ANNALS OF COMMUNICATIONS IN MATHEMATICS
Volume 3, Number 4 (2020), 309-316

DIRECT PRODUCT OF ANTI $\mathcal{N}$-H-IDEALS IN BCK-ALGEBRAS

ANAS AL-MASARWAH

ABSTRACT. In this paper, we introduce the notion of anti $\mathcal{N}$-H-ideals of a BCK-algebra. Then, the notion of the direct product of two anti $\mathcal{N}$-H-ideals by using minimum operation is introduced, and some related properties are studied. Ordinary H-ideals are linked with anti $\mathcal{N}$-H-ideals by means of an anti $\mathcal{N}$-s-level set of the direct product of two $\mathcal{N}$-structures.

1. INTRODUCTION

The study of BCK-algebras was introduced by Imai and Iséki [12] in 1966. BCK-algebras have been applied to many branches of mathematics, such as functional analysis, group theory, topology, probability theory. Since Imai and Iséki [12] introduced the concepts of ideals in BCK-algebras, many types of ideals in BCK-algebras have occurred, for instance, H-ideals, closed ideals, implicative ideals, positive implicative ideals, and so on.

A crisp set C in a universe X is a function $\lambda_C : X \to \{0, 1\}$ yielding the value 0 for elements excluded from the set C and the value 1 for elements belonging to the set C. As a generalization of crisp sets, Zadeh [20] introduced the degree of positive membership in 1965 and defined the concept of fuzzy set theory. This concept was applied to a BCK-algebra by Xi [19]. Jun et al. [14] presented a new function which is called negative-valued function, and developed $\mathcal{N}$-structure as one of the hybrid models of fuzzy set. They applied the idea of $\mathcal{N}$-structure in BCK-algebras and proposed $\mathcal{N}$-subalgebras and $\mathcal{N}$-ideals [14]. In [13], Jun established the definition of doubt fuzzy subalgebras and ideals in BCK-algebras. Al-Masarwah et al. [10] introduced the notions of doubt $\mathcal{N}$-subalgebras and ideals in BCK-algebras, and discussed several properties. After that, many Hybrid models of fuzzy sets were applied in BCK-algebras and other algebraic structures [5, 6, 8, 7, 9, 4, 17, 18, 2, 1, 3, 16, 15].

In this paper, we discuss an $\mathcal{N}$-structure with an application to BCK-algebras. We introduce the notion of anti $\mathcal{N}$-H-ideals in a BCK-algebra. Also, we considered the structure of a BCK-algebra and defined the direct product of two anti $\mathcal{N}$-H-ideals. We present some interesting results about direct product of two anti $\mathcal{N}$-H-ideals of a BCK-algebra.

2010 *Mathematics Subject Classification.* 06F35, 03G25, 03B52, 03B05.
Key words and phrases. BCK-algebra; H-ideal; $\mathcal{N}$-structure; Anti $\mathcal{N}$-ideal; Anti $\mathcal{N}$-H-ideal.
Received: November 30, 2020. Accepted: December 24, 2020. Published: December 31, 2020.

Finally, we proved that the direct product of two anti $\mathcal{N}$-structures becomes an anti $\mathcal{N}$-H-ideal if and only if for any $s \in [-1, 0]$, an anti $\mathcal{N}$-s-level set is an H-ideal of a BCK-algebra $X \times Y$.

2. Preliminaries

In this section, we include some basic definitions and preliminary facts about a BCK-algebra which are essential for our results. By a BCK-algebra, we mean an algebra $(X; *, 0)$ of type (2, 0) satisfying the following axioms for all $x, y, z \in X$:
(I) $((x * y) * (x * z)) * (z * y) = 0$,
(II) $(x * (x * y)) * y = 0$,
(III) $x * x = 0$,
(IV) $0 * x = 0$,
(V) $x * y = 0$ and $y * x = 0$ imply $x = y$.

Any BCK-algebra X satisfies the following axioms for all $x, y, z \in X$:
(I1) $x * 0 = x$,
(I2) $(x * y) * z = (x * z) * y$,
(I3) $x * y \leq x$,
(I4) $(x * y) * z \leq (x * z) * (y * z)$,
(I5) $x \leq y \Rightarrow x * z \leq y * z, z * y \leq z * x$.

A partial ordering $\leq$ on a BCK-algebra X can be defined by $x \leq y$ if and only if $x * y = 0$. A non-empty subset K of a BCK/BCI-algebra X is called:

(1) A subalgebra of a BCK-algebra X [12] if $x * y \in K, \forall x, y \in X$,
(2) An ideal of a BCK-algebra X [12] if $\forall x, y \in X$,
   - $0 \in K$,
   - $x * y \in K$ and $y \in K$ imply $x \in K$.
(3) An H-ideal of a BCK-algebra X [11] if $\forall x, y, z \in X$,
   - $0 \in K$,
   - $((x * y) * z) \in K$ and $y \in K$ imply $x * z \in K$.

Definition 2.1. [21] A fuzzy set $A = \{(x, \mu_A(x)) \mid x \in X\}$ in a BCK-algebra X is called an anti (a doubt) fuzzy H-ideal of X if

(1) $\mu_A(0) \leq \mu_A(x)$,
(2) $\mu_A(x * z) \leq \max\{\mu_A(x * (y * z)), \mu_A(y)\}$, for all $x, y, z \in X$.

Denote by $\mathcal{F}(X, [-1, 0])$ the collection of functions from a set X to the interval $[-1, 0]$. We say that, an element of $\mathcal{F}(X, [-1, 0])$ is a negative-valued function from X to [-1, 0] (briefly, $\mathcal{N}$-function on X). By an $\mathcal{N}$-structure we mean an ordered pair (X, ϕ), where φ is an $\mathcal{N}$-function on X. In what follows, φ is an $\mathcal{N}$-function on X unless otherwise specified.

In [14], Jun et al. introduced the concepts of $\mathcal{N}$-subalgebras and $\mathcal{N}$-ideals in a BCK-algebra as follows:

Definition 2.2. An $\mathcal{N}$-structure (X, φ) is called an $\mathcal{N}$-subalgebra of X if for all $x, y \in X$:

$$\varphi(x * y) \leq \max\{\varphi(x), \varphi(y)\}.$$

Definition 2.3. An $\mathcal{N}$-structure (X, φ) is called an $\mathcal{N}$-ideal of X if for all $x, y \in X$:

(1) $\varphi(0) \leq \varphi(x)$,
(2) $\varphi(x) \leq \max\{\varphi(x * y), \varphi(y)\}$.

3. Direct product of anti $\mathcal{N}$-H-Ideals

In this section, we introduce the concept of an anti $\mathcal{N}$-H-ideal. Then, we give the definition of the direct product of two $\mathcal{N}$-H-ideals of two BCK-algebras X and Y, and we provide some of its properties.

In what follows, X and Y are BCK-algebras, so we use $(X \times Y; *, (0,0))$ to denote a BCK-algebra unless otherwise specified. For the sake of brevity, we call $X \times Y$ a BCK-algebra.

Definition 3.1. An $\mathcal{N}$-structure (X, φ) is called an anti $\mathcal{N}$-H-ideal of X if it satisfies the following conditions for all $x, y, z \in X$:

(1) $\varphi(0) \geq \varphi(x)$,
(2) $\varphi(x * z) \geq \min\{\varphi(x * (y * z)), \varphi(y)\}$.

Example 3.2. Let $X = \{0, a, b, c\}$ be a BCK-algebra with the following Cayley table:

$*$	0	a	b	c
0	0	0	0	0
a	a	0	0	a
b	b	a	0	b
c	c	c	c	0

Let (X, φ) be an $\mathcal{N}$-structure in which φ is given by:

$$\varphi(x) = \begin{cases} -0.1, & \text{if } x = 0 \\ -0.2, & \text{if } x = a,b,c. \end{cases}$$

Then by routine calculation, we know that (X, φ) is an anti $\mathcal{N}$-H-ideal of X.

Definition 3.3. Let $(X, *_X, 0_X)$ and $(Y, *_Y, 0_Y)$ be two BCK-algebras. The direct product of X and Y is defined to be the set $X \times Y = \{(x, y) \mid x \in X, y \in Y\}$. In $X \times Y$ we define the product $*_{X\times Y}$ as follows:

$$(x, y) *_{X\times Y} (u, v) = (x *_X u, y *_Y v) \text{ for all } (x, y), (u, v) \in X \times Y.$$

One can easily verify that the direct product of two BCK-algebras is again a BCK-algebra. Now, we write the following definition.

Definition 3.4. Let X be a BCK-algebra and let (X, φ) and (X, ϕ) be two anti $\mathcal{N}$-H-ideals of X. The direct product of (X, φ) and (X, ϕ) is defined by $(X \times X, \varphi \times \phi)$, where $\varphi \times \phi : X \times X \to [-1, 0]$ is given by

$$(\varphi \times \phi)(x, y) = \min\{\varphi(x), \phi(y)\}$$

for all $(x, y) \in X \times X$.

In the following, we extend the above definition to the direct product of anti $\mathcal{N}$-H-ideals of any BCK-algebras X and Y.

Definition 3.5. Let X and Y be two BCK-algebras and let (X, φ) and (Y, ϕ) be two anti $\mathcal{N}$-H-ideals of X and Y, respectively. Then, the direct product of (X, φ) and (Y, ϕ) is defined by $(X \times Y, \varphi \times \phi)$, where $\varphi \times \phi : X \times Y \to [-1, 0]$ is given by

$$(\varphi \times \phi)(x, y) = \min\{\varphi(x), \phi(y)\}$$

for all $(x, y) \in X \times Y$.

Definition 3.6. An $\mathcal{N}$-structure $(X \times Y, \varphi \times \phi)$ of a BCK-algebra $X \times Y$ is called an anti $\mathcal{N}$-H-ideal of $X \times Y$ if it satisfies the following conditions for all $(x, y), (u, v), (w, z) \in X \times Y$:

(1) $(\varphi \times \phi)(0,0) \geq (\varphi \times \phi)(x,y)$,
(2) $(\varphi \times \phi)((x,y)*(w,z)) \geq \min\{(\varphi \times \phi)((x,y)*((u,v)*(w,z))), (\varphi \times \phi)(u,v)\}$.

Example 3.7. Consider a BCK-algebra $X = \{0,a,b,c\}$ and an anti $\mathcal{N}$-H-ideal (X,φ) of X which are given in Example 3.2. Define an anti $\mathcal{N}$-H-ideal (X,ϕ) in X as follows:

$$\phi(x) = \begin{cases} -0.3, & \text{if } x = 0 \\ -0.4, & \text{if } x = a,b,c. \end{cases}$$

Consider $(X \times X, \varphi \times \phi)$, where $(\varphi \times \phi)(x,y) = \min\{\varphi(x), \phi(y)\}$ is defined as:

$$(\varphi \times \phi)(x,y) = \begin{cases} -0.3, \text{ if } & (x,y) = (0,0), (a,0), (b,0), (c,0) \\ -0.4, & \text{otherwise.} \end{cases}$$

By routine calculations, we know that $(X \times X, \varphi \times \phi)$ is an anti $\mathcal{N}$-H-ideal of $X \times X$.

Theorem 3.1. *Let (X,φ) and (Y,ϕ) be two anti $\mathcal{N}$-H-ideals of BCK-algebras X and Y, respectively. Then, $(X \times Y, \varphi \times \phi)$ is an anti $\mathcal{N}$-H-ideal of $X \times Y$.*

Proof. For any $(x,y) \in X \times Y$, we have

$$(\varphi \times \phi)(0,0) = \min\{\varphi(0), \phi(0)\} \geq \min\{\varphi(x), \phi(y)\} = (\varphi \times \phi)(x,y).$$

Now, for any $(x,y),(u,v),(w,z) \in X \times Y$, we have

$$\begin{aligned}(\varphi \times \phi)((x,y)*(w,z)) &= (\varphi \times \phi)(x*w, y*z) \\ &= \min\{\varphi(x*w), \phi(y*z)\} \\ &\geq \min\{\min\{\varphi(x*(u*w)), \varphi(u)\}, \min\{\phi(y*(v*z)), \phi(v)\}\} \\ &= \min\{\min\{\varphi(x*(u*w)), \phi(y*(v*z))\}, \min\{\varphi(u), \phi(v)\}\} \\ &= \min\{(\varphi \times \phi)((x*(u*w)),(y*(v*z))), (\varphi \times \phi)(u,v)\} \\ &= \min\{(\varphi \times \phi)((x,y)*((u,v)*(w,z))), (\varphi \times \phi)(u,v)\}.\end{aligned}$$

Hence, $(X \times Y, \varphi \times \phi)$ is an anti $\mathcal{N}$-H-ideal of $X \times Y$. □

Proposition 3.2. *Let $(X \times Y, \varphi \times \phi)$ be an anti $\mathcal{N}$-H-ideal of $X \times Y$. If $(x,y) \leq (u,v)$, then $(\varphi \times \phi)(x,y) \geq (\varphi \times \phi)(u,v)$ for all $(x,y),(u,v) \in X \times Y$.*

Proof. Let $(x,y),(u,v) \in X \times Y$ such that $(x,y) \leq (u,v)$. Then, $(x,y)*(u,v) = (0,0)$. Now,

$$\begin{aligned}(\varphi \times \phi)(x,y) &= (\varphi \times \phi)((x,y)*(0,0)) \\ &\geq \min\{(\varphi \times \phi)((x,y)*((u,v)*(0,0))), (\varphi \times \phi)(u,v)\} \\ &= \min\{(\varphi \times \phi)((x,y)*(u,v)), (\varphi \times \phi)(u,v)\} \\ &= \min\{(\varphi \times \phi)(0,0), (\varphi \times \phi)(u,v)\} = (\varphi \times \phi)(u,v).\end{aligned}$$

Therefore, $(\varphi \times \phi)(x,y) \geq (\varphi \times \phi)(u,v)$ for all $(x,y),(u,v) \in X \times Y$. □

Proposition 3.3. *Let $(X \times Y, \varphi \times \phi)$ be an anti $\mathcal{N}$-H-ideal of $X \times Y$ such that*

$$(\varphi \times \phi)((x,y)*(u,v)) \geq (\varphi \times \phi)(u,v)$$

for all $(x,y),(u,v) \in X \times Y$. Then, $(X \times Y, \varphi \times \phi)$ is an $\mathcal{N}$-constant.

Proof. Note that in a BCK-algebra $X \times Y$, $(x,y)*(0,0) = (x,y)$ for all $(x,y) \in X \times Y$, and by using the assumption, we have

$$(\varphi \times \phi)(x,y) = (\varphi \times \phi)((x,y)*(0,0)) \geq (\varphi \times \phi)(0,0).$$

It follows from Definition 3.6, $(\varphi\times\phi)(x,y) = (\varphi\times\phi)(0,0)$ for all $(x,y),(u,v)\in X\times Y$. Therefore, $(X\times Y,\varphi\times\phi)$ is an $\mathcal{N}$-constant. □

Proposition 3.4. *Let (X,φ) and (Y,ϕ) be two anti $\mathcal{N}$-H-ideals of X and Y, respectively. If $(X\times Y,\varphi\times\phi)$ is an anti $\mathcal{N}$-H-ideal of $X\times Y$, then $\varphi(0)\geq\phi(y)$ and $\phi(0)\geq\varphi(x)$ for all $x\in X, y\in Y$.*

Proof. Assume that $\varphi(0)<\phi(y)$ and $\phi(0)<\varphi(x)$ for some $x\in X, y\in Y$. Then,

$$\begin{aligned}(\varphi\times\phi)(x,y) &= \min\{\varphi(x),\phi(y)\}\\ &> \min\{\varphi(0),\phi(0)\}\\ &= (\varphi\times\phi)(0,0),\end{aligned}$$

which is a contradiction. Thus, $\varphi(0)\geq\phi(y)$ and $\phi(0)\geq\varphi(x)$ for all $x\in X, y\in Y$. □

Theorem 3.5. *Let (X,φ) and (Y,ϕ) be two $\mathcal{N}$-structures of X and Y, respectively, such that $(X\times Y,\varphi\times\phi)$ is an anti $\mathcal{N}$-H-ideal of $X\times Y$. Then, either (X,φ) is an anti $\mathcal{N}$-H-ideal of X or (Y,ϕ) is an anti $\mathcal{N}$-H-ideal of Y.*

Proof. Since $(X\times Y,\varphi\times\phi)$ is an anti $\mathcal{N}$-H-ideal of $X\times Y$, then for all $(x,y),(u,v),(w,z)\in X\times Y$, we have

$$(\varphi\times\phi)((x,y)*(w,z))\geq\min\{(\varphi\times\phi)((x,y)*((u,v)*(w,z))),(\varphi\times\phi)(u,v)\}.$$

By putting $y=z=v=0$, we have

$$(\varphi\times\phi)((x,0)*(w,0))\geq\min\{(\varphi\times\phi)((x,0)*((u,0)*(w,0))),(\varphi\times\phi)(u,0)\}. \quad (3.1)$$

Also, we have

$$\begin{aligned}(\varphi\times\phi)((x,0)*(w,0)) &= (\varphi\times\phi)((x*w),(0*0))\\ &= \min\{\varphi(x*w),\phi(0*0)\}\\ &= \varphi(x*w) \qquad (3.2)\end{aligned}$$

and

$$\begin{aligned}(\varphi\times\phi)((x,0)*((u,0)*(w,0))) &= (\varphi\times\phi)((x,0)*((u*w),(0*0)))\\ &= (\varphi\times\phi)((x*(u*w)),(0*(0,0)))\\ &= \min\{\varphi(x*(u*w)),\phi(0*(0,0))\}\\ &= \varphi(x*(u*w)). \qquad (3.3)\end{aligned}$$

Again, by using Proposition 3.4, we have

$$(\varphi\times\phi)(u,0) = \min\{\varphi(u),\phi(0)\} = \varphi(u). \quad (3.4)$$

So, from (3.1), (3.2), (3.3) and (3.4) we get, $\varphi(x*w)\geq\min\{\varphi(x*(u,w)),\varphi(u)\}$. Hence, (X,φ) is an anti $\mathcal{N}$-H-ideal of X. □

Proposition 3.6. *Let $(X\times Y,\varphi\times\phi)$ be an anti $\mathcal{N}$-H-ideal of a BCK-algebra $X\times Y$. Then,*

$$(\varphi\times\phi)((0,0)*((0,0)*(x,y)))\geq(\varphi\times\phi)(x,y)$$

for all $(x,y)\in X\times Y$.

Proof. Note that

$$\begin{aligned}
&(\varphi\times\phi)((0,0)*((0,0)*(x,y)))\\
&\qquad\geq \min\{(\varphi\times\phi)((0,0)*((x,y)*((0,0)*(x,y)))),(\varphi\times\phi)(x,y)\}\\
&\qquad= \min\{(\varphi\times\phi)((0,0)*((x,y)*(0,0))),(\varphi\times\phi)(x,y)\}\\
&\qquad= \min\{(\varphi\times\phi)((0,0)*(x,y)),(\varphi\times\phi)(x,y)\}\\
&\qquad= \min\{(\varphi\times\phi)(0,0),(\varphi\times\phi)(x,y)\}\\
&\qquad= (\varphi\times\phi)(x,y) \text{ for all } (x,y)\in X\times Y.
\end{aligned}$$

Therefore, $(\varphi\times\phi)((0,0)*((0,0)*(x,y)))\geq(\varphi\times\phi)(x,y)$ for all $(x,y)\in X\times Y$. □

Corollary 3.7. *Let $(X\times Y,\varphi\times\phi)$ be an anti $\mathcal{N}$-H-ideal of $X\times Y$. Then, the set*

$$D_{(\varphi\times\phi)}=\{(x,y)\in X\times Y\mid(\varphi\times\phi)(x,y)=(\varphi\times\phi)(0,0)\}$$

is an H-ideal of $X\times Y$.

Proof. Let $(X\times Y,\varphi\times\phi)$ be an anti $\mathcal{N}$-H-ideal of $X\times Y$. Obviously, $(0,0)\in D_{(\varphi\times\phi)}$. Let $(x,y),(u,v),(w,z)\in D_{(\varphi\times\phi)}$ be such that $(x,y)*((u,v)*(w,z)),(u,v)\in D_{(\varphi\times\phi)}$. Then,

$$(\varphi\times\phi)((x,y)*((u,v)*(w,z)))=(\varphi\times\phi)(0,0)=(\varphi\times\phi)(u,v).$$

Now,

$$\begin{aligned}
(\varphi\times\phi)((x,y)*(w,z)) &\geq \min\{(\varphi\times\phi)((x,y)*((u,v)*(w,z))),(\varphi\times\phi)(u,v)\}\\
&= (\varphi\times\phi)(0,0).
\end{aligned}$$

Again, since $(X\times Y,\varphi\times\phi)$ is an anti $\mathcal{N}$-H-ideal of $X\times Y$, so $(\varphi\times\phi)(0,0)\geq(\varphi\times\phi)((x,y)*(w,z))$. Therefore, $(\varphi\times\phi)(0,0)=(\varphi\times\phi)((x,y)*(w,z))$. It follows that $(x,y)*(w,z)\in D_{(\varphi\times\phi)}$ for all $(x,y),(u,v),(w,z)\in X\times Y$. Therefore, $D_{(\varphi\times\phi)}$ is an H-ideal of X. □

Definition 3.8. Let $(X\times Y,\varphi\times\phi)$ be an anti $\mathcal{N}$-H-ideal of $X\times Y$ and $s\in[-1,0]$. Then, an anti $\mathcal{N}$-s-level set of $(X\times Y,\varphi\times\phi)$ is as follows:

$$(\varphi\times\phi)_s=\{(x,y)\in X\times Y\mid(\varphi\times\phi)(x,y)\geq s\}.$$

Theorem 3.8. *Let $(X\times Y,\varphi\times\phi)$ be an $\mathcal{N}$-structure of $X\times Y$. Then, $(X\times Y,\varphi\times\phi)$ is an anti $\mathcal{N}$-H-ideal of $X\times Y$ if and only if $(\varphi\times\phi)_s\neq\emptyset$ is an H-ideal of $X\times Y$ for all $s\in[-1,0]$.*

Proof. Assume that $(X\times Y,\varphi\times\phi)$ is an anti $\mathcal{N}$-H-ideal of $X\times Y$ and $s\in[-1,0]$ such that $(\varphi\times\phi)_s\neq\emptyset$. Let $(c,d)\in(\varphi\times\phi)_s$. Then, we have $(\varphi\times\phi)(c,d)\geq s$. So we deduce that $(\varphi\times\phi)(0,0)\geq(\varphi\times\phi)(c,d)\geq s$. This shows that $(0,0)\in(\varphi\times\phi)_s$. Let $(x',y'),(u',v'),(w',z')\in X\times Y$ such that $(x',y')*((u',v')*(w',z'))\in(\varphi\times\phi)_s$ and $(u',v')\in(\varphi\times\phi)_s$. Then, $(\varphi\times\phi)((x',y')*((u',v')*(w',z')))\geq s$ and $(\varphi\times\phi)(u',v')\geq s$. Since $(X\times Y,\varphi\times\phi)$ is an anti $\mathcal{N}$-H-ideal of $X\times Y$, it follows that

$$\begin{aligned}
&(\varphi\times\phi)((x',y')*(w',z'))\\
&\qquad\geq \min\{(\varphi\times\phi)((x',y')*((u',v')*(w',z'))),(\varphi\times\phi)(u',v')\}\\
&\qquad\geq \min\{s,s\}=s,
\end{aligned}$$

so $(x',y')*(w',z')\in(\varphi\times\phi)_s$. Therefore, $(\varphi\times\phi)_s$ is an H-ideal of $X\times Y$.

Conversely, assume that $(\varphi \times \phi)_s \neq \emptyset$ is an H-ideal of $X \times Y$ for all $s \in [-1, 0]$. Let $(x, y) \in X \times Y$ be such that $(\varphi \times \phi)(0, 0) < (\varphi \times \phi)(x, y)$. By taking

$$s_o = \frac{1}{2}[(\varphi \times \phi)(0, 0) + (\varphi \times \phi)(x, y)],$$

we get $(\varphi \times \phi)(0, 0) < s_o < (\varphi \times \phi)(x, y)$. Therefore, $(0, 0) \notin (\varphi \times \phi)_{s_o}$. This is a contradiction. Hence, $(\varphi \times \phi)(0, 0) \geq (\varphi \times \phi)(x, y)$ for all $(x, y) \in X \times Y$. Again, we assume that $(x, y), (u, v), (w, z) \in X \times Y$ be such that

$$(\varphi \times \phi)((x, y) * (w, z)) < \min\{(\varphi \times \phi)((x, y) * ((u, v) * (w, z))), (\varphi \times \phi)(u, v)\}.$$

By taking

$$s_1 = \frac{1}{2}[(\varphi \times \phi)((x, y) * (w, z)) + \min\{(\varphi \times \phi)((x, y) * ((u, v) * (w, z))), (\varphi \times \phi)(u, v)\}],$$

we have

$$(\varphi \times \phi)((x, y) * (w, z)) < s_1 < \min\{(\varphi \times \phi)((x, y) * ((u, v) * (w, z))), (\varphi \times \phi)(u, v)\}.$$

This shows that, $(x, y) * ((u, v) * (w, z)) \in (\varphi \times \phi)_{s_1}$, $(u, v) \in (\varphi \times \phi)_{s_1}$, but $(x, y) * (w, z) \notin (\varphi \times \phi)_{s_1}$, this is a contradiction. Therefore,

$$(\varphi \times \phi)((x, y) * (w, z)) \geq \min\{(\varphi \times \phi)((x, y) * ((u, v) * (w, z))), (\varphi \times \phi)(u, v)\}$$

for all $(x, y), (u, v), (w, z) \in X \times Y$. Hence, $(X \times Y, \varphi \times \phi)$ is an anti $\mathcal{N}$-H-ideal of $X \times Y$. □

4. CONCLUSIONS

In this work, we introduced the notion of anti $\mathcal{N}$-H-ideals in a BCK-algebra. Also, we considered the structure of a BCK-algebra and defined the direct product of two anti $\mathcal{N}$-H-ideals. We presented some interesting results about the direct product of two anti $\mathcal{N}$-H-ideals of a BCK-algebra. Finally, we proved that the direct product of two anti $\mathcal{N}$-structures becomes an anti $\mathcal{N}$-H-ideal if and only if for any $s \in [-1, 0]$, an anti $\mathcal{N}$-s-level cut set is an H-ideal of a BCK-algebra $X \times Y$.

5. ACKNOWLEDGEMENTS

The author thanks the editors of this journal for their support in designing this paper.

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ANAS AL-MASARWAH
SCHOOL OF MATHEMATICAL SCIENCES, FACULTY OF SCIENCE AND TECHNOLOGY, UNIVERSITI KEBANGSAAN MALAYSIA, 43600 UKM BANGI, SELANGOR DE, MALAYSIA
Email address: almasarwah85@gmail.com