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ON CHARACTERIZATION OF REGULAR ORDERED TERNARY SEMIHYPERGROUPS BY RELATIVE HYPERIDEALS

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ABSTRACT. In the present paper, we introduce the relative left, right, lateral, two-sided hyperideal, relative quasi-hyperideal, relative bi-hyperideal, relative sub-idempotent ordered bi-hyperideal, relative generalized quasi-hyperideal, relative generalized bi-hyperideal, relative regularity of ordered ternary semihypergroups and relative left (right, lateral) simple ordered ternary semihypergroups. We characterize relative regular ordered ternary semihypergroups through relative quasi-hyperideals and relative bi-hyperideals. We also obtain some results based on relative simple ordered ternary semihypergroups, and other results connecting these relative hyperideal-theoretic notions.

1. INTRODUCTION

In 1932, Lehmer [9] introduced the concept of ternary algebraic structure which is also called triplexes. A ternary semigroup is a nonvoid set equipped with an associative ternary multiplication. It is a well known fact that the subset $\mathbb{Z}^+$ of all positive integers of $\mathbb{Z}$ is a semigroup under multiplication. Now if we consider the subset $\mathbb{Z}^-$ of all negative integers of Z, then it is not a semigroup under multiplication. The set $\mathbb{Z}^-$ is a natural example of a ternary semigroup. In 1955, Los [14] proved that every ternary semigroup can be embedded in a semigroup. Ternary semigroups have been studied by several algebraists. In 1965, Sioson [12] started and developed the theory of ideals in ternary semigroups. He also studied regular ternary semigroup characterizing ternary semigroups in terms of quasi-ideals. In 1980, Dudek and Grozdzinska [33] studied the ideals in n-ary semigroups. In 1981, Lyapin [10] studied ternary semigroups out of binary semigroups. In 1990, Santigo [20] studied regular ternary semigroups. In 1995, Dixit and Diwan [32] studied quasi-ideals and bi-ideals in ternary semigroups. In 2008, Dutta et al. [29] studied ideals in regular ternary semigroups and in the same year, Shabir and Bano [21] studied prime bi-ideals in ternary semigroups. In 2011, Sheeja and Bala [13] studied orthodox ternary semigroups and in the same year, Kar and Maity [28] studied different types of ideals in ternary semigroups. In 2014, Dubey et al. [22] introduced the notions of generalised quasi-ideals and generalised

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bi-ideals in a ternary semigroup. They also characterised these notions in terms of minimal quasi-ideals and minimal bi-ideals in a ternary semigroup.

In 2010, Iampan [4] studied ideal extensions of ordered ternary semigroups. In 2012, the concept of ordered ternary semigroups was studied by Daddi and Pawar [31], and in the same year, Yaqoob et al. [23] studied roughness and fuzziness in ordered ternary semigroup while Changphas [30] studied quasi and bi-ideals in ordered ternary semigroup. In 2014, Lekkoksung and Jampachon [25] studied right weakly regular ordered ternary semigroups. In 2019, Pornsura and Pibaljommee [26] studied some regularity in ordered ternary semigroups.

In 1934, Marty [11] defined and studied hyperstructure theory by introducing "hyperoperation" generalizing and extending the classical "binary operation". The hyperoperation is also known by its optimistic name "hope operation" given by some author in the literatue, as this structure has the hope that traditional algebraic structures generally lacks, in the sense that in a traditional algebraic structure, the composition of two elements is an element, while in an algebraic hyperstructure, the composition of two elements is a set, giving rise to its wide scope, applicability within the domain of mathematics and in other disciplines of sciences and social sciences. In 2020, Mahboob, Khan and Davvaz[5] studied (m, n)-hyperideals in ordered semi-hypergroups.

In 2010, Davvaz and Fotea [6], [7], [8] studied binary relations on ternary semihypergroups and some basic properties of compatible relations on them. They also analyzed the ternary hypergroup associated with a binary relation. In 2013 and 2017, Naka et al. and Hila et al.[15], [16], [17] studied some properties of hyperideals in ternary semihypergroup, regularity in ternary semihypergroup and introduced the notion of generalized quasi (bi)-hyperideals in ternary semihypergroups. Farooq et al. [19] applied soft set theory to ordered ternary semihypergroups.

In 1962, the concept of T-ideal (or relative ideal) in a semigroup $S(T \subseteq S)$ was introduced by Wallace [2], [3]. In 1967, Hrmova [27] generalized this notion of relative ideals in a semigroup S by introducing (S_1, S_2)-ideal in semigroup, where $S_1, S_2 \subseteq S$. In 2019, Khan and Ali [24] introduced and studied these relative ideals in ordered semigroups. In 2020, Ali, Khan and Mahboob [18] further studied relative ideals in ordered semigroups. In 2020, the author[1] introduced and studied some relative weakly hyperideals and relative prime bi-hyperideals in ordered hypersemigroups and in involution ordered hypersemigroups. In the present article, we introduce the concepts of relative regularity, relative left(right, lateral), relative m-right, relative n-left, relative (p, q)-lateral, relative two-sided, relative ordered hyperideals, relative S-$(m, (p, q), n)$ quasi-hyperideal, S-$(m, (p, q), n)$ bi-hyperideal, (S_1, S_2, S_3)-$(m, (p, q), n)$-generalized bi-hyperideal and (S_1, S_2, S_3)-$(m, (p, q), n)$-generalized quasi-hyperideal in ordered ternary semihypergroup H, where $S_1, S_2, S_3 \subseteq H$, enriched with new results, study relative simple ordered ternary semihypergroups, and characterize relative regular ordered ternary semihypergroup in terms of relative ternary hyperideals, relative ternary quasi-hyperideal and relative ternary bi-hyperideal of ordered ternary semihypergroups.

2. Terminology and Basic Definitions

In this section, we present the basic definitions and results relating to ordered ternary semihypergroup theory which will be used in the sequel of the present article.
A ternary hyperstructure H is a nonvoid set H equipped with a ternary hyperoperation "$\circ$" on H defined as follows:

$$\circ : H \times H \times H \to \mathcal{P}^*(H) \mid (x_1, x_2, x_3) \to (x_1 \circ x_2 \circ x_3)$$

and a ternary operation " $*$ " on $\mathcal{P}^*(H)$ defined as follows:

$$* : \mathcal{P}^*(H) \times \mathcal{P}^*(H) \times \mathcal{P}^*(H) \to \mathcal{P}^*(H) \mid (X_1, X_2, X_3) \to X_1 * X_2 * X_3$$

such that

$$X_1 * X_2 * X_3 = \bigcup_{(x_1,x_2,x_3) \in X_1 \times X_2 \times X_3} (x_1 \circ x_2 \circ x_3)$$

for any $X_1, X_2, X_3 \in \mathcal{P}^*(H)$, where $\mathcal{P}^*(H)$ denotes the nonempty subsets of H. In this paper, we use infix notations $X_1 * X_2 * X_3$ and $x_1 \circ x_2 \circ x_3$ for functional notations $*(X_1, X_2, X_3)$ and $\circ(x_1, x_2, x_3)$. A ternary hyperoperation " $\circ$ " on H yields a ternary operation " $*$ " on $\mathcal{P}^*(H)$. Conversely, a ternary operation " $*$ " on $\mathcal{P}^*(H)$ gives a ternary hyperoperation " $\circ$ " on H, defined as follows: $x_1 \circ x_2 \circ x_3 = \{x_1\} * \{x_2\} * \{x_3\}$.
A ternary hyperstructure H is called a ternary semihypergroup if for all $a, b, c, d, e \in H$,

$$(a \circ b \circ c) \circ d \circ e = a \circ (b \circ c \circ d) \circ e = a \circ b \circ (c \circ d \circ e).$$

A nonempty subset S of a ternary semihypergroup H is called a ternary subsemihypergroup of H if $S * S * S \subseteq S$. Equivalently, a nonempty subset S of a ternary semihypergroup H is called a ternary subsemihypergroup of H if $s_1 \circ s_2 \circ s_3 \subseteq S$ for all $s_1, s_2, s_3 \in S$. A ternary semihypergroup H equipped with a partial order " $\leq$ " on H that is compatible with ternary semihypergroup operation " $\preceq$ " such that for all $a, b, x, y \in H$,

$$a \leq b \Rightarrow (a \circ x \circ y) \preceq (b \circ x \circ y); (x \circ a \circ y) \preceq (x \circ b \circ y); (x \circ y \circ a) \preceq (x \circ y \circ b),$$

is called an ordered ternary semihypergroups.
For a subset X of an ordered ternary semihypergroup H, we define by $(X]_H$ the subset of H as follows:

$$(X]_H = \{s \in H \mid s \leq x \text{ for some x} \in \text{X}\}.$$

If " $\leq$ " is an order relation on a semihypergroup H, we define the order relation " $\preceq$ " on $\mathcal{P}^*(H)$ as follows:

$$\preceq := \{(X, Y) \mid \forall x \in X \ \exists \, y \in Y \text{ such that x} \leq \text{y}\}.$$

Therefore, for $X, Y \in \mathcal{P}^*(H)$, we denote $X \preceq Y$ if for every $x \in X$, there exists $y \in Y$ such that $x \leq y$.

Example 2.1. Consider $H = \mathbb{Z}^-$, where $\mathbb{Z}^-$ is the set of all negative integers. Then obviously, $\mathbb{Z}^-$ becomes an ordered ternary semihypergroup under partial order relation "$\leq$" and the ternary hyperoperation $\circ : \mathbb{Z}^- \times \mathbb{Z}^- \times \mathbb{Z}^- \to \mathbb{Z}^-$ defined as follows:
$z_1 \circ_2 \circ z_3 = \{z_1 \cdot z_2 \cdot z_3\}$ for all $z_1, z_2, z_3 \in \mathbb{Z}^-$, where "$\cdot$" is an ordinary multiplication of the set $\mathbb{Z}$ of all integers.

Throughout this paper, we will denote the ordered ternary semihypergroup $(H, \circ, *, \leq)$ by H unless otherwise stated. Let H be an ordered ternary semihypergroup, $S \subseteq H$ and let X, Y, Z be nonempty subsets of S, then we easily have the following crucial properties of ternary hyperoperation " $\circ$ " on H and ternary operation " $*$ " on $\mathcal{P}^*(H)$ as follows:

(i) If $x \in X * Y * Z$, then $x \in x^{'} \circ y \circ z$ for some $x^{'} \in X, y \in Y, z \in Z$.
(ii) If $x \in X, y \in Y, z \in Z$, then $x \circ y \circ z \subseteq X * Y * Z$.

Definition 2.2. Suppose that $(H, \circ, \leq)$ is an ordered ternary semihypergroup and $S \subseteq H$. Then a nonempty subset I of H is called a left (resp., lateral, right) S-hyperideal (or respective relative hyperideal) of H if

(i) $S * S * I \subseteq I$(resp., $S * I * S \subseteq I$, and $I * S * S \subseteq I$); and
(iv) if $x \in I$ and $S \ni y \leq x$, then $y \in I$, i. e., $(I]_S = I$.

Moreover, I is called a relative two-sided hyperideal of H if I is both a relative left and a relative right hyperideal of H and a relative hyperideal of H if I is a relative left, a relative right and a relative lateral hyperideal of H. A respective relative hyperideal of an ordered ternary semihypergroup H is called proper if $I \neq H$.

Example 2.3. Consider $H = M_2(Z_0^-)$ as the ternary semihypergroup of the set of all 2×2 square matrices over Z_0^-, the set of all non-positive integers and $S \subseteq H$. Then

$$L = \{\begin{pmatrix} a & 0 \\ b & 0 \end{pmatrix} : a, b \in Z_0^-\}$$

is a left relative hyperideal of H but not a relative lateral hyperideal of H or a relative right hyperideal of H.

$$R = \{\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} : a, b \in Z_0^-\}$$

is a right relative hyperideal of H but not a relative lateral hyperideal of H or a relative left hyperideal of H.

$$I = \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in Z_0^-\}$$

is a relative hyperideal of H.

$$Q = \{\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} : a \in Z_0^-\}$$

is a relative quasi-hyperideal of H but not a relative right hyperideal of H, a relative lateral hyperideal of H or a relative right hyperideal of H.

Definition 2.4. Suppose that H is an ordered ternary semihypergroup, $S \subseteq H$ and m, n, p, q are positive integers, where $p + q$ is an even positive integer. Then

(i) A nonempty subset R of H is called an S-m-right hyperideal (or relative m-right hyperideal) of H if $R * (S * S)^m \subseteq R$; and for $b \in R$, and $s \in S$ such that $s \leq b$, then $s \in R$, i.e., $(R]_S = R$.
(ii) A nonempty subset M of H is called an S-(p, q)-lateral hyperideal (or relative (p, q)-lateral hyperideal of H if $S^p * M * S^q \cup S^p * S * M * S * S^q \subseteq M$; and for $b \in M$, and $s \in S$ such that $s \leq b$, then $s \in M$, i.e., $(M]_S = M$.
(iii) A nonempty subset L of H is called an S-n-left hyperideal(or relative n-left hyperideal) of H if $(S * S)^n * L \subseteq L$; and for $b \in L$, and $s \in S$ such that $s \leq b$, then $s \in L$, i.e., $(L]_S = L$.

Definition 2.5. Suppose that $(H, \circ, \leq)$ is an ordered ternary semihypergroup, S_1, S_2, S_3 are nonempty subsets of H, and $S = S_1 \cup S_2 \cup S_3$. A nonempty subset I of H is called a relative (S_1, S_2, S_3)-left hyperideal(resp. lateral, right) of H if $S_1 * S_1 * I \subseteq I$, or $S_2 * S_2 * I \subseteq I$, or $S_3 * S_3 * I \subseteq I$(resp., $S_1 * I * S_1 \subseteq I$, or $S_2 * I * S_2 \subseteq I$, or $S_3 * I * S_3 \subseteq I$; and $I * S_1 * S_1 \subseteq I$, or $I * S_2 * S_2 \subseteq I$, or $I * S_3 * S_3 \subseteq I$); and

$S_1 \cup S_2 \cup S_3 \ni a \leq b$ for some $b \in S \Rightarrow a \in S$. If $S_1 = \varnothing$, then, accordingly, we have the respective $(\varnothing, S_2, S_3)$- hyperideals of H, if $S_2 = \varnothing$, we have the respective $(S_1, \varnothing, S_3)$-hyperideals of H, and finally, if $S_3 = \varnothing$, we obtain the respective $(S_1, S_2, \varnothing)$- hyperideals of H.

Definition 2.6. Suppose that H is an ordered ternary semihypergroup and $S \subseteq H$. Then a nonempty subset Q of H is called relative ordered S-quasi hyperideal of H if

(i) (a) $(Q * S * S) \cap (S * Q * S \cup S * S * Q * S * S) \cap (S * S * Q) \subseteq Q$, or
(b) $(Q * S * S) \cap (S * Q * S) \cap (S * S * Q) \subseteq Q$ and $(Q * S * S) \cap (S * S * Q * S * S) \cap (S * S * Q) \subseteq Q$; and
(ii) For $q \in Q$ and $s \in S$ such that $s \leq q \Rightarrow s \in Q$, i.e., $(Q]_S = Q$.

For $s \in S$, a relative ordered quasi-hyperideal of H generated by s is denoted by $Q(s)$. It is the smallest ordered quasi-hyperideal (or relative quasi-hyperideal) of H containing s. We have

$$Q(s) = (s \cup (\{(s * S * S) \cap ((S * s * S) \cup (S * S * s * S * S)) \cap (S * S * s)\}]_S]_S.$$

Definition 2.7. Suppose that H is an ordered ternary semihypergroup, $S \subseteq H$ and m, n, p, q are positive integers greater than 0 and $p + q$ =even. Then a nonempty subset Q of H is called an S-generalized quasi-hyperideal or S-$(m, (p, q), n)$-quasi-hyperideal (or relative generalized quasi-hyperideal) of H if

(i) $Q * (S * S)^m \cap (S^p * Q * S^q \cup S^p * S * Q * S * S^q) \cap (S * S)^n * Q \subseteq Q$; and
(ii) For $q \in Q$, and $s \in S$ such that $s \leq q$, then $s \in Q$, i. e., $(Q]_S = Q$.

Definition 2.8. Suppose that H is an ordered ternary semihypergroup and $S_1, S_2, S_3 \subseteq H, S = S_1 \cup S_2 \cup S_3$. Then a nonempty subset Q of H is called relative ordered (S_1, S_2, S_3)-quasi hyperideal of H if

(i) (a) $(Q * S_2 * S_2) \cap (S_3 * Q * S_3 \cup S_3 * S_1 * Q * S_2 * S_3) \cap (S_1 * S_1 * Q) \subseteq Q$, or
(b) $(Q * S_2 * S_2) \cap (S_3 * Q * S_3) \cap (S_1 * S_1 * Q) \subseteq Q$ and $(Q * S_2 * S_2) \cap (S_3 * S_1 * Q * S_2 * S_3) \cap (S_1 * S_1 * Q) \subseteq Q$; and
(ii) For $q \in Q$ and $s \in S_1 \cup S_2 \cup S_3$ such that $s \leq q \Rightarrow s \in Q$, i.e., $(Q]_S = Q$.

Definition 2.9. Suppose that H is an ordered ternary semihypergroup and $S_1, S_2, S_3 \subseteq H, S = S_1 \cup S_2 \cup S_3$. Then a nonempty subset Q of H is called relative ordered (S_1, S_2, S_3)-$(m, (p, q), n)$-quasi hyperideal of H if

(i) (a) $Q * (S_2 * S_2)^m \cap (S_3^p * Q * S_3^q \cup S_3^p * S_1 * Q * S_2 * S_3^q) \cap (S_1 * S_1)^n * Q \subseteq Q$, or
(b) $(Q * (S_2 * S_2)^m) \cap (S_3^p * Q * S_3^q) \cap ((S_1 * S_1)^n * Q) \subseteq Q$ and $(Q * (S_2 * S_2)^m) \cap (S_3^p * S_1 * Q * S_2 * S_3^q) \cap ((S_1 * S_1)^n * Q) \subseteq Q$; and
(ii) For $q \in Q$ and $s \in S_1 \cup S_2 \cup S_3$ such that $s \leq q \Rightarrow s \in Q$, i.e., $(Q]_S = Q$.

Definition 2.10. Suppose that H is an ordered ternary semihypergroup and $S \subseteq H$. Then a nonempty subset B of H is called ordered S-bi hyperideal (or relative bi-hyperideal) of H if

(i) $B * S * B * S * B \subseteq B$; and
(ii) for $b \in B$, $s \in S$ such that $s \leq b \Rightarrow s \in B$, i.e., $(B]_S = B$.

A relative ordered bi-hyperideal of H generated by $s \in S$ is denoted by $B(s)$ which is the smallest relative ordered bi-hyperideal of H containing s. We also have $B(s) = (s \cup (s * S * s * S * s)]_S$.
For $\{s\}$, we denote $L(s)$, $R(s), M(s)$ and $I(s)$ by $L(\{s\})$, $R(\{s\})$, $M(\{s\})$ and $I(\{s\})$ respectively, and we call them the relative ordered principal left hyperideal, the relative

ordered principal right hyperideal, the relative ordered principal lateral hyperideal and relative principal ordered hyperideal of tenary semihypergroup H respectively generated by $s \in S \subseteq H$. Moreover, we have

$L(s) = \{t \in S \mid t \leq s \; or \; \{t\} \preceq (s_1 \circ s_2 \circ s) \text{ for some } s_1, s_2 \in S\} = (s \cup (S*S*s)]_S = (s]_S \cup (S*S*s]_S.$

$R(s) = \{t \in S \mid t \leq s \; or \; \{t\} \preceq (s \circ s_1 \circ s_2) \text{ for some } s_1, s_2 \in S\} = (s \cup (s*S*S)]_S = (s]_S \cup (s*S*S]_S.$

$$\begin{aligned} M(s) &= \{t \in S \mid t \leq s \; or \; \{t\} \preceq (s_1 \circ s \circ s_2) \text{ or } \{t\} \preceq (s_1 \circ s_2 \circ s \circ s_3 \circ s_4) \text{ for some } s_1, s_2, s_3, s_4 \in S\} \\ &= (s \cup (S*s*S) \cup (S*S*s*S*S)]_S \\ &= (s]_S \cup (S*s*S]_S \cup (S*S*s*S*S]_S. \end{aligned}$$

$I(s) = \{t \in S \mid t \leq s \; or \; \{t\} \preceq (s_1 \circ s_2 \circ s),$
or $\{t\} \preceq (s_2 \circ s_1 \circ s) \; or \; \{t\} \preceq (s_1 \circ s \circ s_2)$
or $\{t\} \preceq (s_1 \circ s_2 \circ s \circ s_3 \circ s_4)$ for some $s_1, s_2, s_3, s_4 \in S\}$
$= (s]_S \cup (S*S*s]_S \cup (s*S*S]_S \cup (S*s*S]_S \cup (S*S*s*S*S]_S.$

We represent by $L(X)$, $R(X)$, $M(X)$ and $I(X)$ the relative ordered left hyperideal, relative ordered right hyperideal, relative ordered lateral hyperideal and relative ordered hyperideal of H respectively, generated by a nonempty subset X of $S \subseteq H$. We have

$L(X) = (X \cup (S*S*X)]_S, R(X) = (X \cup (X*S*S)]_S, M(X) = (X \cup (S*X*S) \cup (S*S*X*S*S)]_S,$
$I(X) = (X \cup (S*S*X) \cup (X*S*S) \cup (S*X*S) \cup (S*S*X*S*S)]_S, A \subseteq S.$

Definition 2.11. Suppose that H is an ordered ternary semihypergroup and $S_1, S_2, S_3 \subseteq H$. Then a nonempty subset B of H is called ordered (S_1, S_2, S_3)-bi hyperideal (or relative bi-hyperideal) of H if

((i) $B*(S_1 \cup S_2 \cup S_3)*B*(S_1 \cup S_2 \cup S_3)*B \subseteq B$; and
(ii) for all $b \in B$, $s \in S_1 \cup S_2 \cup S_3$ such that $s \leq b \Rightarrow s \in B$, i.e., $(B]_S = B$.

Definition 2.12. Suppose that H is an ordered ternary semihypergroup, $S \subseteq H$ and m, n, p, q are positive integers greater than 0 and p and q are odd. Then a nonempty subset B of H is called an S-generalized bi-hyperideal or S-$(m, (p, q), n)$-bi-hyperideal (or relative generalized bi-hyperideal) of H if

(i) $B*(S*S)^{m-1}*S^p*B*S^q*(S*S)^{n-1}*B \subseteq B$; and
(ii) For $b \in B$, and $s \in S$ such that $s \leq b$, then $s \in B$, i.e., $(B]_S = B$.

Definition 2.13. Suppose that H is an ordered ternary semihypergroup, $S \subseteq H$ and m, n, p, q are positive integers greater than 0 and p and q are odd. Then a nonempty subset B of H is called an (S_1, S_2, S_3)-$(m, (p, q), n)$-generalized bi-hyperideal of H if

(i) $B*((S_1 \cup S_2)^{m-1} \cup S_3^p)*B*(S_3^q \cup (S_1 \cup S_2)^{n-1})*B \subseteq B$; and
(ii) For all $b \in B$, and $s \in S_1 \cup S_2 \cup S_3$ such that $s \leq b$, then $s \in B$, i.e., $(B]_S = B$.

Definition 2.14. Suppose that H is an ordered ternary semihypergroup and $S \subseteq H$. An element $s \in S$ is called relative regular if there exists $x \in S$ such that $\{a\} \preceq a \circ x \circ a$. If every element of S is relative regular, then H is called S-regular(or relative regular) ordered ternary semihypergroup.

Definition 2.15. An ordered ternary semihypergroup H is called a relative ordered left(right or lateral) simple if it has no proper relative ordered left (right or lateral) hyperideal of H.

Definition 2.16. Suppose that H is an ordered ternary semihypergroup and $S \subset H$. A relative ordered bi-hyperideal B of H is called relative sub-idempotent ordered bi-hyperideal of H if $B^3 = (B * B * B) \subseteq B$.

3. Characterization of ordered ternary semihypergroups by relative bi-hyperideals and relative quasi-hyperideals

We begin with the following:

Lemma 3.1. *Let H be an ordered ternary semihypergroup, $S \subseteq H$ and let X, Y, Z be nonempty subsets of S, then we easily have the following:*

- *(i)* $X \subseteq (X]_S$.
- *(ii) If* $X \subseteq Y \subseteq S$, *then* $(X]_S \subseteq (Y]_S$.
- *(iii)* $(X]_S * (Y]_S * (Z]_S \subseteq (X * Y * Z]_S$.
- *(iv)* $((X]_S]_S = (X]_S$.
- *(v) For any* $s \in S$,$(S * S * s]_S$, $(S * s * S]_S \cup (S * S * s * S * S]_S$ *and* $(s * S * S]_S$ *are relative ordered left, relative ordered lateral and relative ordered right hyperideal of* H *respectively.*
- *(vi) For any* $s \in S$, *the set* $(S * S * s * S * S]_S$ *is a relative ordered hyperideal of* H.
- *(vii)* $(X \cup Y \cup Z]_S = (X]_S \cup (Y]_S \cup (Z]_S$.
- *(viii)* $(X \cap Y \cap Z]_S \subseteq (X]_S \cap (Y]_S \cap (Z]_S$.

Theorem 3.2. *Suppose that H is an ordered ternary semihypergroup. Then, we have the following:*

- *(i) Nonempty intersection of any number of relative ordered quasi-hyperideals of H is a relative ordered quasi- hyperideal of H.*
- *(ii) If S is a relative ordered ternary subsemihypergroup and Q is a relative ordered quasi-hyperideal of H, then $Q \cap S$ is a relative ordered quasi-hyperideal of H.*
- *(iii) If I is a relative ordered hyperideal of H and Q is a relative ordered quasi-hyperideal of H, then $I \cap Q$ is a relative ordered quasi-hyperideal of H.*
- *(iv) Every relative ordered hyperideal of H is a relative ordered bi-hyperideal of H.*
- *(v) Every relative ordered quasi-hyperideal of H is a relative ordered bi-hyperideal of H.*
- *(vi) Nonempty intersection of any number of relative ordered bi-hyperideals of H is a relative ordered bi-hyperideal of H.*
- *(vii) If S is an ordered ternary semihypergroup of H and B is an ordered bi-hyperideal of H, then $B \cap S$ is a relative ordered bi-hyperideal of H.*
- *(viii) If S is an ordered ternary semihypergroup of H and B is a relative ordered quasi-hyperideal of H, then $Q \cap S$ is a relative ordered quasi-hyperideal of H.*

Theorem 3.3. *Suppose that B is a relative ordered (S_1, S_2, S_3)-$(m, (p, q), n)$-bi-hyperideal of an ordered ternary semihypergroup $(H, \circ, \leq)$. Let $B_1, B_2, S_1, S_2, S_3 \subseteq H$, $S = S_1 \cup S_2 \cup S_3$. Then the following hypersets*

$$(B * B_1 * B_2]_S,\ (B_1 * B * B_2]_S, \text{and } (\mathrm{B}_1 * \mathrm{B}_2 * \mathrm{B}]_\mathrm{S}$$

are relative ordered (S_1, S_2, S_3)-$(m, (p, q), n)$-bi-hyperideal of H.

Proof. *Since B is a relative (S_1, S_2, S_3)-$(m, (p, q), n)$-bi hyperideal of H, we have*

$$\begin{aligned}
& (B * B_1 * B_2]_S * ((S_1 \cup S_2)^{m-1} \cup S_3^p) * \\
& (B * B_1 * B_2]_S * (S_3^q \cup (S_1 \cup S_2)^{n-1}) * (B * B_1 * B_2]_S \\
\subseteq \; & ((B * B_1 * B_2) * ((S_1 \cup S_2)^{m-1} \cup S_3^p) * \\
& (B * B_1 * B_2) * (S_3^q \cup (S_1 \cup S_2)^{n-1}) * (B * B_1 * B_2)]_S \\
\subseteq \; & ((B * B_1 * B_2 * (S_1 \cup S_2)^{m-1} \cup S_3^p) * \\
& (B * B_1 * B_2 * S_3^q \cup (S_1 \cup S_2)^{n-1}) * B * B_1 * B_2]_S \\
\subseteq \; & ((B * (S_1 \cup S_2)^{m-1} \cup S_3^p * B * S_3^q \cup (S_1 \cup S_2)^{n-1} * B) * B_1 * B_2]_S \\
\subseteq \; & (B * B_1 * B_2]_S.
\end{aligned}$$

Suppose that $x \in (B * B_1 * B_2]_S$, and $a \in S$. Then, $\{x\} \preceq s_3 \circ s_1 \circ s_2$ for some $s_3 \circ s_1 \circ s_2 \subseteq (B * B_1 * B_2]_S$ such that $a \leq x$. Thus, we have $\{a\} \preceq s_3 \circ s_1 \circ s_2$, and it follows that $a \in (B * B_1 * B_2]_S$. So, $(B * B_1 * B_2]_S$ is a relative ordered (S_1, S_2, S_3)-$(m, (p, q), n)$-bi-hyperideal of H. Similarly, one can show that $(B_1 * B * B_2]_S$, and $(B_1 * B_2 * B]_S$ are relative ordered (S_1, S_2, S_3)-$(m, (p, q), n)$-bi- hyperideals of H.

Theorem 3.4. *Suppose that $(H, \circ, \leq)$ is an ordered ternary semihypergroup. Let B_1, B_2, B_3, $S_1, S_2, S_3 \subseteq H$, $S = S_1 \cup S_2 \cup S_3$ such that B_1, B_2, B_3 are relative ordered (S_1, S_2, S_3)-$(m, (p, q), n)$-bi hyperideals of H. Then $(B_1 * B_2 * B_3]_S$ is a relative ordered (S_1, S_2, S_3)-$(m, (p, q), n)$-bi hyperideals of H.*

Proof. *The proof easily follows from Theorem 3.3 and is, therefore, omitted.*

Theorem 3.5. *Suppose that $(H, \circ, \leq)$ is an ordered ternary semihypergroup. Let $S \subseteq H$ such that L, M, N are relative ordered S-m-right hyperideal, relative ordered S-(p, q)-lateral hyperideal, relative ordered S-n-left hyperideal of H, respectively, where m, p, q, n are positive integers and $p + q$ is an even positive integer. Then H is relative S-regular if and only if $(R * M * L]_S = R \cap M \cap L$.*

Proof. $\Rightarrow$ *Suppose that H is relative S-regular ordered semihypergroup, and L, M, N are relative ordered m-right hyperideal, relative ordered (p, q)- lateral hyperideal, relative ordered S-n-left hyperideal of H, respectively. Then, we have*

$$(R * M * L]_S \subseteq (R * (S_2 * S_3)^m]_S \subseteq (R]_S = R.$$

In a similar way, we have

$$(R*M*L]_S \subseteq (S*M*S]_S \subseteq ((S^p*M*S^q) \cup (S^p*S*M*S*S^q)]_S \subseteq (M]_S = M, \text{and}$$

$$(R * M * L]_S \subseteq ((S * S)^n * L]_S \subseteq (L]_S = L.$$

Therefore, we obtain the following Equation:

$$(R * M * L]_S \subseteq R \cap M \cap L. \tag{3.1}$$

Furthermore, suppose that

$$\{x\} \preceq (x \circ a \circ x) \preceq (x \circ a \circ x \circ a \circ x) \preceq (x \circ a \circ x \circ a \circ x \circ a \circ x) = (x \circ a \circ x) \circ (a \circ x \circ a) \circ a \subseteq (R*M*L]_S.$$

Therefore, $x \in (R * M * L]_S$. So, now we have the following Equation:

$$R \cap M \cap L \subseteq (R * M * L]_S. \tag{3.2}$$

Hence, by Equation (3.1) and Equation (3.2), we obtain the desired result as follows:

$$(R * M * L]_S = R \cap M \cap L.$$

$\Leftarrow$ Let $(R * M * L]_S = R \cap M \cap L$. For any $x \in S, R(x), M(x)$ and $L(x)$ are relative ordered S-m-right, relative ordered S-(p,q)-lateral and relative ordered S-n-left hyperideals of H, respectively. Then by the given hypothesis, we have

$$(R(x) * M(x) * L(x)]_S = R(x) \cap M(x) \cap L(x).$$

Since $x \in R(x)$, $x \in M(x)$, and $x \in L(x)$, we obtain the following:

$$x \in R(x) \cap M(x) \cap L(x) = (R(x) * M(x) * L(x)]_S.$$

Therefore, we receive the following:

$$\{x\} \preceq (x \circ s_1 \circ s_2) \circ (s_3 \circ x \circ s_4) \circ (s_5 \circ s_6 \circ x) = x \circ (s_1 \circ s_2 \circ s_3) \circ x \circ (s_4 \circ s_5 \circ s_6) \circ x = x \circ a \circ x, \text{where}$$

$$\{a\} = (s_1 \circ s_2 \circ s_3) \circ x \circ (s_4 \circ s_5 \circ s_6) \subseteq S.$$

Hence, H is relative S-regular ordered semihypergroup.

Theorem 3.6. *Suppose that $(H, \circ, \leq)$ is an ordered ternary semihypergroup. Let $S, B, S_1, S_2, S_3 \subseteq H$, $S = S_1 \cup S_2 \cup S_3$ and B be a relative (S_1, S_2, S_3)-$(m, (p,q), n)$-bi-hyperideal of H. Then*

$$B = (B * ((S_1 \cup S_2)^{m-1} \cup S_3^p) * B * (S_3^q \cup (S_1 \cup S_2)^{n-1}) * B]_S,$$

where m, p, q, n are positive integers and $p + q$ is an even positive integer.

Proof. *Since B is a relative (S_1, S_2, S_3)-$(m, (p,q), n)$-bi-hyperideal of H, therefore, it follows that*

$$B * ((S_1 \cup S_2)^{m-1} \cup S_3^p) * B * (S_3^q \cup (S_1 \cup S_2)^{n-1}) * B \subseteq B, \text{ and } (\mathrm{B}]_{\mathrm{S}} = \mathrm{B}.$$

Therefore, we have the following equation:

$$(B * ((S_1 \cup S_2)^{m-1} \cup S_3^p) * B * (S_3^q \cup (S_1 \cup S_2)^{n-1}) * B]_S \subseteq (B]_S = B. \tag{3.3}$$

Suppose that $x \in B$, as H is a relative regular, thus there exists $b \in S$ such that

$$\{x\} \preceq (x \circ b \circ x) \preceq (x \circ b \circ x \circ b \circ x) \subseteq (B * ((S_1 \cup S_2)^{m-1} \cup S_3^p) * B * (S_3^q \cup (S_1 \cup S_2)^{n-1}) * B).$$

Thus, we have

$$x \in (B * ((S_1 \cup S_2)^{m-1} \cup S_3^p) * B * (S_3^q \cup (S_1 \cup S_2)^{n-1}) * B]_S.$$

Therefore, we obtain the following equation:

$$B \subseteq (B * ((S_1 \cup S_2)^{m-1} \cup S_3^p) * B * (S_3^q \cup (S_1 \cup S_2)^{n-1}) * B]_S. \tag{3.4}$$

Hence, referring to Equation 3.3 and Equation 3.4, we obtain the desired result as follows:

$$B = (B * ((S_1 \cup S_2)^{m-1} \cup S_3^p) * B * (S_3^q \cup (S_1 \cup S_2)^{n-1}) * B]_S.$$

Theorem 3.7. *An ordered ternary semihypergroup $(H, \circ, \leq)$ is simple if and only if it has no proper relative ordered S-bi-hyperideal of H.*

Proof. *$\Rightarrow$ Suppose that $(H, \circ, \leq)$ is simple ordered semihypergroup and B is a relative ordered S-bi-hyperideal of H. As, $B \neq \varnothing$, the following hypersets:*

$$(B * (S * S)^m]_S, (S^p * B * S^q \cup S^p * S * B * S * S^q]_S, \text{and } ((\mathrm{S} * \mathrm{S})^{\mathrm{n}} * \mathrm{B}]_{\mathrm{S}}$$

are relative ordered S-m-right hyperideal, relative ordered S-(p,q)-lateral hyperideal, and relative ordered S-n-right hyperideal of H, respectively. But H is relative simple. Thus, we have

$$(B * (S * S)^m]_S = S, (S^p * B * S^q \cup S^p * S * B * S * S^q]_S = S, \text{and } ((\mathrm{S} * \mathrm{S})^{\mathrm{n}} * \mathrm{B}]_{\mathrm{S}} = \mathrm{S}.$$

Now, since B is a relative ordered S-bi-hyperideal of H, therefore, we obtain

$$\begin{aligned} S=(B*(S*S)^m]_S &= (B*(S*S)^m*(S*S)^m*(S*S)^m*B]_S \\ &= (B*(S*S)^m*B*(S*S)^m*(S*S)^m*(S*S)^m*B]_S \\ &\subseteq (B*(S*S)^m*B*(S*S)^m*B]_S \\ &\subseteq (B*S*B*S*B]_S \\ &\subseteq (B]_S=B, \end{aligned}$$

Thus, $S\subseteq B$. Therefore, $S=B$. So, B is not proper relative S-bi-hyperideal of H. Hence, H has no proper S–bi-hyperideal of H.
$\Leftarrow$ Suppose that H has no proper relative ordered S-bi-hyperideal. Suppose that L is a relative ordered S-m-left hyperideal of H. Then, by Theorem 3.2(iv), we obtain L is a relative ordered S-bi-hyperideal of H. So, $L=S$. Thus, H is relative left simple. In a similar manner, one can prove that H is a relative S-n-right simple and relative S-(p,q)-lateral simple. Hence, H is relative simple.

Theorem 3.8. *Suppose that $(H,\circ,\leq)$ is a regular ordered ternary semihypergroup, and $B,S\subseteq H$. Then the following assertions are true:*

(i) For every $s\in S$, $B(s)=(R(s)*M(s)*L(s)]_S$.
(ii) $((B*S*S]_S\cap(S*B*S]_S\cap(S*S*B]_S]_S=(B*S*S\cap S*B*S\cap S*S*B]_S$.

Proof. *(i) Suppose that $x\in B(s)$. Since H is a relative regular ordered ternary semihypergroup, there exists $b\in S$ such that $\{s\}\preceq s\circ b\circ s$. So, $x\in B(s)\Rightarrow\{x\}\preceq\{a\}\preceq s\circ b\circ s\preceq s\circ b\circ s\circ b\circ s\preceq(s\circ b\circ s)\circ(b\circ s\circ b)\circ s\subseteq(R(s)*M(s)*L(s))$. Thus, $x\in(R(s)*M(s)*L(s)]_S$. So, we have*

$$B(s)\subseteq(R(s)*M(s)*L(s)]_S. \tag{3.5}$$

Next, $d\in(R(s)*M(s)*L(s)]_S$, $\exists\, s_1,s_2,s_3,s_4,s_5,s_6\in S$ such that

$$\begin{aligned} \{d\} &\preceq (s\circ s_1\circ s_2)\circ(s_3\circ s\circ s_4)\circ(s_5\circ s_6\circ s) \\ &= s\circ(s_1\circ s_2\circ s_3)\circ s\circ(s_4\circ s_5\circ s_6)\circ s \\ &\subseteq (s*S*s*S*s]_S. \end{aligned}$$

So, $d\in(s*S*s*S*s]_S=B(s)$, we have

$$(R(s)*M(s)*L(s)]_S\subseteq B(s). \tag{3.6}$$

From Equation 3.5 and Equation 3.6, we obtain

$$B(s)=(R(s)*M(s)*L(s)]_S.$$

(ii) Suppose that $B\neq\varnothing$, $B\subseteq H$, and $x\in((B*S*S]_S\cap(S*B*S]_S\cap(S*S*B]_S]_S$. Then, there exists $s_1,s_2,s_3,s_4,s_5,s_6\in S$ and $l,m,n\in B$ such that

$$\{x\}\preceq l\circ s_1\circ s_2,\ \{x\}\preceq s_3\circ m\circ s_4),\{x\}\preceq s_5\circ s_6\circ n.$$

Since H is a relative regular ordered ternary semihypergroup, $\exists$ for $x\in S$, $z\in S$ such that

$$\{x\}\preceq(x\circ z\circ x),\ \text{and}\ \{\mathrm{x}\}\preceq(\mathrm{x\circ z\circ x})\preceq(\mathrm{x\circ z\circ x\circ z\circ x})\preceq(\mathrm{x\circ s_1\circ s_2)\circ z\circ(s_3\circ m\circ s_4)\circ z\circ(s_5\circ s_6\circ n}).$$

So,

$$\begin{aligned} &l\circ(s_1\circ s_2\circ z\circ s_3\circ m)\circ(s_4\circ z\circ s_5\circ s_6\circ n)\subseteq B*S*S, \\ &(l\circ s_1\circ s_2\circ z\circ s_3)\circ m\circ(s_4\circ z\circ s_5\circ s_6\circ n)\subseteq S*B*S, \\ &(l\circ s_1\circ s_2\circ z\circ s_3)\circ(m\circ s_4\circ z\circ s_5\circ s_6)\circ n\subseteq S*S*B. \end{aligned}$$

Thus,

$(l \circ s_1 \circ s_2) \circ z \circ (s_3 \circ m \circ s_4) \circ z \circ (s_5 \circ s_6 \circ n) \subseteq (B * S * S) \cap (S * B * S) \cap (S * S * B)$.

Hence, we have

$$x \in ((B * S * S]_S \cap (S * B * S]_S \cap (B * S * S]_S]_S.$$

Therefore,

$((B * S * S) \cap (S * B * S) \cap (B * S * S)]_S \subseteq ((B * S * S]_S \cap (S * B * S]_S \cap (B * S * S]_S]_S$.

Obviously, we have

$((B * S * S]_S \cap (S * B * S]_S \cap (B * S * S]_S]_S \subseteq ((B * S * S) \cap (S * B * S) \cap (B * S * S)]_S$.

Hence, we obtain

$((B * S * S]_S \cap (S * B * S]_S \cap (B * S * S]_S]_S = ((B * S * S) \cap (S * B * S) \cap (B * S * S)]_S$.

Theorem 3.9. *Suppose that $(H, \circ, \leq)$ is a regular ordered ternary semihypergroup, and $S \subseteq H$. Then, every relative S- bi hyperideal B of H is a relative S-quasi hyperideal of H, and conversely.*

Proof. $\Rightarrow$ *One way is straightforward to prove as if B is a relative S-quasi hyperideal of H, then by Theorem 3.2 (v), we have B is a relative S- bi hyperideal of H.*
$\Leftarrow$ *Now, we proceed to prove the other way as follows. Suppose that B is a relative S-bi hyperideal of H. Then by Theorem 3.5, a relative ordered ternary semihypergroup H is relative regular if and only if for every relative ordered right hyperideal R of H, every relative ordered lateral hyperideal M of H, and every relative ordered left hyperideal L of H, we have $(R * M * L]_S = R \cap M \cap L$. Next, we obtain*

$$\begin{aligned}(B * S * S) \cap (S * B * S) \cap (S * S * B) &= ((B * S * S) * (S * B * S) * (S * S * B)]_S \\ &= (B * (S * S * S) * B * (S * S * S) * B]_S \\ &\subseteq (B * S * B * S * B]_S \\ &\subseteq (B]_S = B,\end{aligned}$$

and

$$\begin{aligned}(B * S * S) \cap (S * S * B * S * S) \cap (S * S * B) &= ((B * S * S) * (S * S * B * S * S) * (S * S * B)]_S \\ &= (B * (S * S * S) * (S * B * S) * (S * S * S) * B]_S \\ &\subseteq (B * S * S * B * S * S * B]_S \\ &\subseteq (B]_S = B.\end{aligned}$$

Hence, B is a relative ordered S-quasi hyperideal of H.

Corollary 3.10. *In a regular ordered ternary semihypergroup $(H, \circ, \leq)$, a nonempty subset B of H is a relative S-bi hyperideal of H if and only if $B = (R * M * L]_S = R \cap M \cap L$, where $S \subseteq H$ for a relative ordered right hyperideal R of H, relative ordered lateral hyperideal M of H and relative ordered left hyperideal L of H.*

Proof. *It is consequence of Theorem 3.5 and Theorem 3.9.*

Theorem 3.11. *Suppose that $(H, \circ, \leq)$ is an ordered ternary semihypergroup, and $S \subseteq H$. If H is both relative ordered left and relative ordered right S-simple, then H is relative S-regular.*

Proof. *Suppose that $s \in S$. Then $(S * S * s]_S$ and $(s * S * S]_S$ are relative left and relative right hyperideals of H respectively. But H is relative left S-imple and relative* *right S-simple, thus it follows that $(S*S*s]_S = S$ and $(s*S*S]_S = S$. We have*

$$\begin{aligned} s \in (s*S*S]_S &= (s*S*(S*S*s)]_S \\ &= (s*(S*S*S)*s]_S \\ &\subseteq (s*S*s]_S. \end{aligned}$$

Hence, H is relative S-regular.

Theorem 3.12. *Suppose that $(H, \circ, \leq)$ is an ordered regular ternary semihypergroup and B is a relative ordered S-$(m,(p,q),n)$-bi hyperideal of H. Then B is a sub-idempotent S-$(m,(p,q),n)$-bi-hyperideal of H if and only if H is relative regular, where m,p,q,n are positive integers and $p+q$ is an even positive integer.*

Proof. *Suppose that B is a relative ordered S-$(m,(p,q),n)$-bi hyperideal of H. Then we have*

$$B*(S*S)^{m-1}*S^p*B*S^q*(S*S)^{n-1}*B \subseteq B,$$

and

$$(B*(S*S)^{m-1}*S^p*B*S^q*(S*S)^{n-1}*B]_S \subseteq (B]_S = B.$$

As, H is relative S-$(m,(p,q),n)$ regular, we have

$$\begin{aligned} B &\subseteq (B*S*B]_S \\ &\subseteq (B*(S*S)^{m-1}*S^p]_S*(B*S^q*(S*S)^{n-1}*B]_S \\ &= (B*(S*S)^{m-1}*S^p*B*S^q*(S*S)^{n-1}*B]_S. \end{aligned}$$

Therefore, we have

$$\begin{aligned} B^3 &= (B*B*B]_S \\ &\subseteq ((B*(S*S)^{m-1}*S^p*B*S^q*(S*S)^{n-1}*B]_S)* \\ &\quad ((B*(S*S)^{m-1}*S^p*B*S^q*(S*S)^{n-1}*B]_S)* \\ &\quad ((B*(S*S)^{m-1}*S^p*B*S^q*(S*S)^{n-1}*B]_S)* \\ &\subseteq ((B*(S*S)^{m-1}*S^p*B*S^q*(S*S)^{n-1}*B)* \\ &\quad (B*(B*(S*S)^{m-1}*S^p*B*S^q*(S*S)^{n-1}*B)* \\ &\quad (B*(S*S)^{m-1}*S^p*B*S^q*(S*S)^{n-1}*B)]_S \\ &= (B*((S*S)^{m-1}*S^p*B*S^q*(S*S)^{n-1})*B* \\ &\quad (B*(S*S)^{m-1}*S^p*B*S^q*(S*S)^{n-1}*B)* \\ &\quad B*((S*S)^{m-1}*S^p*B*S^q*(S*S)^{n-1})*B]_S \\ &\subseteq (B*((S*S)^{m-1}*S^p*S^q*(S*S)^{n-1})* \\ &\quad (S*S^{m-1}*S^p)*(B*B*B)*(S^q*(S*S)^{n-1} \\ &\quad *(S*S)^{m-1}*S^p*S^q*(S*S)^{n-1})*B]_S \\ &\subseteq ((B*S*B)*(B*B*S)*B]_S \\ &\subseteq ((B*S*B)*(S*S*S)*B]_S \\ &\subseteq (B*S*B*S*B]_S \\ &\subseteq (B]_S = B. \end{aligned}$$

Theorem 3.13. *Suppose that $(H, \circ, \leq)$ is an ordered ternary semihypergroup and B is a relative ordered S-$(m,(p,q),n)$-bi hyperideal of H, M is relative ordered S-(p,q)-lateral hyperideal of H, and L is relative ordered S-m left hyperideal of H, where m,p,q,n are positive integers and $p+q$ is an even positive integer and $S \subseteq H$. Then H is relative*

regular if and only if $B \cap M \cap L \subseteq (B * M * L]_S$.
Proof.$\Rightarrow$ *Suppose that H is relative regular, then as B is a relative ordered S-$(m,(p,q),n)$-bi hyperideal of H, M is relative ordered S-(p,q)-lateral hyperideal of H, and L is relative ordered S-m-left hyperideal of H for any $s \in B \cap M \cap L$, there exists $a \in S$ such that*

$$\begin{aligned} \{s\} &\preceq (s \circ a \circ s) \\ &\preceq (s \circ a \circ s) \circ s \circ a \\ &\preceq (s \circ a \circ s \circ a \circ s) \circ a \circ (s \circ a \circ s) \\ &\preceq (s \circ a \circ s \circ a \circ s) \circ (a \circ s \circ a) \circ s \\ &\subseteq (B * (S * S)^{m-1} * S^p * B * S^q * (S * S)^{n-1} * B) * ((S * M * S) * L) \\ &\subseteq B * M * L. \end{aligned}$$

Therefore, $s \in (B * M * L]_S$. Hence, $B \cap M \cap L \subseteq (B * M * L]_S$.
$\Leftarrow$ *Suppose that $B \cap M \cap L \subseteq (B * M * L]_S$ for every relative ordered S-$(m,(p,q),n)$-bi hyperideal B of H, relative ordered S-(p,q)-lateral hyperideal M of H, and relative ordered S-m-left hyperideal L of H. Let $s \in S$. Take relative ordered bi-hyperideal $B(s)$, relative ordered lateral hyperideal $M(s)$ and relative ordered left hyperideal $L(s)$ of H generated by s. Then we obtain,*

$$\begin{aligned} \{s\} &\subseteq B(s) \cap M(s) \cap L(s) \\ &\subseteq (B(s) * M(s) * L(s)]_S \\ &\subseteq (s \cup s * S * s * S * s]_S * S * (s \cup S * S * s]_S \\ &\subseteq (s * S * s]_S \cup (s * S * S * S * s]_S \\ &\quad \cup (s * S * s * S * s * S * s]_S \cup (s * S * s * S * s * S * S * S * s]_S \\ &\subseteq (s * S * s]_S. \end{aligned}$$

Hence, H is relative regular.

Theorem 3.14. *For an ordered ternary semihypergroup $(H, \circ, \leq)$, the following assertions are equivalent:*

- *(i) H is relative regular.*
- *(ii) $B \cap I \cap L \subseteq (B * I * L]_S$ for a relative ordered bi-hyperideal B, a relative ordered hyperideal I and a relative ordered left hyperideal L of H.*
- *(iii) $R \cap M \cap B = (R * M * B]_S$ for a relative ordered bi-hyperideal B, a relative ordered lateral hyperideal M and a relative ordered right hyperideal R of H.*
- *(iv) $R \cap I \cap B = (R * I * B]_S$ for a relative ordered bi-hyperideal B, a relative ordered hyperideal I and a relative ordered right hyperideal R of H.*
- *(v) $B \cap I \cap Q \subseteq (B * I * Q]_S$ for a relative ordered bi-hyperideal B, a relative ordered hyperideal I and a relative ordered quasi-hyperideal Q of H.*
- *(vi) $R \cap M \cap L \subseteq (R * M * L]_S$ for a relative ordered right hyperideal R, a relative ordered lateral hyperideal M and a relative ordered left hyperideal L of H.*

Proof. *Similar to Theorem 3.13.*

Theorem 3.15. *Suppose that H is a relative ordered ternary subsemihypergroup and $S \subseteq H$. Then H is relative S-regular if and only if $s \in (s * S * s]_S$ for all $s \in S$.*
Proof. $\Rightarrow$ *Let H be relative S-regular. Then for $s \in S$, $\exists\, x \in S$ such that $\{s\} \preceq s \circ x \circ s$. As, $s \circ x \circ s \subseteq s * S * s$, hence, we obtain $s \in (s * S * s]_S$.*

$\Leftarrow$ *Suppose that* $s \in (s*S*s]_S$ *for* $s \in S$. *Since,* $\{s\} \preceq s \circ x \circ s$ *for some* $x \in S$, *therefore, it is obtained that* s *is relative regular. Hence,* H *is relative* S*-regular.*

Theorem 3.16. *Suppose that* H *is an ordered ternary semihypergroup. If all the relative ordered* S*-bi-hyperideals of* H *are idempotent, then* H *is relative regular.*

Proof. *Suppose that* R, M *and* L *are a relative ordered* S*-right, relative ordered* S*-lateral and relative ordered* S*-left hyperideal of* H *respectively. Then by Theorem 3.2(iv),* $R \cap M \cap L$ *is a relative ordered bi-hyperideal of* H. *But by the assumption, we have*

$$R \cap M \cap L = (R \cap M \cap L)^3 = (R \cap M \cap L) * (R \cap M \cap L) * (R \cap M \cap L) \subseteq R * M * L.$$

Since $R * M * L \subseteq R \cap M \cap L$. *Therefore,* $R * M * L = R \cap M \cap L$. *Hence, by Theorem 3.5,* H *is relative* S*-regular.*

4. CONCLUSION

In this paper, various notions based on the relative left, right, lateral, two-sided hyperideal, relative quasi-hyperideal, relative bi-hyperideal, relative sub-idempotent ordered bi-hyperideal, relative generalized quasi-hyperideal, relative generalized bi-hyperideal, relative regularity of ordered ternary semihypergroups and relative left (right, lateral) simple ordered ternary semihypergroups have been defined and studied. The results based on relative regular ordered ternary semihypergroups by relative quasi-hyperideals and relative bi-hyperideals have been obtained. Some results based on relative simple ordered ternary semihypergroups, and other results on different types of relative hyperideals have also been studied.

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