}(X,Y) = -(1-\epsilon)(n-1)\eta(X)\eta(Y). \tag{7.4}$$

In view of (3.17), (7.4) takes the form

$$\epsilon g(\phi Y,Z) + \lambda g(Y,Z) + [\mu - (1-\epsilon)(n-1)]\eta(Y)\eta(Z) = 0. \tag{7.5}$$

Taking $Y = Z = \xi$ in (7.5) and using (2.1) and (2.2), we obtain

$$\lambda - \epsilon\mu = (1-\epsilon)(n-1). \tag{7.6}$$

Thus we have the following theorem:

Theorem 7.1. *If in an n-dimensional $\epsilon-LP-$Sasakian manifold with a quarter-symmetric metric connection the tensor ϕ is $\eta-$parallel, then either the vector field ξ is timelike or the manifold is a special type of an $\eta-$Einstein manifold of the form* (7.4) *and the scalars λ and μ are related by $\lambda - \epsilon\mu = (1-\epsilon)(n-1)$.*

If μ=0, then (7.6) reduces to $\lambda = (1-\epsilon)(n-1)$. In this case we have

Corollary 7.2. *If (g,ξ,λ) is a Ricci soliton in an n-dimensional $\epsilon - LP-$Sasakian manifold with a quarter-symmetric metric connection with $\eta-$parallel $\phi-$tensor, then the Ricci soliton on M is expanding according to the vector field ξ being spacelike.*

Example. We consider the 3-dimensional manifold $M = \{(x,y,z) \in R^3\}$, where (x,y,z) are the standard coordinates in R^3. Let e_1, e_2 and e_3 be the vector fields on M given by

$$e_1 = coshx_3\frac{\partial}{\partial x_1} + sinhx_3\frac{\partial}{\partial x_2},\ e_2 = sinhx_3\frac{\partial}{\partial x_1} + coshx_3\frac{\partial}{\partial x_2},\ e_3 = \frac{\partial}{\partial x_3} = \xi.$$

Let g be the Lorentzian like (semi-Riemannian) metric defined by

$$g(e_1, e_1) = g(e_2, e_2) = 1,\ g(e_3, e_3) = -\epsilon,\ g(e_1, e_2) = g(e_1, e_3) = g(e_2, e_3) = 0.$$

Let η be the 1-form on M defined by $\eta(X) = \epsilon g(X, e_3) = \epsilon g(X, \xi)$ for all $X \in \chi(M)$. Let ϕ be the $(1, 1)$ tensor field on M defined by

$$\phi e_1 = -\epsilon e_2,\ \phi e_2 = -\epsilon e_1,\ \phi e_3 = 0.$$

Using Koszul's formula for the metric g, we can easily calculate

$$\nabla_{e_1} e_1 = 0,\ \nabla_{e_2} e_1 = -\epsilon e_3,\ \nabla_{e_3} e_1 = 0,\ \nabla_{e_1} e_2 = -\epsilon e_3,\ \nabla_{e_2} e_2 = 0, \tag{7.7}$$

$$\nabla_{e_3} e_2 = 0,\ \nabla_{e_1} e_3 = -e_2,\ \nabla_{e_2} e_3 = -e_1,\ \nabla_{e_3} e_3 = 0.$$

Also, one can easily verify that

$$\nabla_X \xi = \epsilon \phi X \quad \text{and} \quad (\nabla_X \phi)Y = g(X, Y)\xi + \epsilon\eta(Y)X + 2\epsilon\eta(X)\eta(Y)\xi.$$

Thus the manifold M is an $\epsilon - LP-$Sasakian manifold. From (3.3) and (7.7), we obtain

$$\bar{\nabla}_{e_1} e_1 = 0,\ \bar{\nabla}_{e_2} e_1 = (1-\epsilon)e_3,\ \bar{\nabla}_{e_3} e_1 = 0,\ \bar{\nabla}_{e_1} e_2 = (1-\epsilon)e_3 \tag{7.8}$$

$$\bar{\nabla}_{e_2} e_2 = 0,\ \bar{\nabla}_{e_3} e_2 = 0,\ \bar{\nabla}_{e_1} e_3 = 0,\ \bar{\nabla}_{e_2} e_3 = 0,\ \bar{\nabla}_{e_3} e_3 = 0.$$

It is known that

$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z. \tag{7.9}$$

By using the above results, we can easily obtain the components of the curvature tensors as follows:

$$R(e_1, e_2)e_1 = \epsilon e_2,\ R(e_1, e_2)e_2 = -\epsilon e_1,\ R(e_1, e_2)e_3 = 0, \tag{7.10}$$

$$R(e_2, e_3)e_1 = 0,\ R(e_2, e_3)e_2 = -\epsilon e_3,\ R(e_2, e_3)e_3 = -e_2,$$

$$R(e_1, e_3)e_1 = -\epsilon e_3,\ R(e_1, e_3)e_2 = 0,\ R(e_1, e_3)e_3 = -e_1,$$

and

$$\bar{R}(e_1, e_2)e_1 = \epsilon e_2,\ \bar{R}(e_1, e_2)e_2 = 2(1-\epsilon)e_1,\ \bar{R}(e_1, e_2)e_3 = 0, \tag{7.11}$$

$$\bar{R}(e_2, e_3)e_1 = 0,\ \bar{R}(e_2, e_3)e_2 = (1-\epsilon)e_3,\ \bar{R}(e_2, e_3)e_3 = -(1-\epsilon)e_2,$$

$$\bar{R}(e_1, e_3)e_1 = (1-\epsilon)e_3,\ \bar{R}(e_1, e_3)e_2 = 0,\ \bar{R}(e_1, e_3)e_3 = -(1-\epsilon)e_1.$$

From these curvature tensors, we can easily calculate

$$S(e_1, e_1) = S(e_2, e_2) = 0,\ S(e_3, e_3) = -2. \tag{7.12}$$

$$\bar{S}(e_1, e_1) = \bar{S}(e_2, e_2) = (1-\epsilon),\ \bar{S}(e_3, e_3) = -2(1-\epsilon). \tag{7.13}$$

Now from (3.17) and (7.13), we obtain $\lambda = -(1-\epsilon)$ and $\mu = 3(1-\epsilon)$. Therefore the data (g, ξ, λ, μ) for $\lambda = -(1-\epsilon)$ and $\mu = 3(1-\epsilon)$ defines an $\eta-$Ricci soliton on M.

REFERENCES

[1] Ahmad, M., Jun, J. B. and Haseeb, A., Hypersurfaces of almost r-paracontact Riemannian manifold with a quarter-symmetric metric connection, Bull. Korean Math. Soc., 46(2009), no. 3, 477-487.

[2] Bejancu, A. and Duggal, K. L., Real hypersurfaces of indefinite Kaehler manifolds, Int. J. Math. Math. Sci., 16(1993), 545-556.

[3] Cho, J. T. and Kimura, M., Ricci solitons and real hypersurfaces in a complex space form, Tohoku Math. J., 61(2009), no. 2, 205-212.

[4] De, K. and De, U. C., η-Ricci solitons on Kenmotsu 3-manifolds, An. Univ. Vest Timi. Ser. Mat.-Inform., 56 (2018), 51-63.

[5] De, U. C. and Mondal, A. K., Quarter-symmetric metric connection on 3-dimensional quasi-Sasakian manifolds, SUT J. Math., 46(2010), 35-52.

[6] De, U. C. and Sarkar, A., On $\epsilon-$Kenmotsu manifold, Hardonic J. 32(2009), no.2, 231-242.

[7] Golab, S., On semi-symmetric and quarter-symmetric linear connections, Tensor (N.S.), 29(1975), 249-254.

[8] Gray, A., Einstein-like manifolds which are not Einstein, Geom. Dedicata, 7(1978), 259-280.

[9] Haseeb, A. and Prasad, R., η-Ricci solitons on $\epsilon - LP-$Sasakian manifolds with a quarter-symmetric metric connection, Honam Math. J., 41 (2019), 539-558.

[10] Haseeb, A. and Prasad, R., $*-$Conformal $\eta-$Ricci solitons in $\epsilon-LP-$Sasakian manifolds, Publ. De L'Inst. Math., Nouvelle serie, tome 108(122) (2020), 89-100.

[11] Haseeb, A. and De, U. C., η-Ricci solitons in $\epsilon-$Kenmotsu manifolds, J. Geom., 110 (2019), 12 pages.

[12] Kumar, R., Rani, R. and Nagaich, R. K., On sectional curvature of $\epsilon-$Sasakian manifolds, IJMMS, (2007), Article ID: 93562, 10 pages.

[13] Prakasha, D. G. and Hadimani, B. S., η-Ricci solitons on para-Sasakian manifolds, J. Geom., 108 (2017), 383-392.

[14] Prasad, R. and Srivastava, V., On ϵ-Lorentzian para-Sasakian manifolds, Commun. Korean Math. Soc., 27(2012), 297-306.

[15] Singh, A. and Kishor, S., Some types of η-Ricci solitons on Lorentzian para-Sasakian manifolds, Facta Universitatis (NIS), 33(2018), no. 2, 217-230.

[16] Singh, R. N. and Pandey, S. K., On quarter-symmetric metric connection in an LP-Sasakian manifold, Thai J. Math., 12(2014), 357-371.

[17] Venkatesha, S . V. Vishnuvardhana, $\epsilon-$Kenmotsu manifolds admitting a semi-symmetric metric connection, Ital. J. Pure Appl. Math., 38 (2017), 615-623.

[18] Xufeng, X. and Xiaoli, C., Two theorems on ϵ-Sasakian manifolds, Int. J. Math. Math. Sci., 21(1998), 249-54.

[19] Yadav, S. K., Chaubey, S. K. and Suthar, D. L., Some results on η-Ricci solitons on $(LCS)_n$- manifolds, Surveys in Mathematics and its applications., 13 (2018), 237-250.

[20] Yano, K. and Kon, M., Structures on Manifolds, Series in Pure Math., Vol. 3, World Sci., 1984.

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