ANNALS OF COMMUNICATIONS IN MATHEMATICS
Volume 3, Number 1 (2020), 88-96
ISSN: 2582-0818

BIPOLAR FUZZY BCI-IMPLICATIVE IDEALS OF BCI-ALGEBRAS

D. AL-KADI* AND G. MUHIUDDIN

ABSTRACT. The paper introduces the notions of bipolar fuzzy BCI-implicative ideals and bipolar fuzzy closed BCI-implicative ideals of BCI-algebras. It is proved that any bipolar fuzzy BCI-implicative ideal is a bipolar fuzzy ideal but not the converse. Characterizations of bipolar fuzzy BCI-implicative ideals and bipolar fuzzy closed BCI-implicative ideals are given and more properties are studied.

1. INTRODUCTION

The study of BCK/BCI-algebras was introduced by Imai and Iséki [4, 5] in 1966. Both algebras; BCK-algebra and it's extension BCI-algebra are two important classes of logical algebras proposed by Iséki [6, 7]. Hu et al.[3] introduced BCI-algebras satisfying the condition $(x * y) * z = x * (y * z)$.

The notion of fuzzy sets were initially introduced by Zadeh in [23]. Elements of fuzzy sets have degrees of belongings that vary over the unit interval. If the membership degree is one (zero) then this indicates that the element belongs (does not belong) to the corresponding fuzzy set. If it lies over the open unit interval then the element belongs to the fuzzy set partially (see [2, 26] for more information on fuzzy sets). In some cases, the standard fuzzy set illustration cannot tell relevant parts from irrelevant ones.

To solve such a problem, Zhang et al. generalized the concept [23] and initiated the notion bipolar fuzzy sets [24, 25]. Also, bipolar-valued fuzzy sets, which are introduced by Lee [11], and intuitionistic fuzzy sets given by Atanassov in [1] are extensions of fuzzy sets (see [12] for Lee's comparison). The latter author studied bipolar fuzzy subalgebras and ideals in BCK/BCI-algebras [10]. Recently, Muhiuddin et al. studied some concepts on bipolar fuzzy subalgebras and ideals on different algebras (see for e.g., [19], [16]).

Liu et al. [13] studied fuzzy ideals in BCI-algebras. Meng et al. [15] developed fuzzy implicative ideals of BCK-algebras. Jun [8] introduced closed fuzzy ideals in BCI-algebras. The ideal theory of BCK/BCI-algebras has been studied on various aspects (see for e.g., [9, 20, 17, 21, 18, 22]).

2010 *Mathematics Subject Classification.* 03E72, 03G25, 06F35, 08A72, 16D25, 94D05.
Key words and phrases. BCI-algebras; (implicative, closed) ideal; bipolar fuzzy (BCI-implicative, closed BCI-implicative) ideal.
Received: March 26, 2020. Accepted: May 25, 2020.
*Corresponding author.

In this paper, the notions of bipolar fuzzy (closed) BCI-implicative ideals are introduced. Conditions for a bipolar fuzzy (closed) ideal to be a bipolar fuzzy BCI-implicative ideal of BCI-algebras are provided. Also, we show that a bipolar fuzzy BCI-implicative ideal in BCI-algebras is a bipolar fuzzy ideal and that converse is not true.

2. PRELIMINARIES

Definition 2.1. An algebra $(K;*,0)$ of kind $(2,0)$ is a $BCI-algebra$ if it satisfies for all $k,l,m \in K$,
(K_1) $((k*l)*(k*m))*(m*l)=0$,
(K_2) $(k*(k*l))*l=0$,
(K_3) $k*k=0$,
(K_4) $k*l=0$ and $l*k=0 \Rightarrow k=l$.

The following are true in a BCI-algebra K.
(P_1) $k*0=k$
(P_2) $(k*l)*m=(k*m)*l$
(P_3) $k \leq l \Rightarrow k*m \leq l*m$ and $m*l \leq m*k$
(P_4) $0*(k*l)=(0*k)*(0*l)$
(P_5) $0*(0*(k*l))=0*(l*k)$
(P_6) $(k*m)*(l*m) \leq (k*l)$
(P_7) $k*(k*(k*l))=k*l$
for any $k,l,m \in K$ (see [6] for more details).
For brevity, K denotes a BCI-algebra. We remind the reader of the following definitions that are taken from [14, 13].

A nonempty subset L of K is an $ideal$ of K if it satisfies
(J_1) $0 \in L$,
(J_2) $\forall k,l \in K, k*l \in L, l \in L \Rightarrow k \in L$.

A nonempty subset L of K is a $BCI-implicative\ ideal$ of K if it satisfies
(J_1) and
(J_3) $\forall k,l,m \in K, (((k*l)*l)*(0*l))*m \in L, m \in L \Rightarrow k*((l*(l*k))*(0*(0*(k*l)))) \in L$.

A fuzzy set μ in K is a $fuzzy\ ideal$ of K if it satisfies for all $k,l,m \in K$,
(F_1) $\mu(0) \geq \mu(k)$,
(F_2) $\mu(k) \geq \mu(k*l) \wedge \mu(l)$.

A fuzzy set μ in K is a $fuzzy\ BCI-implicative\ ideal$ of K if it satisfies (F_1) and for all $k,l,m \in K$,
(F_3) $\mu(k*((l*(l*k))*(0*(0*(k*l))))) \geq \mu((((k*l)*l)*(0*l))*m) \wedge \mu(m)$.

3. BIPOLAR FUZZY BCI-IMPLICATIVE IDEAL

We remind the reader that K denotes a BCI-algebra unless otherwise specified.

For any set of real numbers $\{a_i \mid i \in \Delta\}$, we define

$$\vee\{a_i \mid i \in \Delta\} := \begin{cases} max\{a_i \mid i \in \Delta\} & if\ \Delta\ isfinite, \\ sup\{a_i \mid i \in \Delta\} & otherwise, \end{cases}$$

$$\wedge\{a_i \mid i \in \Delta\} := \begin{cases} min\{a_i \mid i \in \Delta\} & if\, \Delta\, isfinite, \\ inf\{a_i \mid i \in \Delta\} & otherwise. \end{cases}$$

Moreover, if $\Delta = \{1, 2, ..., n\}$, then $\vee\{a_i \mid i \in \Delta\}$ and $\wedge\{a_i \mid i \in \Delta\}$ are denoted by $a_1 \vee a_2 \vee ... \vee a_n$ and $a_1 \wedge a_2 \wedge ... \wedge a_n$, respectively.

A bipolar fuzzy set in K is denoted by $\mu = (K; \mu_n, \mu_p)$, where μ_n, μ_p are the maps from K to $[-1, 0]$ and from K to $[0, 1]$ respectively.

Definition 3.1. [10] A bipolar fuzzy set $\mu = (K; \mu_n, \mu_p)$ in K is called a $bipolar\ fuzzy\ ideal$ of K if it satisfies the following assertions:
(BF_1) $(\forall k \in K)$ $(\mu_n(0) \leq \mu_n(k),\ \mu_p(0) \geq \mu_p(k))$,
(BF_2) $(\forall k, l \in K)$ $\mu_n(k) \leq \mu_n(k * l) \vee \mu_n(l)$,
(BF_3) $(\forall k, l \in K)$ $\mu_p(k) \geq \mu_p(k * l) \wedge \mu_p(l)$.

Definition 3.2. A bipolar fuzzy set $\mu = (K; \mu_n, \mu_p)$ in a BCI-algebra K is called a $bipolar\ fuzzy\ BCI - implicative\ ideal$ of K if it satisfies (BF_1) and the following assertions:
(BF_4) $(\forall k, l, m \in K)$ $\mu_n(k * ((l * (l * k)) * (0 * (0 * (k * l))))) \leq \mu_n((((k * l) * l) * (0 * l)) * m) \vee \mu_n(m)$,
(BF_5) $(\forall k, l, m \in K)$ $\mu_p(k * ((l * (l * k)) * (0 * (0 * (k * l))))) \geq \mu_p((((k * l) * l) * (0 * l)) * m) \wedge \mu_p(m)$.

Example 3.3. Consider a BCI-algebra $(K; *, 0)$ where $K = \{0, j, k, l\}$ and $*$ is given by the Cayley table:

*	0	j	k	l
0	0	j	k	l
j	j	0	l	k
k	k	l	0	j
l	l	k	j	0

Let $\mu = (K; \mu_n, \mu_p)$ be a bipolar fuzzy set in K represented by:

K	0	j	k	l
μ_n	-0.7	-0.7	-0.4	-0.4
μ_p	0.8	0.8	0.2	0.2

Then, by routine calculations, $\mu = (K; \mu_n, \mu_p)$ is a bipolar fuzzy BCI-implicative ideal of K.

Theorem 3.1. *Any bipolar fuzzy BCI-implicative ideal of K is a bipolar fuzzy ideal of K.*

Proof. Assume that μ is a bipolar fuzzy BCI-implicative ideal of K. Then
$\mu_n(k * ((l * (l * k)) * (0 * (0 * (k * l))))) \leq \mu_n((((k * l) * l) * (0 * l)) * m) \vee \mu_n(m)$
and
$\mu_p(k * ((l * (l * k)) * (0 * (0 * (k * l))))) \geq \mu_p((((k * l) * l) * (0 * l)) * m) \wedge \mu_p(m)$
for all $k, l, m \in K$.

Substitute m by l and l by 0 to get,
$\mu_n(k*((0*(0*k))*(0*(0*(k*0))))) \leq \mu_n((((k*0)*0)*(0*0))*l) \vee \mu_n(l)$
and
$\mu_p(k*((0*(0*k))*(0*(0*(k*0))))) \geq \mu_p((((k*0)*0)*(0*0))*l) \wedge \mu_p(l)$.
i.e., $\mu_n(k) \leq \mu_n(k*l) \vee \mu_n(l)$ and $\mu_p(k) \geq \mu_p(k*l) \wedge \mu_p(l)$.
Hence, μ is a bipolar fuzzy ideal of K. □

The converse of Theorem 3.1 is not true as proved by the next example.

Example 3.4. Consider a BCI-algebra $(K;*,0)$ where $K = \{0, j, k, l\}$ and $*$ is given by the Cayley table:

*	0	j	k	l
0	0	0	0	l
j	j	0	0	l
k	k	k	0	l
l	l	l	l	0

Let $\mu = (K; \mu_n, \mu_p)$ be a bipolar fuzzy ideal in K represented by:

K	0	j	k	l
μ_n	-0.5	-0.2	-0.2	-0.2
μ_p	0.4	0.3	0.3	0.3

Then $\mu = (K; \mu_n, \mu_p)$ is not a bipolar fuzzy BCI-implicative ideal of K as
$\mu_n(j*((k*(k*j))*(0*(0*(j*k))))) = \mu_n(j) = -0.2 \nleq -0.5 = \mu_n((((j*k)*k)*(0*k))*0) \vee \mu_n(0) = \mu_n(0)$.

Lemma 3.2. [10] *A bipolar fuzzy set $\mu = (K; \mu_n, \mu_p)$ in K is a bipolar fuzzy ideal of K if and only if for all $k, l, m \in K$, $(k*l)*m = 0$ implies $\mu_n(k) \leq \mu_n(l) \vee \mu_n(m)$ and $\mu_p(k) \geq \mu_p(l) \wedge \mu_p(m)$.*

Lemma 3.3. [10] *A bipolar fuzzy set $\mu = (K; \mu_n, \mu_p)$ in K is a bipolar fuzzy ideal of K if and only if for all $k, l, m \in K$, $k*l = 0$ implies $\mu_n(k) \leq \mu_n(l)$ and $\mu_p(k) \geq \mu_p(l)$.*

Theorem 3.4. *Let $\mu = (K; \mu_n, \mu_p)$ be a bipolar fuzzy ideal of K. Then the following assertions are equivalent:*
1. $\mu = (K; \mu_n, \mu_p)$ is a bipolar fuzzy BCI-implicative ideal of K.
2. $\mu_n(k*((l*(l*k))*(0*(0*(k*l))))) \leq \mu_n(((k*l)*l)*(0*l))$ and
$\mu_p(k*((l*(l*k))*(0*(0*(k*l))))) \geq \mu_p(((k*l)*l)*(0*l))$.
3. $\mu_n(k*((l*(l*k))*(0*(0*(k*l))))) = \mu_n(((k*l)*l)*(0*l))$ and
$\mu_p(k*((l*(l*k))*(0*(0*(k*l))))) = \mu_p(((k*l)*l)*(0*l))$,
for all $k, l \in K$.

Proof. $(1 \Rightarrow 2)$ Let $\mu = (K; \mu_n, \mu_p)$ be a bipolar fuzzy BCI-implicative ideal of K. Then for any $k, l, m \in K$,
$\mu_n(k*((l*(l*k))*(0*(0*(k*l))))) \leq \mu_n((((k*l)*l)*(0*l))*m) \vee \mu_n(m)$
and
$\mu_p(k*((l*(l*k))*(0*(0*(k*l))))) \geq \mu_p((((k*l)*l)*(0*l))*m) \wedge \mu_p(m)$.
Put $m = 0$ to get
$\mu_n(k*((l*(l*k))*(0*(0*(k*l))))) \leq \mu_n((((k*l)*l)*(0*l))*0) \vee \mu_n(0)$

and
$\mu_p(k*((l*(l*k))*(0*(0*(k*l))))) \geq \mu_p((((k*l)*l)*(0*l))*0) \wedge \mu_p(0)$,
i.e., $\mu_n(k*((l*(l*k))*(0*(0*(k*l))))) \leq \mu_n(((k*l)*l)*(0*l))$ and
$\mu_p(k*((l*(l*k))*(0*(0*(k*l))))) \geq \mu_p(((k*l)*l)*(0*l))$.
Which are the required conditions.
$(2 \Rightarrow 3)$ Assume that for all $k, l \in K$,

$$\mu_n(k*((l*(l*k))*(0*(0*(k*l))))) \leq \mu_n(((k*l)*l)*(0*l)) \tag{3.1}$$

and

$$\mu_p(k*((l*(l*k))*(0*(0*(k*l))))) \geq \mu_p(((k*l)*l)*(0*l)). \tag{3.2}$$

We know (using P_5, P_6, P_1 and K_3) that
$(((k*l)*l)*(0*l))*(k*((l*(l*k))*(0*(0*(k*l))))) \leq 0$.
Therefore by Lemma 3.2, we have

$$\begin{aligned}\mu_n(((k*l)*l)*(0*l)) &\leq \mu_n(k*((l*(l*k))*(0*(0*(k*l))))) \vee \mu_n(0)\\ &= \mu_n(k*((l*(l*k))*(0*(0*(k*l)))))\end{aligned}$$

and

$$\begin{aligned}\mu_p(((k*l)*l)*(0*l)) &\geq \mu_p(k*((l*(l*k))*(0*(0*(k*l))))) \wedge \mu_n(0)\\ &= \mu_p(k*((l*(l*k))*(0*(0*(k*l))))).\end{aligned}$$

That is,

$$\mu_n(((k*l)*l)*(0*l)) \leq \mu_n(k*((l*(l*k))*(0*(0*(k*l))))) \tag{3.3}$$

and

$$\mu_p(((k*l)*l)*(0*l)) \geq \mu_p(k*((l*(l*k))*(0*(0*(k*l))))). \tag{3.4}$$

From the inequalities (3.1) and (3.3), (3.2) and (3.4), we have
$\mu_n(k*((l*(l*k))*(0*(0*(k*l))))) = \mu_n(((k*l)*l)*(0*l))$
and
$\mu_p(k*((l*(l*k))*(0*(0*(k*l))))) = \mu_p(((k*l)*l)*(0*l))$.
$(3 \Rightarrow 1)$ Assume that for all $k, l \in K$.

$$\mu_n(k*((l*(l*k))*(0*(0*(k*l))))) = \mu_n(((k*l)*l)*(0*l)) \tag{3.5}$$

and

$$\mu_p(k*((l*(l*k))*(0*(0*(k*l))))) = \mu_p(((k*l)*l)*(0*l)). \tag{3.6}$$

Consider $(((k*l)*l)*(0*l))*((((k*l)*l)*(0*l))*m)$.
Using (P_6) and (K_1) we have
$(((k*l)*l)*(0*l))*((((k*l)*l)*(0*l))*m) \leq m$.
Therefore by Lemma 3.2, we get

$$\mu_n(((k*l)*l)*(0*l)) \leq \mu_n((((k*l)*l)*(0*l))*m) \vee \mu_n(m) \tag{3.7}$$

and

$$\mu_p(((k*l)*l)*(0*l)) \geq \mu_p((((k*l)*l)*(0*l))*m) \wedge \mu_p(m) \tag{3.8}$$

combine (3.5) and (3.7), (3.6) and (3.8), to get
$\mu_n(k*((l*(l*k))*(0*(0*(k*l))))) \leq \mu_n((((k*l)*l)*(0*l))*m) \vee \mu_n(m)$
and
$\mu_p(k*((l*(l*k))*(0*(0*(k*l))))) \geq \mu_p((((k*l)*l)*(0*l))*m) \wedge \mu_p(m)$.
Hence, $\mu = (K; \mu_n, \mu_p)$ is a bipolar fuzzy BCI-implicative ideal of K. □

Definition 3.5. [24] Let $\mu_1 = (K; \mu_{n_1}, \mu_{p_1})$ and $\mu_2 = (K; \mu_{n_2}, \mu_{p_2})$ be bipolar fuzzy sets in K. Then the "union" denoted by $\mu_1 \bigcup \mu_2$ is $(K; \mu_{n_1} \wedge \mu_{n_2}, \mu_{p_1} \vee \mu_{p_2})$.

Theorem 3.5. *Let $\mu_1 = (K; \mu_{n_1}, \mu_{p_1})$ and $\mu_2 = (K; \mu_{n_2}, \mu_{p_2})$ be bipolar fuzzy BCI-implicative ideals of K. Then $\mu_1 \bigcup \mu_2$ is a bipolar fuzzy BCI-implicative ideal of K.*

Proof. Let $\mu_1 = (K; \mu_{n_1}, \mu_{p_1})$ and $\mu_2 = (K; \mu_{n_2}, \mu_{p_2})$ be bipolar fuzzy BCI-implicative ideals of K.
Then, $\mu_{n_1}(0) \leq \mu_{n_1}(k)$ and $\mu_{p_1}(0) \geq \mu_{p_1}(k)$, $\mu_{n_2}(0) \leq \mu_{n_2}(k)$ and $\mu_{p_2}(0) \geq \mu_{p_2}(k)$.
Therefore, $(\mu_{n_1} \wedge \mu_{n_2})(0) \leq \mu_{n_1}(k) \wedge \mu_{n_2}(k)$ and $(\mu_{p_1} \vee \mu_{p_2})(0) \geq \mu_{p_1}(k) \vee \mu_{p_2}(k)$.
Also, for all $k, l \in K$ we have
$\mu_{n_1}(k * ((l * (l * k)) * (0 * (0 * (k * l))))) = \mu_{n_1}(((k * l) * l) * (0 * l))$,
$\mu_{p_1}(k * ((l * (l * k)) * (0 * (0 * (k * l))))) = \mu_{p_1}(((k * l) * l) * (0 * l))$
and
$\mu_{n_2}(k * ((l * (l * k)) * (0 * (0 * (k * l))))) = \mu_{n_2}(((k * l) * l) * (0 * l))$,
$\mu_{p_2}(k * ((l * (l * k)) * (0 * (0 * (k * l))))) = \mu_{p_2}(((k * l) * l) * (0 * l))$.
Thus,$(\mu_{n_1} \wedge \mu_{n_2})(k * ((l * (l * k)) * (0 * (0 * (k * l)))))$
$= \mu_{n_1}(((k * l) * l) * (0 * l)) \wedge \mu_{n_2}(((k * l) * l) * (0 * l))$
$= (\mu_{n_1} \wedge \mu_{n_2})(((k * l) * l) * (0 * l))$
and
$(\mu_{p_1} \vee \mu_{p_2})(k * ((l * (l * k)) * (0 * (0 * (k * l)))))$
$= \mu_{p_1}(((k * l) * l) * (0 * l)) \vee \mu_{p_2}(((k * l) * l) * (0 * l))$
$= (\mu_{p_1} \vee \mu_{p_2})(((k * l) * l) * (0 * l))$.
That is, $\mu_1 \bigcup \mu_2$ is a bipolar fuzzy BCI-implicative ideal of K by Theorem 3.4. □

Definition 3.6. [24] Let $\mu_1 = (K; \mu_{n_1}, \mu_{p_1})$ and $\mu_2 = (K; \mu_{n_2}, \mu_{p_2})$ be bipolar fuzzy sets in K. Then the "intersection" denoted by $\mu_1 \bigcap \mu_2$ is $(K; \mu_{n_1} \vee \mu_{n_2}, \mu_{p_1} \wedge \mu_{p_2})$.

Theorem 3.6. *If $\mu_1 = (K; \mu_{n_1}, \mu_{p_1})$ and $\mu_2 = (K; \mu_{n_2}, \mu_{p_2})$ are bipolar fuzzy BCI-implicative ideals of K, then so is $\mu_1 \bigcap \mu_2 = (K; \mu_{n_1} \vee \mu_{n_2}, \mu_{p_1} \wedge \mu_{p_2})$.*

Proof. Let $\mu_1 = (K; \mu_{n_1}, \mu_{p_1})$ and $\mu_2 = (K; \mu_{n_2}, \mu_{p_2})$ be bipolar fuzzy BCI-implicative ideals of K.
Then, $\mu_{n_1}(0) \leq \mu_{n_1}(k)$ and $\mu_{p_1}(0) \geq \mu_{p_1}(k)$, $\mu_{n_2}(0) \leq \mu_{n_2}(k)$ and $\mu_{p_2}(0) \geq \mu_{p_2}(k)$.
Therefore, $(\mu_{n_1} \vee \mu_{n_2})(0) \leq \mu_{n_1}(k) \vee \mu_{n_2}(k)$ and $(\mu_{p_1} \wedge \mu_{p_2})(0) \geq \mu_{p_1}(k) \wedge \mu_{p_2}(k)$.
Also for all $k, l \in K$ we have
$\mu_{n_1}(k * ((l * (l * k)) * (0 * (0 * (k * l))))) = \mu_{n_1}(((k * l) * l) * (0 * l))$,
$\mu_{p_1}(k * ((l * (l * k)) * (0 * (0 * (k * l))))) = \mu_{p_1}(((k * l) * l) * (0 * l))$
and
$\mu_{n_2}(k * ((l * (l * k)) * (0 * (0 * (k * l))))) = \mu_{n_2}(((k * l) * l) * (0 * l))$,
$\mu_{p_2}(k * ((l * (l * k)) * (0 * (0 * (k * l))))) = \mu_{p_2}(((k * l) * l) * (0 * l))$.
Thus,$(\mu_{n_1} \vee \mu_{n_2})(k * ((l * (l * k)) * (0 * (0 * (k * l)))))$
$= \mu_{n_1}(((k * l) * l) * (0 * l)) \vee \mu_{n_2}(((k * l) * l) * (0 * l))$
$= (\mu_{n_1} \vee \mu_{n_2})(((k * l) * l) * (0 * l))$
and
$(\mu_{p_1} \wedge \mu_{p_2})(k * ((l * (l * k)) * (0 * (0 * (k * l)))))$
$= \mu_{p_1}(((k * l) * l) * (0 * l)) \wedge \mu_{p_2}(((k * l) * l) * (0 * l))$
$= (\mu_{p_1} \wedge \mu_{p_2})(((k * l) * l) * (0 * l))$.
By Theorem 3.4, $\mu_1 \bigcap \mu_2$ is a bipolar fuzzy BCI-implicative ideal of K. □

Definition 3.7. A bipolar fuzzy set $\mu = (K; \mu_n, \mu_p)$ in K is said to be a $bipolar\ fuzzy\ closed\ BCI-implicative\ ideal$ of K if it satisfies (BF_1), (BF_4), (BF_5) and
(BF_6) $\mu_n(0 * k) \leq \mu_n(k)$ and $\mu_p(0 * k) \geq \mu_p(k)$, for all $k \in K$.

Example 3.8. Consider the BCI-algebra and the bipolar fuzzy set μ given in Example 3.3. Then μ is a bipolar fuzzy BCI-implicative ideal of K and as for all $k \in K$, $0 * k = k$ then we have $\mu_n(0 * k) = \mu_n(k)$ and $\mu_p(0 * k) = \mu_p(k)$. Thus μ is a bipolar fuzzy closed BCI-implicative ideal of K.

From the above example we give the following corollary.

Corollary 3.7. *Let $\mu = (K; \mu_n, \mu_p)$ be a bipolar fuzzy BCI-implicative ideal of K. If $0 * k = k$ for all $k \in K$, then $\mu = (K; \mu_n, \mu_p)$ is a bipolar fuzzy closed BCI-implicative ideal of K.*

Theorem 3.8. *Let $\mu = (K; \mu_n, \mu_p)$ be a bipolar fuzzy closed ideal of K. Then $\mu = (K; \mu_n, \mu_p)$ is a bipolar fuzzy BCI-implicative ideal of K if and only if:*
1. $\mu_n(k * (l * (l * k))) \leq \mu_n(((k * l) * l) * (0 * l))$
2. $\mu_p(k * (l * (l * k))) \geq \mu_p(((k * l) * l) * (0 * l))$,
for all $k, l \in K$.

Proof. Suppose that μ is a bipolar fuzzy BCI-implicative ideal of K. Since μ is a bipolar fuzzy closed ideal of K, so for any $k, l \in K$,

$$\mu_n(0 * (((k * l) * l) * (0 * l))) \leq \mu_n(((k * l) * l) * (0 * l))$$

and

$$\mu_p(0 * (((k * l) * l) * (0 * l))) \geq \mu_p(((k * l) * l) * (0 * l)).$$

By K_1, P_2, P_5 and K_3,

$$(k * (l * (l * k))) * (k * ((l * (l * k)) * (0 * (0 * (k * l))))) \leq 0 * (k * l).$$

Moreover by P_2 and P_5,

$$0 * (((k * l) * l) * (0 * l)) = 0 * (k * l).$$

Hence, by Lemma 3.2,

$$\mu_n(k*(l*(l*k))) \leq \mu_n(k*((l*(l*k))*(0*(0*(k*l))))) \vee \mu_n(0*(((k*l)*l)*(0*l)))$$

and

$$\mu_p(k*(l*(l*k))) \geq \mu_p(k*((l*(l*k))*(0*(0*(k*l))))) \wedge \mu_p(0*(((k*l)*l)*(0*l))).$$

Now by Theorem 3.4,

$$\mu_n(k * (l * (l * k))) \leq \mu_n(((k * l) * l) * (0 * l)) \vee \mu_n(0 * (((k * l) * l) * (0 * l)))$$
$$\mu_n(k * (l * (l * k))) \leq \mu_n(((k * l) * l) * (0 * l))$$

and

$$\mu_p(k * (l * (l * k))) \geq \mu_p(((k * l) * l) * (0 * l)) \wedge \mu_p(0 * (((k * l) * l) * (0 * l)))$$
$$\mu_p(k * (l * (l * k))) \geq \mu_p(((k * l) * l) * (0 * l)).$$

Conversely, suppose that μ is a bipolar fuzzy closed ideal of K satisfying the conditions:

$$\mu_n(k * (l * (l * k))) \leq \mu_n(((k * l) * l) * (0 * l))$$

and

$$\mu_p(k * (l * (l * k))) \geq \mu_p(((k * l) * l) * (0 * l)),$$

for all $k, l \in K$.
Using K_1 and K_2, we know that

$$\begin{aligned}
&(k * ((l * (l * k)) * (0 * (0 * (k * l))))) * (k * (l * (l * k))) \\
&\leq (l * (l * k)) * ((l * (l * k)) * (0 * (0 * (k * l)))) \\
&\leq 0 * (0 * (k * l)).
\end{aligned}$$

By using Lemma 3.2,

$$\begin{aligned}
&\mu_n(k*((l*(l*k))*(0*(0*(k*l)))))\\
&\leq \mu_n(k*(l*(l*k)))\vee\mu_n(0*(0*(k*l)))\\
&\leq \mu_n(((k*l)*l)*(0*l))\vee\mu_n((0*(k*l)))(By\ given\ conditions)\\
&\leq \mu_n(((k*l)*l)*(0*l))\vee\mu_n((0*(((k*l)*l)*(0*l))))\\
&\leq \mu_n(((k*l)*l)*(0*l))
\end{aligned}$$

and

$$\begin{aligned}
&\mu_p(k*((l*(l*k))*(0*(0*(k*l)))))\\
&\geq \mu_p(k*(l*(l*k)))\wedge\mu_p(0*(0*(k*l)))\\
&\geq \mu_p(((k*l)*l)*(0*l))\wedge\mu_p((0*(k*l)))(By\ given\ conditions)\\
&\geq \mu_p(((k*l)*l)*(0*l))\wedge\mu_p((0*(((k*l)*l)*(0*l))))\\
&\leq \mu_p(((k*l)*l)*(0*l)).
\end{aligned}$$

Hence, μ is a bipolar fuzzy BCI-implicative ideal of K, by Theorem 3.4.

□

4. CONCLUSIONS

The notions of bipolar fuzzy (closed) BCI-implicative ideals are introduced. Conditions for a bipolar fuzzy (closed) ideal to be a bipolar fuzzy BCI-implicative ideal of BCI-algebras are provided. It has been shown that a bipolar fuzzy BCI-implicative ideal in BCI-algebras is a bipolar fuzzy ideal and that converse is not true.

5. ACKNOWLEDGEMENTS

The authors would like to thank the referees for their suggestions and comments.

REFERENCES

[1] K. T. Atanassov, Intuitionistic fuzzy sets, Fuzzy sets and systems, 20 (1) August (1986), 87-96. doi:10.1016/S0165-0114(86)80034-3.
[2] D. Dubois and H. Prade, Fuzzy sets and systems: Theory and Applications, Mathematics in Science and Engineering, 144, Academic Press, Inc, New York-London, 1980.
[3] Q. P. Hu and K. Iseki, On BCI-algebras satisfying (x* y)* z= x*(y* z), In Mathematics seminar notes, 8 (3) (1980), 553-555.
[4] Y. Imai and K. Iséki, On axiom systems of propositional calculi. XIV., Proc. Japan Acad., 42 (1966), 19-22. doi:10.3792/pja/1195522169.
[5] K. Iséki, An algebra related with a propositional calculus, Proc. Japan. Acad., 42 (1966), 26-29. doi:10.3792/pja/1195522171.
[6] K. Iséki, On BCI-algebras, Mathematics Seminar Notes (Kobe University), 8 (1980), 125-130.
[7] K. Iséki and S. Tanaka, An introduction to the theory of BCK-algebras, Math. Japon., 23 (1) (1978/79), 1-26.
[8] Y. B. Jun, Closed fuzzy ideals in BCI-algebras, Math. Japon., 38 (1) (1993), 199-202.
[9] Y. B. Jun, G. Muhiuddin, M. A. Ozturk, and E. H. Roh, Cubic soft ideals in BCK/BCI-algebras, Journal of Computational Analysis and Applications, 22 (5) (2017), 929-940.
[10] K. J. Lee, Bipolar fuzzy subalgebras and bipolar fuzzy ideals of BCK/BCI-algebras, Bull. Malays. Math. Sci. Soc. (2), 32 (3) (2009), 361-373.
[11] K. M. Lee, Bipolar-valued fuzzy sets and their operations, Proc. Int. Conf. on Intelligent Technologies, Bangkok, Thailand, 2000, 307-312.
[12] K. M. Lee, Comparison of interval-valued fuzzy sets, intuitionistic fuzzy sets, and bipolar-valued fuzzy sets, J. Fuzzy Logic Intelligent Systems, 14 (2004), 125-129. doi:10.5391/JKIIS.2004.14.2.125.

[13] Y. L. Liu and J. Meng, Fuzzy ideals in BCI-algebras, Fuzzy Sets and Systems, 123 (2) (2001), 227-237. doi:10.1016/S0165-0114(00)00047-6.
[14] Y. L. Liu, Y. Xu, and J. Meng, BCI-implicative ideals of BCI-algebras, Information Sciences, 177 (22) (2007), 4987-4996. doi:10.1016/j.ins.2007.07.003.
[15] J. Meng, Y. B. Jun, and H. S. Kim, Fuzzy implicative ideals of BCK-algebras, Fuzzy sets and systems, 89 (1997), 243-248. doi:10.1016/S0165-0114(96)00096-6.
[16] G. Muhiuddin, Bipolar fuzzy KU-subalgebras/ideals of KU-algebras, Annals of Fuzzy Mathematics and Informatics, 8 (3) (September 2014), 409-418.
[17] G. Muhiuddin and S. Aldhafeeri, N-Soft p-ideal of BCI-algebras, European Journal of Pure and Applied Mathematics, 12 (1) (2019), 79-87. doi:10.29020/nybg.ejpam.v12i1.3343.
[18] G. Muhiuddin and A. M. Al-roqi, Classifications of (α, β)-fuzzy ideals in BCK/BCI-algebras, J. Math. Anal., 7 (6) (2016), 75-82.
[19] G. Muhiuddin, H. Harizavi, and Y. B. Jun, Bipolar-valued fuzzy soft hyper BCK-ideals in hyper BCK-algebras, Discrete Mathematics Algorithms and Applications, 12 (2) (2020). doi:10.1142/S1793830920500184
[20] G. Muhiuddin, S. J. Kim and Y. B. Jun, Implicative N-ideals of BCK-algebras based on neutrosophic N-structures, Discrete Mathematics Algorithms and Applications, 11 (1) (2019), 17 pages. doi:10.1142/S1793830919500113.
[21] G. Muhiuddin, H. S. Kim, S. Z. Song, and Y. B. Jun, Hesitant fuzzy translations and extensions of subalgebras and ideals in BCK/BCI-algebras, Journal of Intelligent and Fuzzy Systems, 32 (1) (2017), 43-48. doi:10.3233/JIFS-151031.
[22] T. Senapati, Y. B. Jun, G. Muhiuddin, and K. P. Shum, Cubic intuitionistic structures applied to ideals of BCI-algebras, Analele Stiintifice ale Universitatii Ovidius Constanta-Seria Matematica, 27 (2) (2019), 213-232. doi:10.2478/auom-2019-0028.
[23] L. A. Zadeh, Fuzzy sets, Inf. Control, 8 (1965), 338-353. doi:10.1016/S0019-9958(65)90241-X.
[24] W. R. Zhang, Bipolar fuzzy sets, Proceedings of Fuzzy-IEEE, (1998), 835-840. doi:10.1109/FUZZY.1998.687599.
[25] W. R. Zhang, L. Zhang, and Y. Yang, bipolar logic and bipolar fuzzy logic, Inform. Sci., 165 (3-4) (2004), 265-287. doi:10.1016/j.ins.2003.05.010.
[26] H. J. Zimmermann, Fuzzy Set Theory and its Applications, Kluwer-Nijhoff Publishing, Boston, MA, 1985.

D. AL-KADI
TAIF UNIVERSITY, DEPARTMENT OF MATHEMATICS AND STATISTIC, TAIF, 21974, SAUDI ARABIA
Email address: `dak12le@hotmail.co.uk`

G. MUHIUDDIN
UNIVERSITY OF TABUK, DEPARTMENT OF MATHEMATICS, TABUK, 71491, SAUDI ARABIA
Email address: `chishtygm@gmail.com`.