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COMMON FIXED POINT RESULT FOR GENERALIZED α_*-ψ-CONTRACTION FOR C-CLASS FUNCTIONS IN b-METRIC SPACES

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ABSTRACT. In this paper, we prove common fixed point theorems for α_*-ψ-contraction in b-metric spaces, which generalizes the result of S. Aleksic et al [Remarks on common fixed point results for generalized α_*-ψ-contraction multivalued mappings in b-metric spaces ,Adv. Fixed Point Theory, 9 (2019), No. 1, 1-16] using the concept of C class function in b-in metric spaces. An example is given to support our results.

1. INTRODUCTION

The Banach contraction principle [9] is the simplest and one of the important results in fixed point theory, that is, every contractive mapping T from a complete metric space (X, d) into itself has a unique fixed point z of the mapping T ($Tz = z$). Sufficient number of authors extended and generalize the concept of a metric space as b-metric spaces, fuzzy metric spaces, Menger metric spaces, quasi metric spaces...

Metric type spaces (or b-metric spaces) is one of the important generalization of metric spaces. This concept was introduced by Bakhtin 1989 [8] and Czerwik 1993 [11].

On the other hand, Nadler [22] introduced the notion of a multi-valued contractive mapping in a complete metric space and also proved Banach's fixed point theorem for a multi-valued mapping in a complete metric space. Moreover many authors proved some fixed point theorems for single-valued and multi-valued mappings in b-metric spaces, we refer the reader to ([1], [3], [6], [7], [10], [12], [13], [14], [15], [16], [17], [19], [20], [21], [23]).

Motivated by the results and notions mentioned above. In this paper, we present new type contractions involving C-class functions and establish several common fixed point theorems for this class of mappings defined on b-metric spaces.Our main result is essentially inspired by S. Aleksic et al [2].

2. PRELIMINARIES

We appeal the following notions and preliminaries.

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Definition 2.1. Let X be a (nonempty) set and $s > 1$ be a given real number. A function $d : X \times X \to [0, 1)$ is a b-metric on X if for all $x, y, z \in X$, the following conditions hold:

(b1) $d(x, y) = 0$ if and only if $x = y$,

(b2) $d(x, y) = d(y, x)$,

(b3) $d(x, z) \leq s[d(x, y) + d(y, z)]$.

A triplet (X, d, s) is called a b-metric space.

Also, every metric space is a b-metric space but the converse is not necessarily true.

Lemma 2.1. *Let (X, d) be a b-metric space. The following properties are satisfied.*

1) $D(x, B) \leq d(x, b)$ for all $x \in X$, $b \in B$ and $B \in CB(X)$.

2) $D(x, B) \leq H(A, B)$ for all $x \in X$ and $A, B \in CB(X)$.

3) $D(x, A) \leq s(d(x, y) + D(y, B))$ for all $x, y \in X$ and $A, B \in CB(X)$.

Let (X, d) be a metric space and $\alpha\colon X \times X \to [0, +\infty)$ be a given mapping. A mapping $T\colon X \to CL(X)$ is an

(1)α_*-admissible, if $\alpha(x, y) \geq 1$ implies $\alpha_*(Tx, Ty) \geq 1$, where $\alpha_*(Tx, Ty) = \inf \{\alpha(a, b) : a \in Tx, b \in Ty\}$;

(2) α-admissible, if for each $x \in X$ and $y \in Tx$ with $\alpha(x, y) \geq 1$, we have $\alpha(y, z) \geq 1$ for all $z \in Ty$.

Definition 2.2. [4] Let $S, T : X \to CB(X)$ be two multi-valued mappings and $\alpha : X \times X \to [0, \infty)$ a function.

The pair (S, T) is said to be triangular α_* -admissible if the following conditions hold:

1) (S, T) is α_* -admissible, that is, $\alpha(x, y) \geq 1$ implies that $\alpha_*(Sx, Ty) \geq 1$ and $\alpha_*(Tx, Sy) \geq 1$; where $\alpha_*(A, B) = \inf \{a(x, y)/x \in A, y \in B\}$,

2) $\alpha(x, u) \geq 1$ and $\alpha(u, y) \geq 1$ imply $\alpha(x, y) \geq 1$:

Definition 2.3. [4] Let $S, T : X \to CB(X)$ be two multi-valued mappings and $\alpha : X \times X \to [0, \infty)$ a function. The pair (S, T) is said to be α_*-orbital admissible if the conditions $\alpha_*(x, Sx) \geq 1$ and $\alpha_*(x, Tx) \geq 1$ imply $\alpha_*(Sx, T^2x) \geq 1$ and $\alpha_*(Tx, S^2x) \geq 1$.

Definition 2.4. [4] Let $S, T : X \to CB(X)$ be two multi-valued mappings and $\alpha : X \times X \to [0, \infty)$ a function. The pair (S, T) is said to be triangular α_*orbital admissible if the following conditions are satisfied :

1) (S, T) is α_* -orbital admissible,

2) $\alpha(x, y) \geq 1, \alpha_*(y, Sy) \geq 1$ and $\alpha_*(y, Ty) \geq 1$ implies that $\alpha_*(x, Sy) \geq 1$ and $\alpha_*(x, Ty) \geq 1$

Lemma 2.2. [22] *If $A, B \in CB(X)$ and $k > 1$, then for each $a \in A$, there exists $b \in B$ such that $d(a, b) \leq kH(A, B)$.*

A. H. Ansari [5] introduced the concept of a C-class functions which covers a large class of contractive conditions

Definition 2.5. [5] A continuous function $F : [0, \infty)^2 \to \mathbb{R}$ is called C-class function if for any $s, t \in [0, \infty)^2$; the following conditions hold

$c1$ $F(s, t) \leq s$;

$c2$ $F(s, t) = s$ implies that either $s = 0$ or $t = 0$.

An extra condition on F that $F(0, 0) = 0$ could be imposed in some cases if required. The letter C will denote the class of all C- functions.

Example 2.6. The following examples shows that the class C is nonempty:
1. $F(s,t) = s - t$.
2. $F(s,t) = ms$, for some $m \in (0,1)$.
3. $F(s,t) = \frac{s}{(1+t)^r}$, for some $r \in (0,1)$.
4.$F(s,t) = \frac{\log(t+a^s)}{(1+t)}$, for some $a > 1$.

Let u denote the class of the functions $\varphi : [0,\infty) \to [0,\infty)$ which satisfy the following conditions:
a) φ is continuous ;
b) $\varphi(t) > 0, t > 0$ and $\varphi(0) \geq 0$.
In 1984, Khan et al. [18] introduced altering distance function as follows:

Definition 2.7. [18] A function $\psi : [0,\infty) \to [0,\infty)$ is called an altering distance function if the following properties are satisfied:
i) ψ is non-decreasing and continuous,
ii) $\psi(t) = 0$ if and only if $t = 0$.

Let us suppose that Ψ denote the class of the altering distance functions.

Definition 2.8. A tripled (ψ, φ, F) where $\psi \in \Psi$; $\varphi \in \Phi_u$ and $F \in C$ is said to be a monotone if for any $x, y \in [0,\infty)$;

$$x \leq y \Rightarrow F(\psi(x), \varphi(x)) \leq F(\psi(y), \varphi(y)).$$

Example 2.9. Let $F(s,t) = s - t$, $\varphi(x) = \sqrt{x}$

$$\psi(x) = \begin{cases} \sqrt{x} & if \quad 0 \leq x \leq 1 \\ x^2 & if \quad x > 1 \end{cases} ,$$

then (ψ, φ, F) is monotone.

Example 2.10. Let $F(s,t) = s - t$, $\varphi(x) = x^2$

$$\psi(x) = \begin{cases} \sqrt{x} & if \quad 0 \leq x \leq 1 \\ x^2 & if \quad x > 1 \end{cases} ,$$

then (ψ, φ, F) is monotone.

Example 2.11. Let $F(s,t) = \frac{s}{1+t}$, $\varphi(x) = \sqrt[3]{x}$

$$\psi(x) = \begin{cases} \sqrt[3]{x} & if \quad 0 \leq x \leq 1 \\ x^3 & if \quad x > 1 \end{cases} ,$$

then (ψ, φ, F) is monotone.

Example 2.12. Let $F(s,t) = \log\left(\frac{t+e^s}{1+t}\right)$, $\varphi(x) = e^x$ and $\psi(x) = x$
then (ψ, φ, F) is monotone.

Example 2.13. Let $F(s,t) = s - t$, $\varphi(x) = x^3$

$$\psi(x) = \begin{cases} \sqrt[3]{x} & if \quad 0 \leq x \leq 1 \\ x^3 & if \quad x > 1 \end{cases} ,$$

then (ψ, φ, F) is monotone.

3. Main Results

Firstly, in this section we assume ψ is altering distance function, φ is ultra altering distance function and F is a C-class function.

Theorem 3.1. *Let $(X, d, s > 1)$ be a b-metric space, $\alpha : X \times X \to [0, \infty)$ a function and $\varepsilon > 1$. Let $S, T : X \to CB(X)$ be two multi-valued mappings such that for $x, y \in X$; with $\alpha(x, y) \geq 1$;*

the pair (S, T) satisfies the inequality

$$\psi\left(s^{\varepsilon}.H(Sx, Ty)\right) \leq F\left(\psi\left(M(x, y)\right), \varphi\left(M(x, y)\right)\right) \tag{3.1}$$

where

$$M(x, y) = \max\left\{d(x, y), D(x, Sx), D(y, Ty), \frac{D(x, Ty) + D(y, Sx)}{2s}\right\}.$$

Suppose that the following conditions are satisfied.

(i) (X, d) is an a-complete b-metric space.

(ii) (S, T) is triangular α_* -orbital admissible.

(iii) There exists $x_0 \in X$ such that $\alpha_*(x_0, Sx_0) \geq 1$.

(iv) S and T are a-continuous multi-valued mappings, or, if $\{x_n\}$ a sequence in X which converges to $x \in X$ such that $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N}$, then there exists a subsequence $\{x_{n_k}\}$ satisfies $\alpha(x_{n_k}, x)$ for all $k \in \mathbb{N}$.

Then S and T have a common fixed point.

Proof. From (iii), there exists $x_1 \in Sx_0$ such that $\alpha_*(x_0, x_1) \geq 1$ and $x_1 \neq x_0$. By the inequality (3.1) and (2) in Lemma (2.1), we have

$$0 < \psi\left(s^{\varepsilon} D(x_1, Tx_1)\right) \leq \psi\left(s^{\varepsilon}.H(Sx_0, Tx_1)\right) \leq F\left(\psi\left(M(x_0, x_1)\right), \varphi\left(M(x_0, x_1)\right)\right). \tag{3.2}$$

Using Lemma (2.2) for

$$k = s^{\varepsilon},$$

there exists $x_2 \in Tx_1$ such that

$$\psi\left(d(x_1, x_2)\right) \leq \psi\left(s^{\varepsilon}.H(Sx_0, Tx_1)\right) \leq F\left(\psi\left(M(x_0, x_1)\right), \varphi\left(M(x_0, x_1)\right)\right), \tag{3.3}$$

where

$$\begin{aligned} M(x_0, x_1) &= \max\left\{d(x_0, x_1), D(x_0, Sx_0), D(x_1, Tx_1), \frac{D(x_0, Tx_1) + D(x_1, Sx_0)}{2s}\right\} \\ &= \max\left\{d(x_0, x_1), D(x_1, Tx_1), \frac{D(x_0, Tx_1)}{2s}\right\} \\ &= \max\left\{d(x_0, x_1), D(x_1, Tx_1)\right\}. \end{aligned}$$

If $M(x_0, x_1) = D(x_1, Tx_1)$, then from (3.1), we obtain

$$\begin{aligned} 0 &< \psi\left(D(x_1, Tx_1)\right) \leq \psi\left(s^{\varepsilon} D(x_1, Tx_1)\right) \leq \psi\left(s^{\varepsilon}.H(Sx_0, Tx_1)\right) \\ &\leq F\left(\psi\left(D(x_1, Tx_1)\right), \varphi\left(D(x_1, Tx_1)\right)\right) \\ &\leq \psi\left(D(x_1, Tx_1)\right). \end{aligned}$$

Since ψ is nondecreasing function, so $D(x_1, Tx_1) = 0$ which is a contradiction. Then

$$\max\left\{d(x_0, x_1), D(x_1, Tx_1)\right\} = d(x_0, x_1).$$

According to the inequality (3.1)

$$\begin{aligned} \psi\left(d(x_1, x_2)\right) &\leq \psi\left(s^{\varepsilon}.H(Sx_0, Tx_1)\right) \leq F\left(\psi\left(d(x_0, x_1)\right), \varphi\left(d(x_0, x_1)\right)\right) \\ &\leq \psi\left(d(x_0, x_1)\right). \end{aligned}$$

Since ψ is nondecreasing, we get

$$d(x_1, x_2) \leq d(x_0, x_1).$$

Similarly, for $x_2 \in Tx_1$, there exists $x_3 \in Sx_2$ such that

$$\psi(d(x_2, x_3)) \leq s^{\varepsilon} H(Tx_1, Sx_2) F\left(\psi\left(d(x_1, x_2)\right), \varphi\left(d(x_1, x_2)\right)\right) \leq \psi\left(d(x_1, x_2)\right).$$

Repeating this process, we can construct a sequence $\{x_n\} \subset X$ as follows.

$$\begin{cases} x_{2k+1} \in Sx_{2k}, \\ x_{2k+2} \in Tx_{2k+1}, \quad \text{otherwise.} \end{cases}$$

for $k = 1, 2, ...$ Since (S, T) is is triangular $\alpha_{\grave{a}st}$ orbital admissible, so from lemma 2.2 we get $\alpha(x_n, x_{n+1}) \geq 1$. Hence we have

$$\begin{aligned} \psi\left(d(x_{2k+1}, x_{2k+2})\right) &\leq \psi(s^{\varepsilon} H(Tx_{2k}, Sx_{2k+1}) \\ &\leq F\left(\psi\left(M(x_{2k}, x_{2k+1})\right), \varphi\left(M(x_{2k}, x_{2k+1})\right)\right) \end{aligned} \tag{3.4}$$

where

$$\begin{aligned} M(x_{2k}, x_{2k+1}) &= \max\left\{ \begin{array}{c} d(x_{2k}, x_{2k+1}), D(x_{2k}, Sx_{2k}), D(x_{2k+1}, Tx_{2k+1}), \\ \frac{D(x_{2k}, Tx_{2k+1}) + D(x_{2k+1}, Sx_{2k})}{2s} \end{array} \right\} \\ &= \max\left\{ d(x_{2k}, x_{2k+1}), D(x_{2k+1}, Tx_{2k+1}), \frac{D(x_{2k}, Tx_{2k+1})}{2s} \right\} \\ &= \max\left\{ d(x_{2k}, x_{2k+1}), D(x_{2k+1}, Tx_{2k+1}) \right\} \end{aligned}$$

If $M(x_{2k}, x_{2k+1}) = D(x_{2k+1}, Tx_{2k+1})$, using (3.1), we get

$$\begin{aligned} 0 &< \psi\left(s^{\varepsilon} D(x_{2k+1}, Tx_{2k+1})\right) \leq \psi\left(s^{\varepsilon}.H(Sx_{2k}, Tx_{2k+1})\right) \\ &\leq F\left(\psi\left(D(x_{2k+1}, Tx_{2k+1})\right), \varphi\left(D(x_{2k+1}, Tx_{2k+1})\right)\right) \\ &\leq \psi\left(D(x_{2k+1}, Tx_{2k+1})\right), \end{aligned}$$

which is a contradiction. Hence

$$\begin{aligned} \psi\left(d(x_{2k+1}, x_{2k+2})\right) &\leq \psi\left(s^{\varepsilon}.H(Sx_{2k}, Tx_{2k+1})\right) \leq F\left(\psi\left(d(x_{2k}, x_{2k+1})\right), \varphi\left(d(x_{2k}, x_{2k+1})\right)\right) \\ &\leq \psi\left(d(x_{2k}, x_{2k+1})\right), \end{aligned} \tag{3.5}$$

which implies

$$d(x_{2k+1}, x_{2k+2}) \leq d(x_{2k}, x_{2k+1}).$$

Thus for all $n \in \mathbb{N}$, we have

$$d(x_{n+1}, x_{n+2}) \leq d(x_n, x_{n+1}),$$

this yields $\{d(x_n, x_{n+1})\}$ is decreasing sequence.

The non increasing of ψ hold the decreasing the sequence $\{d(x_{n+1}, x_n)\}$ such that $\lim_{n\to\infty} d(x_n, x_{n+1}) = r$ Letting $n \to \infty$ in (3.5), we have

$$\begin{aligned} \lim_{n\to\infty} \psi\left(s^{1-\varepsilon} d(x_{n+2}, x_{n+1})\right) &\leq \lim_{n\to\infty} \psi\left(d(x_{n+2}, x_{n+1})\right) \\ &\leq \lim_{n\to\infty} F\left(\psi\left(s^{1-\varepsilon} d(x_n, x_{n-1})\right), \varphi\left(s^{1-\varepsilon} d(x_n, x_{n+1})\right)\right) \\ &\leq \lim_{n\to\infty} \psi\left(s^{1-\varepsilon} d(x_n, x_{n+1})\right). \end{aligned}$$

So, we conclud that

$$\psi\left(s^{1-\varepsilon} r\right) \leq \lim_{n\to\infty} F\left(\psi\left(s^{1-\varepsilon} r\right), \varphi\left(s^{1-\varepsilon} r\right)\right) \leq \lim_{n\to\infty} \psi\left(s^{1-\varepsilon} r\right).$$

For $s^{1-\varepsilon} \neq 0$, implies $r = 0$, a contradiction. Hence, we conclude

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = 0$$

Now, we prove that the sequence $\{x_n\}$ is a Cauchy sequence. Suppose that the $\{x_{2n}\}$ is not a Cauchy sequence. Then there exists an $\varepsilon > 0$ for which we can find two sequences of positive integers $\{2m(k)\}$ and $\{2n(k)\}$ such that for all positive integers k, $2n(k) > 2m(k) > k$ and $d(x_{m(k)}, x_{n(k)}) \geq \varepsilon$.

Let $2n(k)$ be the smallest such positive integer $2n(k) > 2m(k) > k$ such that

$$d(x_{2m(k)}, x_{2n(k)}) \geq \varepsilon, \qquad d(x_{2m(k)}, x_{2n(k)-1}) < \varepsilon$$

by (3.1) , we have

$$\begin{aligned} \psi\left(s^{\varepsilon-1} d(x_{2m(k)}, x_{2n(k)}))\right) &\leq F\left(\psi\left(M(x_{2n(k)-1}, x_{2m(k)})\right), \varphi\left(M(x_{2n(k)-1}, x_{2m(k)})\right)\right) \\ &= F\left(\psi\left(d(x_{2n(k)-1}, x_{2m(k)})\right), \varphi\left(d(x_{2n(k)-1}, x_{2m(k)})\right)\right) \end{aligned}$$

Letting $n \to \infty$ we have

$$\psi(\varepsilon) \leq \psi\left(s^{\varepsilon-1}\varepsilon\right) \leq F\left(\psi(\varepsilon), \varphi(\varepsilon)\right) \leq \psi(\varepsilon)$$

Then $\psi(\varepsilon) = 0$ contradiction with $\varepsilon > 0$. Thus (x_n) is a b-Cauchy sequence in X. α-completeness of (X, d) implies the existence of $z \in X$ such that

$$\lim_{n\to\infty} d(x_n, z) = 0$$

which implies $\lim_{n\to\infty} d(x_{2k+1}, z) = 0$ and $\lim_{n\to\infty} d(x_{2k+2}, z) = 0$.
If S, T are α-continuous, this gives $\lim_{n\to\infty} H(Tx_{2k+1}, z) = 0$.
Using triangular inequality e get

$$D(z, Tz) \leq s(d(x_{2k+1}, Tx_{2k}) + H(Tx_{2k+1}, Tz))$$

passing to limit we obtain $D(z, Tz) = 0$, this yields $z \in Tz$. Similarly we can show easily $z \in Tz$ and z is a common fixed point for S and T.
If for a sequence $\{x_n\}$ in X which converges to $x \in X$ such that $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N}$, then there exists a subsequence $\{x_{n_k}\}$ satisfies $\alpha(x_{n_k}, x)$ for all $k \in \mathbb{N}$.
Since $x_n \to z$, then there exists a sequence $\{x_{n_k}\}$ such that $\alpha(x_{n_k}, x)$, so we have

$$D(z, Tz) \leq s(d(x_{2k+1}, Tx_{2k}) + H(Tx_{2k+1}, Tz)).$$

This complete the proof. □

The following results are a special case of the main result

Corollary 3.2. *Let (X, d) be a b-metric space, $\alpha : X \times X \to [0, \infty)$ a function and $\varepsilon > 1$. Let $S, T : X \to CB(X)$ be two multi-valued mappings such that for $x, y \in X$ with $\alpha(x, y) \geq 1$ we have*

$$\psi(s^{\varepsilon} H(Sx, Ty)) \leq \psi(M(x, y)) - \varphi(M(x, y)), \tag{3.6}$$

where

$$M(x, y) = \max\left\{d(x, y), D(x, Sx), D(y, Ty), \frac{D(x, Ty) + D(y, Sx)}{2s}\right\}.$$

Then S and T have a common fixed point.

Corollary 3.3. *Let (X, d) be a b-metric space, $\alpha : X \times X \to [0, \infty)$ a function and $\varepsilon > 1$. Let $S, T : X \to CB(X)$ be two multi-valued mappings such that for $x, y \in X$ with $\alpha(x, y) \geq 1$ we have*

$$s^{\varepsilon} H(Sx, Ty)) \leq \phi(M(x, y)), \tag{3.7}$$

where $\phi : [0, +\infty) \to [0, +\infty)$ is a lower semi continuous function satisfying $\phi(t) = 0$ if and only if $t = 0$.

$$M(x, y) = \max\left\{d(x, y), D(x, Sx), D(y, Ty), \frac{D(x, Ty) + D(y, Sx)}{2s}\right\}.$$

Then S and T have a common fixed point.

Proof. It suffices to taking $\psi = I$ and $\phi = I - \varphi$ in Theorem 3.1. □

Example 3.1. Let $X = \{1, 2, 3\}$ and $d(x, y) = |x - y|^2$. Define $S, T \colon X \to CB(X)$ and $\alpha : X \times X \to [0, \infty)$ by

$$Sx = \begin{cases} \{2\}, & x \in \{1, 2\} \\ \{1\}, & x = 3 \end{cases}$$

,

$$Tx = \begin{cases} \{3\}, & x = 1 \\ \{2\}, & x \in \{2, 3\} \end{cases}$$

and

$$\alpha(x, y) = \begin{cases} 0, & (x, y) \in \{(3, 1)\} \\ 1, & \text{otherwise.} \end{cases}$$

We claim that S and T satisfy (3.1), by taking $s = 2$, $F(m, t) = \frac{1}{\sqrt{2}}m$, $\varepsilon = \frac{1}{2}$ and $\psi(t) = t$. For that, we need to show that

$$2^{\varepsilon} H(Sx, Ty) \leq \frac{1}{\sqrt{2}} M_s(x, y).$$

Note that $H(Tx, Ty) > 0$ and $\alpha(x, y) \geq 1$ if and only if $(x, y) \in X^2 - \{(2, 2), (1, 2), (1, 3), (2, 3), (3, 1)\}$.

(1) For $x = 1$ and $y = 1$, we have

$$H(S1, T1) = 1 \leq \frac{1}{\sqrt{2}} D(1, T1) = 2\sqrt{2}.$$

which implies

$$2^{\frac{1}{2}} H(S1, T1) \leq \frac{1}{\sqrt{2}} M(1, 1).$$

(2) For $x = 2$ and $y = 1$, we have

$$H(S2, T1) = 1 \leq \frac{1}{\sqrt{2}} D(1, T1) = 2\sqrt{2},$$

which implies

$$2^{\frac{1}{2}} H(S2, T1) \leq \frac{1}{\sqrt{2}} M(2, 1).$$

(3) For $x = 3$ and $y = 2$, we have

$$H(S3, T2) = 1 \leq \frac{1}{\sqrt{2}} D(3, S3) = 2\sqrt{2},$$

which implies

$$2^{\frac{1}{2}} H(S3, T2) \leq \frac{1}{\sqrt{2}} M(3, 2).$$

(4) For $x = 3$ and $y = 3$, we have

$$H(S3, T3) = 1 \leq \frac{1}{\sqrt{2}} D(3, S3) = 2\sqrt{2},$$

which implies

$$2^{\frac{1}{2}} H(S3, T3) \leq \frac{1}{\sqrt{2}} M(3, 3).$$

Consequently, S and T satisfy (3.1). Moreover, it is easy to see that (S, T) is triangular α_*-orbital admissible. Indeed
For $(x, y) \in \{1, 2\}$, we have

$$\alpha_*(x, Sx) = \alpha_*(x, Tx) = 1 \geq 1, and\, \alpha_*(Sx, T^2x) = \alpha_*(Tx, S^2x) = 1 \geq 1,$$

then (S, T) is α_*-orbital admissible.

For $(x, y) \in \{1, 2\}$, we have

$$\alpha(x, y) = 1, and\, \alpha_*(y, Sy) = \alpha_*(y, Ty) = 1 \geq 1.$$

and

$$\alpha_*(x, Sy) = \alpha_*(x, Ty) = 1 \geq 1.$$

Hence (S, T) is triangular α_*-orbital admissible, there exists $x_0 = 1 \in X$ such that $\alpha_*(1, S1) \geq 1$. Also, For $x_n = 1$, we have x_n converges to $1 \in X$, such that
$\forall n \geq 0, \alpha(x_n, x_{n+1}) = \alpha(1, 1) = 1 \geq 1$, and $\alpha(x_n, 1) = 1 \geq 1$, so it suffices to choose $x_{n_k} = x_n$, $\forall \varepsilon > 0$. Consequently, all conditions of Theorem 3.1 are satisfied. Then T, S have a fixed point which is 2.

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