ANNALS OF COMMUNICATIONS IN MATHEMATICS
Volume 4, Number 3 (2021), 278-283
ISSN: 2582-0818
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ON BASIC PROPERTIES OF RELATIVE Γ-IDEALS IN Γ-NEAR RINGS

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ABSTRACT. The algebraic system Γ-near rings was introduced by Satyanarayana. Tamizh and Ganesan introduced the concept of bi-ideals in near-rings [On bi-ideals of near-rings, Indian J. Pure Appl. Math., 18(11), 1002-1005(1987)]. Tamizh and Meenakumari defined the concept of bi-ideals in Γ-near-rings and characterized Γ-near-fields [Bi-Ideals of Gamma Near-Rings, Southeast Asian Bulletin of Mathematics(2004), 27: 983-988]. Satyanarayana, Yahya, Basar and Kuncham studied abstract affine Γ-nearrings [Some Results on Abstract Affine Gamma-Near-Rings, International Journal of Pure and Applied Mathematical Sciences, 7(1) (2014), 43-49]. Recently, Basar, Satyanarayana, Kuncham, Kumar and Yahya studied some relative ideals in Γ-nearrings [A note on relative Γ-ideals in abstract affine Γ-nearrings, GIS Science Journal, 8(10)(2021), 9-13]. In this paper, we study relative quasi-Γ-ideals and relative bi-Γ-ideals in Γ-near rings. We also proved nice characterizations of Γ-near rings by these basic relative Γ-ideals.

1. PRELIMINARIES

It is well known that a near ring is similar to a ring but satisfying fewer axioms. Both right and left near rings exist. Pilz [5] studied right near rings, and Clay [6] studied left near rings. The interesting consequence from the one sided right distributive law of a right near ring N is that $0 \cdot a = 0$ but it is not necessary that $a \cdot 0 = 0$ for $a \in N$. Also, it is derived from this is that $(-a) \cdot b = -(a \cdot b)$ for $a, b \in N$ but it is not necessary that $a \cdot (-b) = -(a \cdot b)$. A near ring is a ring, not necessarily with unity, if and only if addition is commutative and multiplication is also distributive over addition on the left. Near rings arise naturally in many ways. It arises with mappings from a group to itself, i.e., endomorphisms of a group or a cogroup object of a category [5].

Murty [4] introduced the concept of generalized near-field in near-rings. Also, Dheena [8] studied near fields. The concept of Γ-near-fields was introduced by Tamizh and Meenakumari [11]. The best known application of near field is balanced incomplete block design [5] using planar near rings. Planar near rings have many applications to various branches in

2010 *Mathematics Subject Classification.* 16Y30, 16Y60, 16A99.
Key words and phrases. near rings, Γ-near rings, regular Γ-near rings, relative Γ-ideal, relative quasi-Γ-ideal, relative bi-Γ-ideal.
Received: November 18, 2021. Accepted: December 27, 2021. Published: December 31, 2021.
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mathematics. It also has applications to coding theory, cryptography, the design of statistical experiments, mutually orthogonal Latin squares and constructing planes with circles having radius and centre [5].

The concept of Γ-near rings was given by Satyanarayana [See in [2], and in [3]] as follows:

Definition 1.1. Let $(N, +)$ be a group (not necessarily Abelian) and Γ be a nonempty set. Then the triple $(N, +, \Gamma)$ is said to be a right gamma near-ring (denoted as Γ-near-ring) if there exists a mapping $N \times \Gamma \times N \to N$ (the image of (a, α, b) is denoted by $a\alpha b$ for $a, b \in N$ and $\alpha \in \Gamma$ satisfying the following conditions:

(1): $(a + b)\alpha c = a\alpha c + b\alpha c$,
(2): $(a\alpha b)\beta c = a\alpha(b\beta c)$,

for all $a, b, c \in N$ and for all $\alpha, \beta \in \Gamma$.

If A and B are two non-empty subsets of N, then as defined by Satyanarayana [see in [9]],

$$A\Gamma B = \{a\gamma b \mid \gamma \in \Gamma, a \in A, \text{ and b} \in \text{B}\},$$

$A \star B = \sum(a_k(a_k^{'} + b_k) - a_k a_k^{'})$, where $a_k, a_k^{'} \in A$ and $b_k \in B$, and

the other binary operation is as defined by Chelvam and Meenakumari [9] below:

$$A\Gamma \star B = \{a\gamma(a^{'} + b) - a\gamma a^{'} \mid a^{'}, a \in A, \gamma \in \Gamma, b \in B\}.$$

A Γ-near ring is called zero symmetric if $n\gamma 0 = 0$ for all $n \in N$ and $\gamma \in \Gamma$.

N. M. Khan and M. F. Ali [7] introduced relative ideals in ordered semigroups, and obtained nice results providing the motivation and the importance of the considering the present study over the existing studies. Thereafter, Basar [1] introduced relative weakly hyperideals and relative prime bi-hyperideals in ordered semihypergroups and in involution ordered semihypergroups. Then, Basar, Bhavanari, Kuncham, Kumar and Yahya [2] introduced relative weak quasi-Γ-ideals in abstract affine Γ-near rings. In this paper, we study relative quasi-Γ-ideals and relative bi-Γ-ideals in Γ-near rings. Our results generalizes those in near rings and in Γ-near rings. The class of relative ordered quasi-Γ-ideals and relative ordered bi-Γ-ideals generalizes ordered quasi-Γ-ideals, ordered bi-Γ-ideals, quasi-Γ-ideals, bi-Γ-ideals, quasi-ideals and bi-ideals, ordered left(right) Γ-ideals, left(right) Γ-ideals, left(right) ideals, ordered Γ-ideals, Γ-ideals and ideals.

Definition 1.2. [2] Suppose that $S \subseteq N$. A normal subgroup $(I, +)$ of $(N, +)$ is called a relative left (respectively, right) ideal if $a\alpha(b + i) - a\alpha b \in I$ (respectively $i\alpha a \in I$) for all $a, b \in S, \alpha \in \Gamma$, and $i \in I$. I is called a relative ideal if it is both a relative left and a relative right ideal. If $S = N$, then the relative left(right) ideal just coincides with respective ideals of N.

Definition 1.3. [2] Suppose that $S \subseteq N$. A Γ-subgroup B of $(N, +)$ is called a relative bi-Γ-ideal of N if $(B\Gamma S\Gamma B) \cap (B\Gamma S)\Gamma \star B \subseteq B$.

Definition 1.4. [2] Suppose that $S \subseteq N$. A subgroup Q of $(N, +)$ is called a relative quasi-Γ-ideal of N if $(Q\Gamma S) \cap (S\Gamma Q) \cap (S\Gamma \star Q) \subseteq Q$.

Definition 1.5. Suppose that N is a Γ-near ring, and $n \in S \subseteq N$. Then n is called relative regular if there exists $m \in S$ satisfying $n = n\alpha m\beta n$ for $\alpha, \beta \in \Gamma$. Moreover, N is called relative regular if every element of N is relative regular.

Definition 1.6. Suppose that B is a relative bi-Γ-ideal of Γ-near ring N. Then N is called relative B-simple if it has no proper relative bi-Γ-ideal.

Definition 1.7. [11] A Γ-near ring N is called a generalized Γ-near field if for each $n \in N$ there exists a unique $n^{'} \in N$ satisfying $n\alpha n^{'}\beta n = n$ and $n^{'}\alpha n\beta n^{'} = n^{'}$ for $\alpha, \beta \in \Gamma$.

An element $n \in S \subseteq N$ is called relative idempotent if $n\gamma n = n$ for $\gamma \in \Gamma$.

2. Basic Results on Relative Γ-Ideals

Chelvam and Ganesan [10] introduced bi-ideals in near rings. Chelvam and Meenakumari [9] introduced bi-ideals in Γ-near rings. We introduce relative bi-Γ-ideals in Γ-near rings.

Theorem 2.1. *The system of all relative bi-Γ-ideals of N makes N a relative moore set.*

Proof. Let $S \subseteq N$. Suppose that the set of all relative bi-Γ-ideals of $N = \{B_i\}_{i\in I}$, and $B = \bigcap_{i\in I} B_i$. Therefore, we now have the following:

$$(B\Gamma S\Gamma B) \cap (B\Gamma S)\Gamma \star B \subseteq (B_i\Gamma S\Gamma B_i) \cap (B_i\Gamma S)\Gamma \star B_i \subseteq B_i,$$

for every $i \in I$. Hence B is a relative bi-Γ-ideal of N. □

Theorem 2.2. *Suppose that $S \subseteq N$, where N is a zero symmetric Γ-near ring. A Γ-subgroup B of N is a relative bi-Γ-ideal of N if and only if $B\Gamma S\Gamma B \subseteq B$.*

Proof. Suppose that B is a relative bi-Γ-ideal of N and $S \subseteq N$. Therefore, we receive the following:

$$(B\Gamma S\Gamma B) \cap (B\Gamma S)\Gamma \star B \subseteq B.$$

As N is zero symmetric, we have $n\gamma b = n\gamma(0+b) - n\gamma 0 \in S\Gamma \star B$ for $n \in S$ and $\gamma \in \Gamma$. This implies that $S\Gamma B \subseteq S\Gamma \star B$. Next, we have

$$B\Gamma S\Gamma B \subseteq (B\Gamma S\Gamma B) \cap (B\Gamma S)\Gamma \star B \subseteq B.$$

This implies that $B\Gamma S\Gamma B \subseteq B$.
The converse is straightforward. □

Theorem 2.3. *Suppose that N is a zero symmetric Γ-near ring. Let B be a relative bi-Γ-ideal of N. Then $B\Gamma m$ and $m^{'}\Gamma B$ are relative bi-Γ-ideals of N for $m, m^{'} \in S \subseteq N$, where $m^{'}$ is a distributive element.*

Proof. Let $S \subseteq N$. Indeed, $B\Gamma m$ is a Γ-subgroup of $(N, +)$. Moreover,

$$(B\Gamma m)\Gamma S\Gamma(B\Gamma m) \subseteq B\Gamma S\Gamma(B\Gamma m) \subseteq B\Gamma m.$$

Therefore, we receive $B\Gamma m$ is a relative bi-Γ-ideal of N. Also, we see that $m^{'}\Gamma B$ is a Γ-subgroup of $(N, +)$, where $m^{'}$ is a distributive element. Hence $m^{'}\Gamma B$ is a relative bi-Γ-ideal of N.

□

Corollary 2.4. *Suppose that B is a relative bi-Γ-ideal of a zero symmetric Γ-near ring N. Let $x \in S \subseteq N$ be a distributive element. The $x\Gamma B\Gamma y$ is a relative bi-Γ-ideal of N for $y \in S \subseteq N$.*

3. Main Results On Basic Characterization of Regular Γ-Near Rings by Relative Bi-Γ-Ideals and Relative Quasi-Γ-Ideals

In this section, we characterize regular Γ-near rings by relative bi-Γ-ideals and relative quasi-Γ-ideals when the underlying Γ-near ring is zero symmetric. Our results improve and generalize both those in [10] and in [9].

Theorem 3.1. *Suppose that B is a relative bi-Γ-ideal of zero symmetric Γ-near ring N. Then a relative bi-Γ-ideal of B is a relative bi-Γ-ideal of N given that B is a relative regular Γ-near ring.*

Proof. Let $S \subseteq N$. Suppose that I is a relative bi-Γ-ideal of B, where B is a relative regular Γ-near ring. Then the relative Γ-regularity of B gives us $i = i\alpha j\beta i$ for $i \in I$, $\alpha, \beta \in \Gamma$, and $j \in B$. It follows that $I \subseteq (I\Gamma B) \cap (B\Gamma I)$. Therefore, we receive

$$\begin{aligned} I\Gamma S\Gamma I &\subseteq (I\Gamma B)\Gamma S\Gamma(B\Gamma I) \\ &\subseteq I\Gamma(B\Gamma S\Gamma B)\Gamma I \\ &\subseteq I\Gamma B\Gamma I \\ &\subseteq I. \end{aligned}$$

□

Hence I is a relative bi-Γ-ideal of N.

Theorem 3.2. *Suppose that N is a relative regular Γ-near ring such that the relative idempotent commutes in it. Then every relative quasi-Γ-ideal of N is a relative idempotent.*

Proof. Suppose that Q is a relative quasi-Γ-ideal of N, and $q \in Q$. Let $S \subseteq N$. As Q is a sub-Γ-near ring of N, we have $Q\Gamma Q \subseteq Q$. Let $q \in Q$. As N is relative regular, we find $q = q\alpha n\beta q$ for $\alpha, \beta \in \Gamma$, and $n \in S$. By the following identity:

$$\begin{aligned} Q\Gamma(S\Gamma S)\Gamma Q &= (Q\Gamma S)\Gamma(S\Gamma Q) \\ &\subseteq (Q\Gamma S) \cap (S\Gamma Q) \\ &\subseteq Q, \end{aligned}$$

we receive the following:

$$\begin{aligned} q &= q\alpha n\beta q \\ &= (q\alpha n)\beta(q\alpha n\beta q) \\ &= (q\alpha n)\beta q\alpha(n\beta q) \\ &= (q\alpha n)\beta(n\beta q)\alpha q \\ &= (q\alpha(n\beta n)\beta q)\alpha q \\ &\in (Q\Gamma(S\Gamma S)\Gamma Q)\Gamma Q \\ &\subseteq Q\Gamma Q. \end{aligned}$$

Therefore, we have $q \in Q\Gamma Q$. Hence Q is a relative idempotent. □

Theorem 3.3. *Suppose that a relative bi-Γ-ideal B of N is a multiplicative Γ-subgroup of a Γ-near ring N, and $S \subseteq N$. Then B is a relative minimal bi-Γ-ideal of N.*

Proof. Suppose that $0 \neq I$ is a relative bi-Γ-ideal of N such that $I \subseteq B$. Let $S \subseteq N$. We now have $I\Gamma S\Gamma I \subseteq I$. It then follows that

$$I\Gamma B\Gamma I \subseteq I\Gamma S\Gamma I \subseteq I.$$

Therefore, I is a relative bi-Γ-ideal of B. Suppose that $0 \neq i \in I$. As B is a multiplicative sub-Γ-group of I, we have $B = B\Gamma i = i\Gamma B$. Furthermore, we have

$$\begin{aligned} B = B^2 &= B\Gamma B \\ &= (i\Gamma B)\Gamma(B\Gamma i) \\ &\subseteq i\Gamma B\Gamma i \\ &\subseteq I. \end{aligned}$$

Hence $I = B$. □

4. CONCLUSION

The algebraic structure Γ-near rings, defined and introduced by Bhavanari Satyanarayana, represent an interesting field of algebra, important both from the theoretical point of view and also for their applications. The relative results in Γ-near rings can be associated with results of other algebraic structures but with certain constraints and conditions like that quasi-ideals and bi-ideals were introduced both in semigroups and in rings, and then these simple ideals were further generalized in many other algebraic structures by several mathematicians. Dixit and Diwan [12] studied quasi-ideals and bi-ideals in ternary semigroups. By analyzing these, we have studied relative quasi-Γ-ideals, relative bi-Γ-ideals in Γ-near rings and described some characterization of Γ-near rings by these relative Γ-ideals. The abstract concept of relative ideals can be further generalized in near rings, Γ-near rings and in other algebraic system, viz., ternary semigroups, ordered ternary semigroups, semirings, Γ-semirings and ordered semirings, etc. with potentially and impactfully applicable scope for future work directions.

5. ACKNOWLEDGEMENTS

The authors would like to thank the referees, editors and the Editor-in-Chief of "Annals of Communications in Mathematics(ACM)" for their valuable suggestions and corrections for the improvement of this paper.

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