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BOUNDARY VALUE PROBLEMS FOR PANTOGRAPH EQUATIONS UNDER GENERALIZED FRACTIONAL DERIVATIVE

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ABSTRACT. The sufficient conditions are estabilished for the existence of solutions for a class of boundary value problems for pantograph equations involving the ψ-type fractional deerivative in Caputo sense.

1. INTRODUCTION

Fractional calculus is a generalization of the ordinary differentiation and integration to arbitrary non-integer order. The subject is as old as the differential calculus and it has been developed up to nowadays, for example [6, 7, 8, 9]. Fractional differential and integral equations have recently been applied in various areas of Engineering, Mathematics, Physics and Bio-engineering and so on, one can refer to [3, 4].

This paper deals with the existence and uniqueness of solutions for boundary value problems (BVP for short), for ψ-type fractional pantograph equations

$$^{c}\mathscr{D}^{\alpha;\psi}u(t) = f(t, u(t), u(\lambda t)), \quad \text{for each} \quad t \in J = [0, T],\ 2 < \alpha \leq 3, \tag{1.1}$$

$$u(0) = u_0, \quad u^{'}(0) = u_0^*, \quad u^{''}(T) = u_T, \tag{1.2}$$

where $^{c}\mathscr{D}^{\alpha;\psi}$ is the ψ-type Caputo fractional derivative, $0 < \lambda < 1$, $f : J \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a continuous function and u_0, u_0^*, u_T are real constants.

The problem (1.1)-(1.2) is equivalent to the following Volterra integral equations

$$u(t) = \begin{cases} \frac{1}{\Gamma(\alpha)} \int_0^t \psi^{'}(s) \left(\psi(t) - \psi(s)\right)^{\alpha-1} f(s, u(s), u(\lambda s))ds \\ -\frac{(\psi(t))^2}{2\Gamma(\alpha-3)} \int_0^T \psi^{'}(s) \left(\psi(T) - \psi(s)\right)^{\alpha-1} f(s, u(s), u(\lambda s))ds \\ +u_0 + u_0^*(\psi(t)) + \frac{u_T}{2}(\psi(t))^2. \end{cases} \tag{1.3}$$

The pantograph type equations have been studied extensively owing to the numerous applications in which these equations arise. The name pantograph originated from the work of Ockendon and Tayler on the collection of current by the pantograph head of an electric locomotive, this equations are appeared in modeling of various problems in engineering and sciences such as biology, economy, control and electrodynamics. For some

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applications of this equation involving the fractional-order derivative, we refer the interested reader to [2, 10, 11, 12].

2. Preliminary tools

In this part,we give basic definitions and features of the ψ-type fractional derivative presented by Almeida [1]. We denote by $C(J,\mathbb{R})$ the Banach space of all continuous functions from J into $\mathbb{R}$ with the norm

$$\|u\|_{\infty} := \sup\{|u(t)| : t \in J\}.$$

Definition 2.1. The ψ-type fractional order integral of the function $w \in L^1([a,b],\mathbb{R}_+)$ of order $\alpha \in \mathbb{R}_+$ is defined by

$$\mathscr{I}_a^{\alpha;\psi} w(t) = \frac{1}{\Gamma(\alpha)} \int_a^t \psi^{'}(s)\left(\psi(t)-\psi(s)\right)^{\alpha-1} w(s)ds,$$

where Γ is the gamma function.

Definition 2.2. For a function w given on the interval $[a,b]$ the αth ψ-type Riemann-Liouville fractional derivative of w is defined by

$$(\mathscr{D}_{a_+}^{\alpha;\psi} w)(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^n \int_a^t \psi^{'}(s)\left(\psi(t)-\psi(s)\right)^{n-\alpha-1} w(s)ds.$$

Here $n = [\alpha]+1$ and $[\alpha]$ denotes the integer part of α.

Definition 2.3. For a function w given on the interval $[a,b]$ the ψ-type Caputo fractional-order derivative of w is defined by

$$({}^c\mathscr{D}_{a^+}^{\alpha;\psi} w)(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t \psi^{'}(s)\left(\psi(t)-\psi(s)\right)^{n-\alpha-1} w^{(n)}(s)ds.$$

3. Existence and uniqueness of solution

In this part using fixed point theory, we will show the existence of the model of solutions and then give uniqueness of the solutions. Let us start by defining what we mean by a solution of the problem (1.1)-(1.2).

Definition 3.1. A function $u \in C^2(J,\mathbb{R})$ with its ψ-type α-derivative existing on J is said to be a solution of (1.1)-(1.2) if u satisfies the equation ${}^c\mathscr{D}^{\alpha;\psi}u(t) = f(t,u(t),u(\lambda t))$ on J and the conditions $u(0)=u_0, \quad u^{'}(0)=u_0^*, \quad u^{''}(T)=u_T$.

Now, we are ready to present our existence and uniqueness results via fixed point methods. At beginning, we give the following assumptions:

(A1) There exists a constant $k>0$ such that

$$|f(t,x_1,x_2)-f(t,\overline{x_1},\overline{x_2})| \le k\left(|x_1-\overline{x_1}|+|x_2-\overline{x_2}|\right)$$

for each $t \in J$, and for all $x_1, x_2, \overline{x_1}, \overline{x_2} \in \mathbb{R}$.

(A2) There exists a constant $M>0$ such that

$$|f(t,x_1,x_2)| \le M, \quad \text{for each } t \in J \text{ and all } x_1, x_2 \in \mathbb{R}.$$

Theorem 3.1. *Assume that (A1) is satisfied. If*

$$2k(\psi(T))^{\alpha}\left[\frac{1}{\Gamma(\alpha+1)}+\frac{1}{2\Gamma(\alpha-1)}\right] < 1, \tag{3.1}$$

then the BVP (1.1)-(1.2) *has a unique solution on* J.

Proof. Transform the problem (1.1)-(1.2) into a fixed point problem. Consider the operator $F : C(J,\mathbb{R}) \to C(J,\mathbb{R})$ defined by

$$F(u)(t) = \begin{cases} \frac{1}{\Gamma(\alpha)} \int_0^t \psi^{'}(s) \left(\psi(t) - \psi(s)\right)^{\alpha-1} f(s,u(s),u(\lambda s))ds \\ -\frac{(\psi(t))^2}{2\Gamma(\alpha-2)} \int_0^T \psi^{'}(s) \left(\psi(T) - \psi(s)\right)^{\alpha-1} f(s,u(s),u(\lambda s))ds \\ +u_0 + u_0^*(\psi(t)) + \frac{u_T}{2}(\psi(t))^2. \end{cases}$$

Clearly, the fixed points of the operator F are solutions of the problem (1.1)-(1.2). We shall use the Banach contraction principle to prove that F has a fixed point. We shall show that F is a contraction.

Ler $u, v \in C(J,\mathbb{R})$. Then for each $t \in J$ we have

$$\begin{aligned} &|F(u)(t) - F(v)(t)| \\ &\le \frac{1}{\Gamma(\alpha)} \int_0^t \psi^{'}(s) \left(\psi(t) - \psi(s)\right)^{\alpha-1} |f(s,u(s),u(\lambda s)) - f(s,v(s),v(\lambda s))|\, ds \\ &+ \frac{(\psi(T))^\alpha}{2\Gamma(\alpha-2)} \int_0^T \psi^{'}(s) \left(\psi(T) - \psi(s)\right)^{\alpha-3} |f(s,u(s),u(\lambda s)) - f(s,v(s),v(\lambda s))|\, ds \\ &\le \frac{2k\|u-v\|_\infty}{\Gamma(\alpha)} \int_0^t \psi^{'}(s) \left(\psi(t) - \psi(s)\right)^{\alpha-1} ds \\ &+ \frac{(\psi(T))^2 2k\|u-v\|_\infty}{2\Gamma(\alpha-2)} \int_0^T \psi^{'}(s) \left(\psi(T) - \psi(s)\right)^{\alpha-3} ds \\ &\le 2k(\psi(T))^\alpha \left[\frac{1}{\Gamma(\alpha+1)} + \frac{1}{2\Gamma(\alpha-1)}\right] \|u-v\|_\infty . \end{aligned}$$

Thus

$$\|F(u) - F(v)\|_\infty \le 2k(\psi(T))^\alpha \left[\frac{1}{\Gamma(\alpha+1)} + \frac{1}{2\Gamma(\alpha-1)}\right] \|u-v\|_\infty .$$

Consequently, by (3.1) F is contraction. As a consequence of the Banach fixed point theorem, we denote that F has a fixed point which is a solution of the problem (1.1)-(1.2). □

The second result is based on Schaefer's fixed point theorem.

Theorem 3.2. *Assume that (A2) is satisfied. Then the BVP* (1.1)-(1.2) *has at least one solution on J.*

Proof. We shall use Schaefer's fixed point theorem to prove that F has a fixed point. The proof will be given in several steps.

Claim 1. F is continuous.

Let $\{u_n\}$ be a sequence such that $u_n \to u$ in $C(J,\mathbb{R})$. Then for each $t \in J$

$$\begin{aligned} &|F(u_n)(t) - F(u)(t)| \\ &\le \frac{1}{\Gamma(\alpha)} \int_0^t \psi^{'}(s) \left(\psi(t) - \psi(s)\right)^{\alpha-1} |f(s,u_n(s),u_n(\lambda s)) - f(s,v_n(s),v_n(\lambda s))|\, ds \\ &+ \frac{(\psi(T))^2}{2\Gamma(\alpha-2)} \int_0^T \psi^{'}(s) \left(\psi(T) - \psi(s)\right)^{\alpha-3} |f(s,u_n(s),u_n(\lambda s)) - f(s,v_n(s),v_n(\lambda s))|\, ds. \end{aligned}$$

Since f is a continuous function, we have

$$\|F(u_n) - F(u)\|_\infty \to 0 \quad \text{as} \quad n \to \infty.$$

Claim 2. F maps the bounded sets into the bounded sets in $C(J,\mathbb{R})$.

Indeed, it is enough to show that for any $\eta^* > 0$ there exists a positive constant l such that for each $u \in B_{\eta^*} = \{u \in C(J,\mathbb{R}) : \|u\|_\infty \leq \eta^*\}$ we have $\|F(u)\|_\infty \leq l$. By (A2) we have for each $t \in J$,

$$\begin{aligned}|F(u)(t)| &\leq \frac{1}{\Gamma(\alpha)}\int_0^t \psi^{'}(s)\left(\psi(t)-\psi(s)\right)^{\alpha-1}|f(s,u(s),u(\lambda s))|\,ds \\ &+ \frac{(\psi(T))^2}{2\Gamma(\alpha-2)}\int_0^T \psi^{'}(s)\left(\psi(T)-\psi(s)\right)^{\alpha-3}|f(s,u(s),u(\lambda s))|\,ds \\ &+ |u_0| + |u_0^*|\left(\psi(T)\right) + \frac{|u_T|}{2}(\psi(T))^2 \\ &\leq \frac{M}{\Gamma(\alpha+1)}(\psi(T))^\alpha + \frac{M}{\Gamma(\alpha-1)}(\psi(T))^\alpha + |u_0| + |u_0^*|\left(\psi(T)\right) + \frac{|u_T|}{2}(\psi(T))^2.\end{aligned}$$

Thus

$$\begin{aligned}\|F(u)\|_\infty \quad &\leq \frac{M}{\Gamma(\alpha+1)}(\psi(T))^\alpha + \frac{M}{\Gamma(\alpha-1)}(\psi(T))^\alpha + |u_0| + |u_0^*|\left(\psi(T)\right) + \frac{|u_T|}{2}(\psi(T))^2 \\ &:= l.\end{aligned}$$

Claim 3. F maps the bounded sets into the equicontinuous sets of $C(J,\mathbb{R})$.

Let $t_1, t_2 \in J$, $t_1 < t_2$, B_{η^*} be a bounded set of $C(J,\mathbb{R})$ like in Claim 2, and let $u \in B_{\eta^*}$. Then

$$\begin{aligned}&|F(u)(t_2) - F(u)(t_1)| \\ &= \left|\frac{1}{\Gamma(\alpha)}\int_0^{t_1}\left[\psi^{'}(s)\left(\psi(t_2)-\psi(s)\right)^{\alpha-1} - \psi^{'}(s)\left(\psi(t_1)-\psi(s)\right)^{\alpha-1}\right]f(s,u(s),u(\lambda s))ds\right. \\ &\left.+\frac{\left(\psi(t_2)-\psi(t_1)\right)^2}{\Gamma(\alpha-2)}\int_{t_1}^{t_2}\psi^{'}(s)\left(\psi(t_2)-\psi(s)\right)^{\alpha-3}f(s,u(s),u(\lambda s))ds\right| \\ &\leq \frac{M}{\Gamma(\alpha+1)}\left[(\psi(t_2)-\psi(t_1))^\alpha + (\psi(t_1))^\alpha - (\psi(t_2))^\alpha\right] + \frac{M}{2\Gamma(\alpha-1)}\left(\psi(t_2)-\psi(t_1)\right)^\alpha \\ &+ |u_0| + |u_0^*|\left(\psi(t_2)-\psi(t_1)\right) + \frac{|u_T|}{2}\left(\psi(t_2)-\psi(t_1)\right)^2.\end{aligned}$$

As $t_1 \to t_2$, the right-hand side of the above inequality tends to zero. As a consequence of Claim 1 to Claim 3 together with Arzela-Ascoli theorem, we can conclude that $F : C(J,\mathbb{R}) \to C(J,\mathbb{R})$ is completely continuous.

Claim 4. A priori bounds.

Now it remains to show that the set

$$\zeta = \{u \in C(J,\mathbb{R}) : u \in \delta F(u) \quad \text{for some } 0 < \delta < 1\}$$

is bounded.

Let $u \in \zeta$, then $u \in \delta F(u)$ for some $0 < \delta < 1$. Thus for each $t \in J$ we have

$$u(t) = \begin{cases}\frac{\delta}{\Gamma(\alpha)}\int_0^t \psi^{'}(s)\left(\psi(t)-\psi(s)\right)^{\alpha-1}f(s,u(s),u(\lambda s))ds \\ -\frac{\delta(\psi(t))^2}{2\Gamma(\alpha-2)}\int_0^T \psi^{'}(s)\left(\psi(T)-\psi(s)\right)^{\alpha-1}f(s,u(s),u(\lambda s))ds \\ +\delta u_0 + \delta u_0^*(\psi(t)) + \delta\frac{u_T}{2}(\psi(t))^2.\end{cases}$$

This implies by (A2) that for each $t \in J$ we have

$$|u(t)| \leq \frac{M}{\Gamma(\alpha+1)}(\psi(T))^{\alpha} + \frac{M}{\Gamma(\alpha-1)}(\psi(T))^{\alpha} + |u_0| + |u_0^*|(\psi(T)) + \frac{|u_T|}{2}(\psi(T))^2$$
$$:= R.$$

This shows that the set ζ is bounded. As a consequence of Schaefer's fixed point theorem, we deduce that F has a fixed point which is a solution of the problem (1.1)-(1.2). $\square$

4. An example

In this section, we present an example to illustrate the usefulness of our main results. Let us consider the following ψ-type fractional BVP

$$^{c}\mathscr{D}^{\alpha;\psi}u(t) = \frac{1}{20}\left[\frac{|u(t)|}{1+|u(t)|} + \frac{|u(\lambda t)|}{1+|u(\lambda t)|}\right], \quad t \in J := [0,1], \quad 2 < \alpha \leq 3, \tag{4.1}$$

$$u(0) = 0, \quad u^{'}(0) = 1, \quad u^{''}(1) = 0, \tag{4.2}$$

where $0 < \lambda < 1$.
Set

$$f(t,x,y) = \frac{1}{20}\left[\frac{x}{1+x} + \frac{y}{1+y}\right], \quad \text{for any} \quad x, y \in [0,\infty).$$

Let $x, y, \overline{x}, \overline{y} \in [0,\infty)$ and $t \in J$. Then we have

$$|f(t,x,y) - f(t,\overline{x},\overline{y})| \leq \frac{1}{20}\left(|x-\overline{x}| + |y-\overline{y}|\right).$$

Hence the condition (A1) holds with $k = \frac{1}{20}$. We shall check the condition (3.1) is satisfied with $\psi(T) = 1$. Indeed,

$$2k(\psi(T))^{\alpha}\left[\frac{1}{\Gamma(\alpha+1)} + \frac{1}{2\Gamma(\alpha-1)}\right] < 1,$$

Then by Theorem 3.1 the problem (4.1)-(4.2) has a unique solution on $[0,1]$.

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Authors contributions

All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

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