ANNALS OF COMMUNICATIONS IN MATHEMATICS
Volume 4, Number 1 (2021), 63-72
ISSN: 2582-0818

ON PYTHAGOREAN NORMAL SUBBISEMIRING OF BISEMIRING

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ABSTRACT. We discuss the notion of Pythagorean subbisemiring, level sets of Pythagorean subbisemirings and Pythagorean normal subbisemiring of a bisemiring. Also, we investigate some of the properties related to subbisemirings. The fuzzy subset $L = (\pi_L^{\mathscr{P}}, \omega_L^{\mathscr{P}})$ is a Pythagorean subbisemiring if and only if all non-empty level set $L_{(t,s)}(t, s \in (0,1])$ is a subbisemiring. The cartesian product of two Pythagorean subbisemiring is also Pythagorean subbisemiring. The homomorphic image and preimage of Pythagorean subbisemiring is also Pythagorean subbisemiring. To illustrate our results and examples are given.

1. INTRODUCTION

The study of semirings was started by the German Mathematician Dedekind in connection with ideals of commutative rings. Later on semiring were studied by the American Mathematician Vandever in 1935 [11], who accepted semiring as a fundamental algebraic structure. It was basically the generalization of rings and distributive lattices. However the developments of the theory in semirings have been taking place since 1950. L.A Zadeh [18] proposed by fuzzy set theory in 1965 suggests that decision makers are to be solving uncertain problems by considering membership degree. In 1986, Atanassov [7] introduced the notion of intuitionistic fuzzy sets and is characterized by a degree of membership and non-membership satisfying the condition that sum of its membership degree and non membership degree is not exceeding one [7]. However, we may interact a problem in decision making events where the sum of the degree of membership and non-membership of a particular attribute is exceeding one. So Yager was introduced by the concept of Pythagorean sets [16]. It has been to extended the intuitionistic fuzzy sets and characterized by the condition that square sum of its degree of membership and non membership is not exceeding one. In 1993, J. Ahsan, K. Saifullah, and F. Khan [2] introduced the notion of fuzzy semirings. In 2001, M.K Sen and S. Ghosh was introduced in bisemirings. A bisemiring $(S, +, \circ, \times)$ is an algebraic structure in which $(S, +, \circ)$ and $(S, \circ, \times)$ are semirings in which $(S, +)$, $(S, \circ)$ and $(S, \times)$ are semigroups such that (i) $l_1 \circ (l_2 + l_3) = (l_1 \circ l_2) + (l_1 \circ l_3)$, (ii) $(l_2 + l_3) \circ l_1 = (l_2 \circ l_1) + (l_3 \circ l_1)$ and (iii) $l_1 \times (l_2 \circ l_3) = (l_1 \times l_2) \circ (l_1 \times l_3)$, (iv) $(l_2 \circ l_3) \times l_1 = (l_2 \times l_1) \circ (l_3 \times l_1), \forall\, l_1, l_2, l_3 \in S$ [15]. A non-empty subset B of S is a subbisemiring if and only if $l_1 + l_2, l_1 \circ l_2, l_1 \times l_2 \in B$ for all $l_1, l_2 \in B$

2010 *Mathematics Subject Classification.* 16Y60, 05C60, 03E72.
Key words and phrases. Subbisemiring; bisemiring; Homomorphism; Normal.
Received: December 28, 2020. Accepted: March 11, 2021. Published: March 31, 2021.
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[10]. The purpose of this paper is to extend the concept of intuitionistic subsemiring of semiring to Pythagorean subbisemiring of bisemiring. Also we obtain Pythagorean normal subbisemiring of bisemiring.

2. PRELIMINARIES

Notations: (i) Pythagorean subsemirings, Pythagorean subbisemirings, Pythagorean normal subsemirings, Pythagorean normal subbisemirings shortly **PSS, PSBS, PNSS, PNSBS** respectively.
(ii) fuzzy subsemirings, subbisemirings, fuzzy subbisemirings shortly **FSS, SBS, FSBS** respectively.

Definition 2.1. Let $(S, +, \cdot)$ be semiring. A fuzzy subset L of S is said to be **FSS** if
(i) $\pi_L(u_1 + u_2) \geq \min\{\pi_L(u_1), \pi_L(u_2)\}$, (ii) $\pi_L(u_1 \cdot u_2) \geq \min\{\pi_L(u_1), \pi_L(u_2)\}$ for all $u_1, u_2 \in S$.

Definition 2.2. An intuitionistic fuzzy L in non-empty set U is defined, the form
$L = \{< u, \pi_L(u), \omega_L(u) | u \in U >\}$ where $\pi_L : U \to [0, 1]$(define the degree of membership) and $\omega_L : U \to [0, 1]$ (degree of non-membership) for every $u \in U$ satisfying $0 \leq \pi_L(u) + \omega_L(u) \leq 1$.

Definition 2.3. A Pythagorean fuzzy subset L in non-empty set U is defined, the form $L = \{< u, \pi^{\mathscr{P}}_L(u), \omega^{\mathscr{P}}_L(u) | u \in U >\}$ where $\pi^{\mathscr{P}}_L : U \to [0, 1]$(define the degree of membership) and $\omega^{\mathscr{P}}_L : U \to [0, 1]$ (degree of non-membership) for every $u \in U$ satisfying $0 \leq (\pi^{\mathscr{P}}_L(u))^2 + (\omega^{\mathscr{P}}_L(u))^2 \leq 1$.

Definition 2.4. Let L and M be any two Pythagorean fuzzy subsets of a set U.
(i) $L \cap M = \left\{ \left\langle u, \min\{\pi^{\mathscr{P}}_L(u), \pi^{\mathscr{P}}_M(u)\}, \max\{\omega^{\mathscr{P}}_L(u), \omega^{\mathscr{P}}_M(u)\} \right\rangle \right\}$
(ii) $L \cup M = \left\{ \left\langle u, \max\{\pi^{\mathscr{P}}_L(u), \pi^{\mathscr{P}}_M(u)\}, \min\{\omega^{\mathscr{P}}_L(u), \omega^{\mathscr{P}}_M(u)\} \right\rangle \right\}$
(iii) $\Box L = \left\{ \left\langle u, \pi^{\mathscr{P}}_L(u), 1 - \pi^{\mathscr{P}}_L(u) \right\rangle | u \in U \right\}$
(iv) $\Diamond L = \left\{ \left\langle u, 1 - \omega^{\mathscr{P}}_L(u), \omega^{\mathscr{P}}_L(u) \right\rangle | u \in U \right\}$ for all $u \in U$.

Definition 2.5. Let $(S, +, \cdot)$ be semiring. A Pythagorean fuzzy subset L of S is said to be **PSS** of S if
(i) $\pi^{\mathscr{P}}_L(u_1 + u_2) \geq \min\{\pi^{\mathscr{P}}_L(u_1), \pi^{\mathscr{P}}_L(u_2)\}$ and $\pi^{\mathscr{P}}_L(u_1 \cdot u_2) \geq \min\{\pi^{\mathscr{P}}_L(u_1), \pi^{\mathscr{P}}_L(u_2)\}$
(ii) $\omega^{\mathscr{P}}_L(u_1 + u_2) \leq \max\{\omega^{\mathscr{P}}_L(u_1), \omega^{\mathscr{P}}_L(u_2)\}$ and $\omega^{\mathscr{P}}_L(u_1 \cdot u_2) \leq \max\{\omega^{\mathscr{P}}_L(u_1), \omega^{\mathscr{P}}_L(u_2)\}$ for $u_1, u_2 \in S$.

Definition 2.6. Let $(S, +, \cdot)$ be semiring. A Pythagorean fuzzy subset L of S is said to be **PNSS** of S if
(i) $\pi^{\mathscr{P}}_L(u_1 + u_2) = \pi^{\mathscr{P}}_L(u_2 + u_1)$ and $\pi^{\mathscr{P}}_L(u_1 \cdot u_2) = \pi^{\mathscr{P}}_L(u_2 \cdot u_1)$
(ii) $\omega^{\mathscr{P}}_L(u_1 + u_2) = \omega^{\mathscr{P}}_L(u_2 + u_1)$ and $\omega^{\mathscr{P}}_L(u_1 \cdot u_2) = \omega^{\mathscr{P}}_L(u_2 \cdot u_1)$ for all $u_1, u_2 \in S$.

Definition 2.7. Let L and M be Pythagorean fuzzy subsets of G and H respectively. The product of L and M denoted by $L \times M$ is defined as
$L \times M = \{\langle (u_1, u_2), \pi^{\mathscr{P}}_{L\times M}(u_1, u_2), \omega^{\mathscr{P}}_{L\times M}(u_1, u_2) \rangle \mid for\ all\ u_1 \in G\ and\ u_2 \in H\}$,
where $\pi^{\mathscr{P}}_{L\times M}(u_1, u_2) = \min\{\pi^{\mathscr{P}}_L(u_1), \pi^{\mathscr{P}}_M(u_2)\}$ and
$\omega^{\mathscr{P}}_{L\times M}(u_1, u_2) = \max\{\omega^{\mathscr{P}}_L(u_1), \omega^{\mathscr{P}}_M(u_2)\}$.

Definition 2.8. [10] Let $(S_1, +, \cdot, \times)$ and $(S_2, \oplus, \circ, \otimes)$ be two bisemirings. A function $\theta : S_1 \to S_2$ is said to be a homomorphism if
(i) $\theta(u_1 + u_2) = \theta(u_1) \oplus \theta(u_2)$,
(ii) $\theta(u_1 \cdot u_2) = \theta(u_1) \circ \theta(u_2)$,
(iii) $\theta(u_1 \times u_2) = \theta(u_1) \otimes \theta(u_2)$ for all $u_1, u_2 \in S_1$.

3. Pythagorean Subbisemiring

Here S denotes a bisemiring unless otherwise mentioned.

Definition 3.1. A fuzzy subset L of S is said to be a **FSBS** if

$$\begin{cases} \pi_L(u_1 \star_1 u_2) \geq \min\{\pi_L(u_1), \pi_L(u_2)\} \\ \pi_L(u_1 \star_2 u_2) \geq \min\{\pi_L(u_1), \pi_L(u_2)\} \\ \pi_L(u_1 \star_3 u_2) \geq \min\{\pi_L(u_1), \pi_L(u_2)\} \end{cases}$$

$\forall\ u_1, u_2 \in S$.

Example 3.2. Let $S = \{s_1, s_2, s_3, s_4\}$ be the bisemirings sith the Cayley tables:

$\star_1$	s_1	s_2	s_3	s_4
s_1	s_1	s_2	s_3	s_4
s_2	s_2	s_2	s_2	s_2
s_3	s_3	s_2	s_3	s_4
s_4	s_4	s_2	s_4	s_4

$\star_2$	s_1	s_2	s_3	s_4
s_1	s_1	s_1	s_1	s_1
s_2	s_1	s_2	s_2	s_2
s_3	s_1	s_3	s_3	s_3
s_4	s_1	s_2	s_2	s_2

$\star_3$	s_1	s_2	s_3	s_4
s_1	s_1	s_1	s_1	s_1
s_2	s_1	s_1	s_1	s_3
s_3	s_1	s_1	s_1	s_3
s_4	s_1	s_1	s_1	s_3

Then $\pi_L(s_1) = 0.95,\ \pi_L(s_2) = 0.75\ , \pi_L(s_3) = 0.65\ , \pi_L(s_4) = 0.55$. Clearly L is a **FSBS**.

Definition 3.3. A Pythagorean fuzzy subset $L = (\pi_L^{\mathscr{P}}, \omega_L^{\mathscr{P}})$ of S is said to be a **PSBS** if

$$\begin{cases} \pi_L^{\mathscr{P}}(u_1 \star_1 u_2) \geq \min\{\pi_L^{\mathscr{P}}(u_1), \pi_L^{\mathscr{P}}(u_2)\} \\ \pi_L^{\mathscr{P}}(u_1 \star_2 u_2) \geq \min\{\pi_L^{\mathscr{P}}(u_1), \pi_L^{\mathscr{P}}(u_2)\} \\ \pi_L^{\mathscr{P}}(u_1 \star_3 u_2) \geq \min\{\pi_L^{\mathscr{P}}(u_1), \pi_L^{\mathscr{P}}(u_2)\} \end{cases}$$

and

$$\begin{cases} \omega_L^{\mathscr{P}}(u_1 \star_1 u_2) \leq \max\{\omega_L^{\mathscr{P}}(u_1), \omega_L^{\mathscr{P}}(u_2)\} \\ \omega_L^{\mathscr{P}}(u_1 \star_2 u_2) \leq \max\{\omega_L^{\mathscr{P}}(u_1), \omega_L^{\mathscr{P}}(u_2)\} \\ \omega_L^{\mathscr{P}}(u_1 \star_3 u_2) \leq \max\{\omega_L^{\mathscr{P}}(u_1), \omega_L^{\mathscr{P}}(u_2)\} \end{cases}$$

$\forall\ u_1, u_2 \in S$.

Example 3.4. By the Example 3.2.

$$\pi_L^{\mathscr{P}}(s) = \begin{cases} 0.70\ \ if\ \ s = s_1 \\ 0.65\ \ if\ \ s = s_2 \\ 0.55\ \ if\ \ s = s_3 \\ 0.50\ \ if\ \ s = s_4 \end{cases} \quad \omega_L^{\mathscr{P}}(s) = \begin{cases} 0.50\ \ if\ \ s = s_1 \\ 0.60\ \ if\ \ s = s_2 \\ 0.75\ \ if\ \ s = s_3 \\ 0.85\ \ if\ \ s = s_4 \end{cases}$$

Clearly L is a **PSBS**.

Theorem 3.1. *The intersection of a family of* **PSBS** *is a* **PSBS** *of* S.

Proof. Let $\{V_i : i \in I\}$ be a family of **PSBS** and $L = \bigcap_{i\in I} V_i$. Let $s_1, s_2 \in S$. Then

$$\begin{aligned}
\pi_L^{\mathscr{P}}(s_1 \star_1 s_2) &= \inf_{i\in I} \pi_{V_i}^{\mathscr{P}}(s_1 \star_1 s_2)\\
&\geq \inf_{i\in I}\ \min\{\pi_{V_i}^{\mathscr{P}}(s_1), \pi_{V_i}^{\mathscr{P}}(s_2)\}\\
&= \min\left\{\inf_{i\in I} \pi_{V_i}^{\mathscr{P}}(s_1), \inf_{i\in I} \pi_{V_i}^{\mathscr{P}}(s_2)\right\}\\
&= \min\{\pi_L^{\mathscr{P}}(s_1), \pi_L^{\mathscr{P}}(s_2)\}
\end{aligned}$$

Similarly, $\pi_L^{\mathscr{P}}(s_1 \star_2 s_2) \geq \min\{\pi_L^{\mathscr{P}}(s_1), \pi_L^{\mathscr{P}}(s_2)\}$ and $\pi_L^{\mathscr{P}}(s_1 \star_3 s_2) \geq \min\{\pi_L^{\mathscr{P}}(s_1), \pi_L^{\mathscr{P}}(s_2)\}$.

$$\begin{aligned}
\omega_L^{\mathscr{P}}(s_1 \star_1 s_2) &= \sup_{i\in I} \omega_{V_i}^{\mathscr{P}}(s_1 \star_1 s_2)\\
&\leq \sup_{i\in I}\ \max\{\omega_{V_i}^{\mathscr{P}}(s_1), \omega_{V_i}^{\mathscr{P}}(s_2)\}\\
&= \max\left\{\sup_{i\in I} \omega_{V_i}^{\mathscr{P}}(s_1), \sup_{i\in I} \omega_{V_i}^{\mathscr{P}}(s_2)\right\}\\
&= \max\{\omega_L^{\mathscr{P}}(s_1), \omega_L^{\mathscr{P}}(s_2)\}
\end{aligned}$$

Similarly, $\omega_L^{\mathscr{P}}(s_1 \star_2 s_2) \leq \max\{\omega_L^{\mathscr{P}}(s_1), \omega_L^{\mathscr{P}}(s_2)\}$ and $\omega_L^{\mathscr{P}}(s_1 \star_3 s_2) \leq \max\{\omega_L^{\mathscr{P}}(s_1), \omega_L^{\mathscr{P}}(s_2)\}$. Hence L is a **PSBS**.

Theorem 3.2. *If L and M are any two* **PSBS** *of S_1 and S_2 respectively, then $L \times M$ is a* **PSBS**.

Proof. Let L and M be two **PSBS** of S_1 and S_2 respectively. Let $l_1, l_2 \in S_1$ and $m_1, m_2 \in S_2$. Then (l_1, l_2) and (m_1, m_2) are in $S_1 \times S_2$. Now

$$\begin{aligned}
\pi_{L\times M}^{\mathscr{P}}[(l_1, m_1) \star_1 (l_2, m_2)] &= \pi_{L\times M}^{\mathscr{P}}(l_1 \star_1 l_2, m_1 \star_1 m_2)\\
&= \min\{\pi_L^{\mathscr{P}}(l_1 \star_1 l_2), \pi_M^{\mathscr{P}}(m_1 \star_1 m_2)\}\\
&\geq \min\{\min\{\pi_L^{\mathscr{P}}(l_1), \pi_L^{\mathscr{P}}(l_2)\}, \min\{\pi_M^{\mathscr{P}}(m_1), \pi_M^{\mathscr{P}}(m_2)\}\}\\
&= \min\{\min\{\pi_L^{\mathscr{P}}(l_1), \pi_M^{\mathscr{P}}(m_1)\}, \min\{\pi_L^{\mathscr{P}}(l_2), \pi_M^{\mathscr{P}}(m_2)\}\}\\
&= \min\{\pi_{L\times M}^{\mathscr{P}}(l_1, m_1), \pi_{L\times M}^{\mathscr{P}}(l_2, m_2)\}
\end{aligned}$$

Also $\pi_{L\times M}^{\mathscr{P}}[(l_1, m_1) \star_2 (l_2, m_2)] \geq \min\{\pi_{L\times M}^{\mathscr{P}}(l_1, m_1), \pi_{L\times M}^{\mathscr{P}}(l_2, m_2)\}$, $\pi_{L\times M}^{\mathscr{P}}[(l_1, m_1) \star_3 (l_2, m_2)] \geq \min\{\pi_{L\times M}^{\mathscr{P}}(l_1, m_1), \pi_{L\times M}^{\mathscr{P}}(l_2, m_2)\}$. Similarly,

$$\begin{aligned}
\omega_{L\times M}^{\mathscr{P}}[(l_1, m_1) \star_1 (l_2, m_2)] &= \omega_{L\times M}^{\mathscr{P}}(l_1 \star_1 l_2, m_1 \star_1 m_2)\\
&= \max\{\omega_L^{\mathscr{P}}(l_1 \star_1 l_2), \omega_M^{\mathscr{P}}(m_1 \star_1 m_2)\}\\
&\leq \max\{\max\{\omega_L^{\mathscr{P}}(l_1), \omega_L^{\mathscr{P}}(l_2)\}, \max\{\omega_M^{\mathscr{P}}(m_1), \omega_M^{\mathscr{P}}(m_2)\}\}\\
&= \max\{\max\{\omega_L^{\mathscr{P}}(l_1), \omega_M^{\mathscr{P}}(m_1)\}, \max\{\omega_L^{\mathscr{P}}(l_2), \omega_M^{\mathscr{P}}(m_2)\}\}\\
&= \max\{\omega_{L\times M}^{\mathscr{P}}(l_1, m_1), \omega_{L\times M}^{\mathscr{P}}(l_2, m_2)\}
\end{aligned}$$

Also $\omega_{L\times M}^{\mathscr{P}}[(l_1, m_1) \star_2 (l_2, m_2)] \leq \max\{\omega_{L\times M}^{\mathscr{P}}(l_1, m_1), \omega_{L\times M}^{\mathscr{P}}(l_2, m_2)\}$, $\omega_{L\times M}^{\mathscr{P}}[(l_1, m_1) \star_3 (l_2, m_2)] \leq \max\{\omega_{L\times M}^{\mathscr{P}}(l_1, m_1), \omega_{L\times M}^{\mathscr{P}}(l_2, m_2)\}$. Hence $L \times M$ is a **PSBS**.

Theorem 3.3. *Let L be fuzzy subset of S. Then $L = (\pi_L^{\mathscr{P}}, \omega_L^{\mathscr{P}})$ is a* **PSBS** *if and only if all non-empty level set $L_{(t,s)}(t, s \in (0, 1])$ is a* **SBS**.

Proof. Assume that $L = (\pi_L^{\mathscr{P}}, \omega_L^{\mathscr{P}})$ is a **PSBS**. For each $t, s \in (0,1]$ and $l_1, l_2 \in L_{(t,s)}$. We have $\pi_L^{\mathscr{P}}(l_1) \geq t$, $\pi_L^{\mathscr{P}}(l_2) \geq t$, $\omega_L^{\mathscr{P}}(l_1) \leq s$ and $\omega_L^{\mathscr{P}}(l_2) \leq s$. Since L is a **PSBS**, $\pi_L^{\mathscr{P}}(l_1 \star_1 l_2) \geq \min\{\pi_L^{\mathscr{P}}(l_1), \pi_L^{\mathscr{P}}(l_2)\} \geq t, \pi_L^{\mathscr{P}}(l_1 \star_2 l_2) \geq \min\{\pi_L^{\mathscr{P}}(l_1), \pi_L^{\mathscr{P}}(l_2)\} \geq t$ and $\pi_L^{\mathscr{P}}(l_1 \star_1 l_2) \geq \min\{\pi_L^{\mathscr{P}}(l_1), \pi_L^{\mathscr{P}}(l_2)\} \geq t$. Similarly we prove that, $\omega_L^{\mathscr{P}}(l_1 \star_1 l_2) \leq \max\{\omega_L^{\mathscr{P}}(l_1), \omega_L^{\mathscr{P}}(l_2)\} \leq s, \omega_L^{\mathscr{P}}(l_1 \star_2 l_2) \leq \max\{\omega_L^{\mathscr{P}}(l_1), \omega_L^{\mathscr{P}}(l_2)\} \leq s$ and $\omega_L^{\mathscr{P}}(l_1 \star_1 l_2) \leq \max\{\omega_L^{\mathscr{P}}(l_1), \omega_L^{\mathscr{P}}(l_2)\} \leq s$. This implies that $l_1 \star_1 l_2 \in L_{(t,s)}, l_1 \star_2 l_2 \in L_{(t,s)}$ and $l_1 \star_3 l_2 \in L_{(t,s)}$. Therefore $L_{(t,s)}$ is a **SBS** for each $t, s \in (0, 1]$.
Conversely, assume that $L_{(t,s)}$ is a **SBS** for each $t, s \in (0, 1]$. Let $l_1, l_2 \in S$. Then $l_1, l_2 \in L_{(t,s)}$,where $t = \min\{\pi_L^{\mathscr{P}}(l_1), \pi_L^{\mathscr{P}}(l_2)\}$ and $s = \max\{\omega_L^{\mathscr{P}}(l_1), \omega_L^{\mathscr{P}}(l_2)\}$. Thus, $l_1 \star_1 l_2 \in L_{(t,s)}, l_1 \star_2 l_2 \in L_{(t,s)}$ and $l_1 \star_3 l_2 \in L_{(t,s)}$ imply $\pi_L^{\mathscr{P}}(l_1 \star_1 l_2) \geq t = \min\{\pi_L^{\mathscr{P}}(l_1), \pi_L^{\mathscr{P}}(l_2)\}, \pi_L^{\mathscr{P}}(l_1 \star_2 l_2) \geq t = \min\{\pi_L^{\mathscr{P}}(l_1), \pi_L^{\mathscr{P}}(l_2)\}$ and $\pi_L^{\mathscr{P}}(l_1 \star_3 l_2) \geq t = \min\{\pi_L^{\mathscr{P}}(l_1), \pi_L^{\mathscr{P}}(l_2)\}$. Similarly, $\omega_L^{\mathscr{P}}(l_1 \star_1 l_2) \leq s = \max\{\omega_L^{\mathscr{P}}(l_1), \omega_L^{\mathscr{P}}(l_2)\}, \omega_L^{\mathscr{P}}(l_1 \star_2 l_2) \leq s = \max\{\omega_L^{\mathscr{P}}(l_1), \omega_L^{\mathscr{P}}(l_2)\}$ and $\omega_L^{\mathscr{P}}(l_1 \star_3 l_2) \leq s = \max\{\omega_L^{\mathscr{P}}(l_1), \omega_L^{\mathscr{P}}(l_2)\}$. Hence $L = (\pi_L^{\mathscr{P}}, \omega_L^{\mathscr{P}})$ is a **PSBS**.

Theorem 3.4. *Let L be the* **PSBS** *and V be the strongest Pythagorean relation of S. Then L is the* **PSBS** *if and only if V is a* **PSBS** *of $S \times S$.*

Proof. Let L be an **PSBS** and V be the strongest Pythagorean relation of S.
For $l = (l_1, l_2)$ and $m = (m_1, m_2)$ are in $S \times S$. We have

$$\begin{aligned}
\pi_V^{\mathscr{P}}(l \star_1 m) &= \pi_V^{\mathscr{P}}[((l_1, l_2) \star_1 (m_1, m_2)] \\
&= \pi_V^{\mathscr{P}}(l_1 \star_1 m_1, l_2 \star_1 m_2) \\
&= \min\{\pi_L^{\mathscr{P}}(l_1 \star_1 m_1), \pi_L^{\mathscr{P}}(l_2 \star_1 m_2)\} \\
&\geq \min\{\min\{\pi_L^{\mathscr{P}}(l_1), \pi_L^{\mathscr{P}}(m_1)\}, \min\{\pi_L^{\mathscr{P}}(l_2), \pi_L^{\mathscr{P}}(m_2)\}\} \\
&= \min\{\min\{\pi_L^{\mathscr{P}}(l_1), \pi_L^{\mathscr{P}}(l_2)\}, \min\{\pi_L^{\mathscr{P}}(m_1), \pi_L^{\mathscr{P}}(m_2)\}\} \\
&= \min\{\pi_V^{\mathscr{P}}(l_1, l_2), \pi_V^{\mathscr{P}}(m_1, m_2)\} \\
&= \min\{\pi_V^{\mathscr{P}}(l), \pi_V^{\mathscr{P}}(m)\}
\end{aligned}$$

Also $\pi_V^{\mathscr{P}}(l \star_2 m) \geq \min\{\pi_V^{\mathscr{P}}(l), \pi_V^{\mathscr{P}}(m)\}, \pi_V^{\mathscr{P}}(l \star_3 m) \geq \min\{\pi_V^{\mathscr{P}}(l), \pi_V^{\mathscr{P}}(m)\}$.
Similarly, $\omega_V^{\mathscr{P}}(l \star_1 m) \leq \max\{\omega_V^{\mathscr{P}}(l), \omega_V^{\mathscr{P}}(m)\}, \omega_V^{\mathscr{P}}(l \star_2 m) \leq \max\{\omega_V^{\mathscr{P}}(l), \omega_V^{\mathscr{P}}(m)\}$, $\omega_V^{\mathscr{P}}(l \star_3 m) \leq \max\{\omega_V^{\mathscr{P}}(l), \omega_V^{\mathscr{P}}(m)\}$. Hence V is a **PSBS** of $S \times S$.
Conversely assume that V is a **PSBS** of $S \times S$, $l = (l_1, l_2)$ and $m = (m_1, m_2)$ are in $S \times S$. Now, $\min\{\pi_L^{\mathscr{P}}(l_1 \star_1 m_1), \pi_L^{\mathscr{P}}(l_2 \star_1 m_2)]$

$$\begin{aligned}
&= \pi_V^{\mathscr{P}}(l_1 \star_1 m_1, l_2 \star_1 m_2) \\
&= \pi_V^{\mathscr{P}}[(l_1, l_2) \star_1 (m_1, m_2)] \\
&= \pi_V^{\mathscr{P}}(l \star_1 m) \\
&\geq \min\{\pi_V^{\mathscr{P}}(l), \pi_V^{\mathscr{P}}(m)\} \\
&= \min\{\pi_V^{\mathscr{P}}(l_1, l_2)\}, \pi_V^{\mathscr{P}}(m_1, m_2)\} \\
&= \min\{\min\{\pi_L^{\mathscr{P}}(l_1), \pi_L^{\mathscr{P}}(l_2)\}, \min\{\pi_L^{\mathscr{P}}(m_1), \pi_L^{\mathscr{P}}(m_2)\}\}
\end{aligned}$$

If $\pi_L^{\mathscr{P}}(l_1 \star_1 m_1) \leq \pi_L^{\mathscr{P}}(l_2 \star_1 m_2)$, then $\pi_L^{\mathscr{P}}(l_1) \leq \pi_L^{\mathscr{P}}(l_2)$ and $\pi_L^{\mathscr{P}}(m_1) \leq \pi_L^{\mathscr{P}}(m_2)$. We get $\pi_L^{\mathscr{P}}(l_1 \star_1 m_1) \geq \min\{\pi_L^{\mathscr{P}}(l_1), \pi_L^{\mathscr{P}}(m_1)\}$ for all $l_1, m_1 \in S$, and
$\min\{\pi_L^{\mathscr{P}}(l_1 \star_2 m_1), \pi_L^{\mathscr{P}}(l_2 \star_2 m_2)\} \geq \min\{\min\{\pi_L^{\mathscr{P}}(l_1), \pi_L^{\mathscr{P}}(l_2)\}$,
$\min\{\pi_L^{\mathscr{P}}(m_1), \pi_L^{\mathscr{P}}(m_2)\}\}$
If $\pi_L^{\mathscr{P}}(l_1 \star_2 m_1) \leq \pi_L^{\mathscr{P}}(l_2 \star_2 m_2)$, then $\pi_L^{\mathscr{P}}(l_1 \star_2 m_1) \geq \min\{\pi_L^{\mathscr{P}}(l_1), \pi_L^{\mathscr{P}}(m_1)\}$.

$\min\{\pi_L^{\mathscr{P}}(l_1 \star_3 m_1), \pi_L^{\mathscr{P}}(l_2 \star_3 m_2)\} \geq \min\{\min\{\pi_L^{\mathscr{P}}(l_1), \pi_L^{\mathscr{P}}(l_2)\},$
$\min\{\pi_L^{\mathscr{P}}(m_1), \pi_L^{\mathscr{P}}(m_2)\}\}$
If $\pi_L^{\mathscr{P}}(l_1 \star_3 m_1) \leq \pi_L^{\mathscr{P}}(l_2 \star_3 m_2)$, then $\pi_L^{\mathscr{P}}(l_1 \star_3 m_1) \geq \min\{\pi_L^{\mathscr{P}}(l_1), \pi_L^{\mathscr{P}}(m_1)\}$.
Similarly
$\max\{\omega_L^{\mathscr{P}}(l_1 \star_1 m_1), \omega_L^{\mathscr{P}}(l_2 \star_1 m_2)\} \leq \max\{\max\{\omega_L^{\mathscr{P}}(l_1), \omega_L^{\mathscr{P}}(l_2)\},$
$\max\{\omega_L^{\mathscr{P}}(m_1), \omega_L^{\mathscr{P}}(m_2)\}\}$
If $\omega_L^{\mathscr{P}}(l_1 \star_1 m_1) \geq \omega_L^{\mathscr{P}}(l_2 \star_1 m_2)$, then $\omega_L^{\mathscr{P}}(l_1) \geq \omega_L^{\mathscr{P}}(l_2)$ and $\omega_L^{\mathscr{P}}(m_1) \geq \omega_L^{\mathscr{P}}(m_2)$.
We get $\omega_L^{\mathscr{P}}(l_1 \star_1 m_1) \leq \max\{\omega_L^{\mathscr{P}}(l_1), \omega_L^{\mathscr{P}}(m_1)\}$.
$\max\{\omega_L^{\mathscr{P}}(l_1 \star_2 m_1), \omega_L^{\mathscr{P}}(l_2 \star_2 m_2)\} \leq \max\{\max\{\omega_L^{\mathscr{P}}(l_1), \omega_L^{\mathscr{P}}(l_2)\},$
$\max\{\omega_L^{\mathscr{P}}(m_1), \omega_L^{\mathscr{P}}(m_2)\}\}$
If $\omega_L^{\mathscr{P}}(l_1 \star_2 m_1) \geq \omega_L^{\mathscr{P}}(l_2 \star_2 m_2)$, then $\omega_L^{\mathscr{P}}(l_1 \star_2 m_1) \leq \max\{\omega_L^{\mathscr{P}}(l_1), \omega_L^{\mathscr{P}}(m_1)\}$.
$\max\{\omega_L^{\mathscr{P}}(l_1 \star_3 m_1), \omega_L^{\mathscr{P}}(l_2 \star_3 m_2)\} \leq \max\{\max\{\omega_L^{\mathscr{P}}(l_1), \omega_L^{\mathscr{P}}(l_2)\},$
$\max\{\omega_L^{\mathscr{P}}(m_1), \omega_L^{\mathscr{P}}(m_2)\}\}$
If $\omega_L^{\mathscr{P}}(l_1 \star_3 m_1) \geq \omega_L^{\mathscr{P}}(l_2 \star_3 m_2)$, then $\omega_L^{\mathscr{P}}(l_1 \star_3 m_1) \leq \max\{\omega_L^{\mathscr{P}}(l_1), \omega_L^{\mathscr{P}}(m_1)\}$.
Hence L is a **PSBS**.

Theorem 3.5. *(i) If L is the* **PSBS***, then* $H_1 = \{s | s \in S : \pi_L^{\mathscr{P}}(s) = 1, \omega_L^{\mathscr{P}}(s) = 0\}$ *is either empty or is a* **SBS**.
(ii) If L is the **PSBS***, then* $H_2 = \{\langle s, \pi_L^{\mathscr{P}}(s)\rangle : 0 < \pi_L^{\mathscr{P}}(s) \leq 1, \omega_L^{\mathscr{P}}(s) = 0\}$ *is either empty or* **SBS**.
(iii) If L is the **PSBS***, then* $H_3 = \{\langle s, \pi_L^{\mathscr{P}}(s)\rangle : 0 < \pi_L^{\mathscr{P}}(s) \leq 1\}$ *is either empty or* **SBS**.
(iv) If L is the **PSBS***, then* $H_4 = \{\langle s, \omega_L^{\mathscr{P}}(s)\rangle : 0 < \omega_L^{\mathscr{P}}(s) \leq 1\}$ *is either empty or* **SBS**.

Theorem 3.6. *If L is a* **PSBS** *of* $(S, \star_1, \star_2, \star_3)$*, then* $\Box L$ *is a* **PSBS**.

Proof. Let L be an **PSBS** of a bisemiring S. Consider $L = \{\langle s, \pi_L^{\mathscr{P}}(s), \omega_L^{\mathscr{P}}(s)\rangle\}$, for all $s \in S$. Take $\Box L = M = \{\langle s, \pi_M^{\mathscr{P}}(s), \omega_M^{\mathscr{P}}(s)\rangle\}$,where $\pi_M^{\mathscr{P}}(s) = \pi_L^{\mathscr{P}}(s)$, $\omega_M^{\mathscr{P}}(s) = 1 - \pi_L^{\mathscr{P}}(s)$. Clearlt $\pi_M^{\mathscr{P}}(s \star_1 t) \geq \min\{\pi_M^{\mathscr{P}}(s), \pi_M^{\mathscr{P}}(t)\}$, $\pi_M^{\mathscr{P}}(s \star_2 t) \geq \min\{\pi_M^{\mathscr{P}}(s), \pi_M^{\mathscr{P}}(t)\}, \pi_M^{\mathscr{P}}(s \star_3 t) \geq \min\{\pi_M^{\mathscr{P}}(s), \pi_M^{\mathscr{P}}(t)\}$, $\forall s$ and t in S. Since L is an **PSBS**. Then $\pi_L^{\mathscr{P}}(s \star_1 t) \geq \min\{\pi_L^{\mathscr{P}}(s), \pi_L^{\mathscr{P}}(t)\}$ implies that $1 - \omega_M^{\mathscr{P}}(s \star_1 t) \geq \min\{(1 - \omega_M^{\mathscr{P}}(s)), (1 - \omega_M^{\mathscr{P}}(t))\}$. Thus $\omega_M^{\mathscr{P}}(s \star_1 t) \leq 1 - \min\{(1 - \omega_M^{\mathscr{P}}(s)), (1 - \omega_M^{\mathscr{P}}(t))\} = \max\{\omega_M^{\mathscr{P}}(s), \omega_M^{\mathscr{P}}(t)\}$. Therefore $\omega_M^{\mathscr{P}}(s \star_1 t) \leq \max\{\omega_M^{\mathscr{P}}(s), \omega_M^{\mathscr{P}}(t)\}$. Similarly, $\omega_M^{\mathscr{P}}(s \star_2 t) \leq \max\{\omega_M^{\mathscr{P}}(s), \omega_M^{\mathscr{P}}(t)\}$ and $\omega_M^{\mathscr{P}}(s \star_3 t) \leq \max\{\omega_M^{\mathscr{P}}(s), \omega_M^{\mathscr{P}}(t)\}$, for all $s, t \in S$. Hence $\Box L$ is a **PSBS** .

The reverse of the Theorem 3.6 fails by the Example 3.2.

$$\Box L = \begin{cases} \langle 0.85, 0.15\rangle & if\ s = s_1 \\ \langle 0.75, 0.25\rangle & if\ s = s_2 \\ \langle 0.65, 0.35\rangle & if\ s = s_3 \\ \langle 0.60, 0.40\rangle & if\ s = s_4 \end{cases} \quad L = \begin{cases} \langle 0.85, 0.45\rangle & if\ s = s_1 \\ \langle 0.75, 0.55\rangle & if\ s = s_2 \\ \langle 0.65, 0.70\rangle & if\ s = s_3 \\ \langle 0.60, 0.65\rangle & if\ s = s_4 \end{cases}$$

Clearly $\Box L$ is a **PSBS**, but L is not a **PSBS** .

Theorem 3.7. *If L is a* **PSBS** *of* $(S, \star_1, \star_2, \star_3)$ *then* $\Diamond L$ *is a* **PSBS**.

Proof. Let L be an **PSBS**. Consider $L = \{\langle s, \pi_L^{\mathscr{P}}(s), \omega_L^{\mathscr{P}}(s)\rangle\}$, $\forall s \in S$. Take $\Diamond L = M = \{\langle s, \pi_M^{\mathscr{P}}(s), \omega_M^{\mathscr{P}}(s)\rangle\}$ where $\pi_M^{\mathscr{P}}(s) = 1 - \omega_L^{\mathscr{P}}(s), \omega_M^{\mathscr{P}}(s) = \omega_L^{\mathscr{P}}(s)$ and the process of Theorem 3.6. Hence $\Diamond L$ is a **PSBS**.

The inversion of Theorem 3.7 fails as in the Example 3.2,

$$\Diamond L = \begin{cases} \langle 0.45, 0.55\rangle & if\ s = s_1 \\ \langle 0.30, 0.70\rangle & if\ s = s_2 \\ \langle 0.25, 0.75\rangle & if\ s = s_3 \\ \langle 0.15, 0.85\rangle & if\ s = s_4 \end{cases} \qquad L = \begin{cases} \langle 0.60, 0.65\rangle & if\ s = s_1 \\ \langle 0.50, 0.70\rangle & if\ s = s_2 \\ \langle 0.55, 0.75\rangle & if\ s = s_3 \\ \langle 0.40, 0.85\rangle & if\ s = s_4 \end{cases}$$

Clearly $\Diamond L$ is a **PSBS**, but L is not a **PSBS**.

Definition 3.5. Let $(S_1, \oplus_1, \oplus_2, \oplus_3)$ and $(S_2, \odot_1, \odot_2, \odot_3)$ be any two bisemirings Let $\Delta : S_1 \to S_2$ be any function and L be the **PSBS** in S_1, V be the **PSBS** in $\Delta(S_1) = S_2$, defined by $\pi_V^{\mathscr{P}}(s_2) = \sup_{s_1 \in \Delta^{-1} s_2} \pi_L^{\mathscr{P}}(s_1)$ and $\omega_V^{\mathscr{P}}(s_2) = \inf_{s_1 \in \Delta^{-1} s_2} \omega_L^{\mathscr{P}}(s_1)$, for all $s_1 \in S_1$ and $s_2 \in S_2$. Then L is called preimage of V under Δ and is denoted by $\Delta^{-1}(V)$.

Theorem 3.8. *Let $(S_1, \oplus_1, \oplus_2, \oplus_3)$ and $(S_2, \odot_1, \odot_2, \odot_3)$ be any two bisemirings. The homomorphic image of* **PSBS** *of S_1 is a* **PSBS** *of S_2.*

Proof. Let $\Delta : S_1 \to S_2$ be a homomorphism. Then $\Delta(s_1 \oplus_1 s_2) = \Delta(s_1) \odot_1 \Delta(s_2), \Delta(s_1 \oplus_2 s_2) = \Delta(s_1) \odot_2 \Delta(s_2)$ and $\Delta(s_1 \oplus_3 s_2) = \Delta(s_1) \odot_3 \Delta(s_2)$ for all $s_1, s_2 \in S_1$. Let $V = \Delta(L)$, L is the **PSBS** of S_1. Let $\Delta(s_1), \Delta(s_2) \in S_2$,

$$\begin{aligned} \pi_V^{\mathscr{P}}(\Delta(s_1) \odot_1 \Delta(s_2)) &\geq \pi_L^{\mathscr{P}}(s_1 \oplus_1 s_2) \\ &\geq \min\{\pi_L^{\mathscr{P}}(s_1), \pi_L^{\mathscr{P}}(s_2)\} \\ &= \min\{\pi_V^{\mathscr{P}}\Delta(s_1), \pi_V^{\mathscr{P}}\Delta(s_2)\} \\ \pi_V^{\mathscr{P}}(\Delta(s_1) \odot_2 \Delta(s_2)) &\geq \pi_L^{\mathscr{P}}(s_1 \oplus_2 s_2) \\ &\geq \min\{\pi_L^{\mathscr{P}}(s_1), \pi_L^{\mathscr{P}}(s_2)\} \\ &= \min\{\pi_V^{\mathscr{P}}\Delta(s_1), \pi_V^{\mathscr{P}}\Delta(s_2)\} \\ \pi_V^{\mathscr{P}}(\Delta(s_1) \odot_3 \Delta(s_2)) &\geq \pi_L^{\mathscr{P}}(s_1 \oplus_3 s_2) \\ &\geq \min\{\pi_L^{\mathscr{P}}(s_1), \pi_L^{\mathscr{P}}(s_2)\} \\ &= \min\{\pi_V^{\mathscr{P}}\Delta(s_1), \pi_V^{\mathscr{P}}\Delta(s_2)\} \end{aligned}$$

and

$$\begin{aligned} \omega_V^{\mathscr{P}}(\Delta(s_1) \odot_1 \Delta(s_2)) &\leq \omega_L^{\mathscr{P}}(s_1 \oplus_1 s_2) \\ &\leq \max\{\omega_L^{\mathscr{P}}(s_1), \omega_L^{\mathscr{P}}(s_2)\} \\ &= \max\{\omega_V^{\mathscr{P}}\Delta(s_1), \omega_V^{\mathscr{P}}\Delta(s_2)\} \\ \omega_V^{\mathscr{P}}(\Delta(s_1) \odot_2 \Delta(s_2)) &\leq \omega_L^{\mathscr{P}}(s_1 \oplus_2 s_2) \\ &\leq \max\{\omega_L^{\mathscr{P}}(s_1), \omega_L^{\mathscr{P}}(s_2)\} \\ &= \max\{\omega_V^{\mathscr{P}}\Delta(s_1), \omega_V^{\mathscr{P}}\Delta(s_2)\} \\ \omega_V^{\mathscr{P}}(\Delta(s_1) \odot_3 \Delta(s_2)) &\leq \omega_L^{\mathscr{P}}(s_1 \oplus_3 s_2) \\ &\leq \max\{\omega_L^{\mathscr{P}}(s_1), \omega_L^{\mathscr{P}}(s_2)\} \\ &= \max\{\omega_V^{\mathscr{P}}\Delta(s_1), \omega_V^{\mathscr{P}}\Delta(s_2)\}. \end{aligned}$$

Hence V is a **PSBS** of S_2.

Theorem 3.9. *Let $(S_1, \oplus_1, \oplus_2, \oplus_3)$ and $(S_2, \odot_1, \odot_2, \odot_3)$ be any two bisemirings. The homomorphic preimage of* **PSBS** *of S_2 is* **PSBS** *of S_1.*

Proof. Let $\Delta : S_1 \to S_2$ be a homomorphism. Then $\Delta(s_1 \oplus_1 s_2) = \Delta(s_1) \odot_1 \Delta(s_2)$, $\Delta(s_1 \oplus_2 s_2) = \Delta(s_1) \odot_2 \Delta(s_2)$ and $\Delta(s_1 \oplus_3 s_2) = \Delta(s_1) \odot_3 \Delta(s_2)$ $\forall\ s_1, s_2 \in S_1$.
Let $V = \Delta(L)$, where V is an **PSBS** of S_2. Now,

$$\begin{aligned}
\pi_L^{\mathscr{P}}(s_1 \oplus_1 s_2) &= \pi_V^{\mathscr{P}}(\Delta(s_1) \odot_1 \Delta(s_2)) \\
&\geq \min\{\pi_V^{\mathscr{P}}\Delta(s_1), \pi_V^{\mathscr{P}}\Delta(s_2)\} \\
&= \min\{\pi_L^{\mathscr{P}}(s_1), \pi_L^{\mathscr{P}}(s_2)\} \\
\pi_L^{\mathscr{P}}(s_1 \oplus_2 s_2) &= \pi_V^{\mathscr{P}}(\Delta(s_1) \odot_2 \Delta(s_2)) \\
&\geq \min\{\pi_V^{\mathscr{P}}\Delta(s_1), \pi_V^{\mathscr{P}}\Delta(s_2)\} \\
&= \min\{\pi_L^{\mathscr{P}}(s_1), \pi_L^{\mathscr{P}}(s_2)\} \\
\pi_L^{\mathscr{P}}(s_1 \oplus_3 s_2) &= \pi_V^{\mathscr{P}}(\Delta(s_1) \odot_3 \Delta(s_2)) \\
&\geq \min\{\pi_V^{\mathscr{P}}\Delta(s_1), \pi_V^{\mathscr{P}}\Delta(s_2)\} \\
&= \min\{\pi_L^{\mathscr{P}}(s_1), \pi_L^{\mathscr{P}}(s_2)\}
\end{aligned}$$

and

$$\begin{aligned}
\omega_L^{\mathscr{P}}(s_1 \oplus_1 s_2) &= \omega_V^{\mathscr{P}}(\Delta(s_1) \odot_1 \Delta(s_2)) \\
&\leq \max\{\omega_V^{\mathscr{P}}\Delta(s_1), \omega_V^{\mathscr{P}}\Delta(s_2)\} \\
&= \max\{\omega_L^{\mathscr{P}}(s_1), \omega_L^{\mathscr{P}}(s_2)\} \\
\omega_L^{\mathscr{P}}(s_1 \oplus_2 s_2) &= \omega_V^{\mathscr{P}}(\Delta(s_1) \odot_2 \Delta(s_2)) \\
&\leq \max\{\omega_V^{\mathscr{P}}\Delta(s_1), \omega_V^{\mathscr{P}}\Delta(s_2)\} \\
&= \max\{\omega_L^{\mathscr{P}}(s_1), \omega_L^{\mathscr{P}}(s_2)\} \\
\omega_L^{\mathscr{P}}(s_1 \oplus_3 s_2) &= \omega_V^{\mathscr{P}}(\Delta(s_1) \odot_3 \Delta(s_2)) \\
&\leq \max\{\omega_V^{\mathscr{P}}\Delta(s_1), \omega_V^{\mathscr{P}}\Delta(s_2)\} \\
&= \max\{\omega_L^{\mathscr{P}}(s_1), \omega_L^{\mathscr{P}}(s_2)\}
\end{aligned}$$

Hence L is a **PSBS** of S_1.

Theorem 3.10. *Let L be a* **PSBS** *of S, then $L_{(\alpha,\beta)}$ is a* **SBS**, *for $\alpha, \beta \in [0, 1]$.*

Theorem 3.11. *Let $(S_1, \oplus_1, \oplus_2, \oplus_3)$ and $(S_2, \odot_1, \odot_2, \odot_3)$ be any two bisemirings. If $\Delta : S_1 \to S_2$ is a homomorphism, then $\Delta(L_{(\alpha,\beta)})$ is a level* **SBS** *of an* **PSBS** *V of S_2.*

Proof. Let $\Delta : S_1 \to S_2$ be a homomorphism. Then $\Delta(s_1 \oplus_1 s_2) = \Delta(s_1) \odot_1 \Delta(s_2), \Delta(s_1 \oplus_2 s_2) = \Delta(s_1) \odot_2 \Delta(s_2)$ and $\Delta(s_1 \oplus_3 s_2) = \Delta(s_1) \odot_3 \Delta(s_2)$ for all $s_1, s_2 \in P$. Let $V = \Delta(L)$, L is a **PSBS** of S_1. By Theorem 3.8, V is a **PSBS** of S_2. Let $L_{(\alpha,\beta)}$ be a level **SBS** of L. Suppose $s_1, s_2 \in L_{(\alpha,\beta)}$. Then $\Delta(s_1 \oplus_1 s_2), \Delta(s_1 \oplus_2 s_2)$ and $\Delta(s_1 \oplus_3 s_2) \in L_{(\alpha,\beta)}$. Now, $\pi_V^{\mathscr{P}}(\Delta(s_1)) \geq \pi_L^{\mathscr{P}}(s_1) \geq \alpha, \pi_V^{\mathscr{P}}(\Delta(s_2)) \geq \pi_L^{\mathscr{P}}(s_2) \geq \alpha$. Then $\pi_V^{\mathscr{P}}(\Delta(s_1) \odot_1 \Delta(s_2)) \geq \pi_L^{\mathscr{P}}(s_1 \oplus_1 s_2) \geq \alpha$, $\pi_V^{\mathscr{P}}(\Delta(s_1) \odot_2 \Delta(s_2)) \geq \pi_L^{\mathscr{P}}(s_1 \oplus_2 s_2) \geq \alpha$ and $\pi_V^{\mathscr{P}}(\Delta(s_1) \odot_3 \Delta(s_2)) \geq \pi_L^{\mathscr{P}}(s_1 \oplus_3 s_2) \geq \alpha$, for all $\Delta(s_1), \Delta(s_2) \in S_2$. Now, $\omega_V^{\mathscr{P}}(\Delta(s_1)) \leq \omega_L^{\mathscr{P}}(s_1) \leq \beta, \omega_V^{\mathscr{P}}(\Delta(s_2)) \leq \omega_L^{\mathscr{P}}(s_2) \leq \beta$. Then $\omega_V^{\mathscr{P}}(\Delta(s_1) \odot_1 \Delta(s_2)) \leq \omega_L^{\mathscr{P}}(s_1 \oplus_1 s_2) \leq \beta$, $\omega_V^{\mathscr{P}}(\Delta(s_1) \odot_2 \Delta(s_2)) \leq \omega_L^{\mathscr{P}}(s_1 \oplus_2 s_2) \leq \beta$ and $\omega_V^{\mathscr{P}}(\Delta(s_1) \odot_3 \Delta(s_2)) \leq \omega_L^{\mathscr{P}}(s_1 \oplus_3 s_2) \leq \beta$, for all $\Delta(s_1), \Delta(s_2) \in S_2$. Hence $\Delta(L_{(\alpha,\beta)})$ is a level **SBS** of a **PSBS** V of S_2.

Theorem 3.12. *Let $(S_1, \oplus_1, \oplus_2, \oplus_3)$ and $(S_2, \odot_1, \odot_2, \odot_3)$ be any two bisemirings. If $\Delta : S_1 \to S_2$ is a homomorphism, then $L_{(\alpha,\beta)}$ is a level* **SBS** *of an* **PSBS** *L of S_1.*

Proof. Let $\Delta : S_1 \to S_2$ be a homomorphism. Then $\Delta(s_1 \oplus_1 s_2) = \Delta(s_1) \odot_1 \Delta(s_2), \Delta(s_1 \oplus_2 s_2) = \Delta(s_1) \odot_2 \Delta(s_2)$ and $\Delta(s_1 \oplus_3 s_2) = \Delta(s_1) \odot_3 \Delta(s_2)$ for all $s_1, s_2 \in S_1$. Let $V = \Delta(L)$, V is a **PSBS** of S_2. By Theorem 3.9, L is an **PSBS** of S_1. Let $\Delta(L_{(\alpha,\beta)})$ be a level **SBS** of V. Suppose $\Delta(s_1), \Delta(s_2) \in \Delta(L_{(\alpha,\beta)})$. Then $\Delta(s_1 \oplus_1 s_2), \Delta(s_1 \oplus_2 s_2)$ and $\Delta(s_1 \oplus_3 s_2) \in \Delta(L_{(\alpha,\beta)})$. Now, $\pi_L^{\mathscr{P}}(s_1) = \pi_V^{\mathscr{P}}(\Delta(s_1)) \geq \alpha, \pi_L^{\mathscr{P}}(s_2) = \pi_V^{\mathscr{P}}(\Delta(s_2)) \geq \alpha$. Then $\pi_L^{\mathscr{P}}(s_1 \oplus_1 s_2) \geq \alpha, \pi_L^{\mathscr{P}}(s_1 \oplus_2 s_2) \geq \alpha$ and $\pi_L^{\mathscr{P}}(s_1 \oplus_3 s_2) \geq \alpha$. Now, $\omega_L^{\mathscr{P}}(s_1) = \omega_V^{\mathscr{P}}(\Delta(s_1)) \leq \beta, \omega_L^{\mathscr{P}}(s_2) = \omega_V^{\mathscr{P}}(\Delta(s_2)) \leq \beta$. Then $\omega_L^{\mathscr{P}}(s_1 \oplus_1 s_2) = \omega_V^{\mathscr{P}}(\Delta(s_1) \odot_1 \Delta(s_2)) \leq \beta, \omega_L^{\mathscr{P}}(s_1 \oplus_2 s_2) = \omega_V^{\mathscr{P}}(\Delta(s_1) \odot_2 \Delta(s_2)) \leq \beta$ and $\omega_L^{\mathscr{P}}(s_1 \oplus_3 s_2) = \omega_V^{\mathscr{P}}(\Delta(s_1) \odot_3 \Delta(s_2)) \leq \beta$, for all $s_1, s_2 \in S_1$. Hence $L_{(\alpha,\beta)}$ is a level **SBS** of a **PSBS** L of S_1.

4. Pythagorean Normal Subbisemiring

Definition 4.1. A fuzzy subset L of S is said to be a **FNSBS** if

$$\begin{cases} \pi_L(u_1 \star_1 u_2) = \pi_L(u_2 \star_1 u_1) \\ \pi_L(u_1 \star_2 u_2) = \pi_L(u_2 \star_2 u_1) \\ \pi_L(u_1 \star_3 u_2) = \pi_L(u_2 \star_3 u_1) \end{cases}$$

$\forall \ u_1, u_2 \in S$.

Definition 4.2. A Pythagorean fuzzy subset L of S is said to be a **PNSBS** if

$$\begin{cases} \pi_L^{\mathscr{P}}(u_1 \star_1 u_2) = \pi_L^{\mathscr{P}}(u_2 \star_1 u_1) \\ \pi_L^{\mathscr{P}}(u_1 \star_2 u_2) = \pi_L^{\mathscr{P}}(u_2 \star_2 u_1) \\ \pi_L^{\mathscr{P}}(u_1 \star_3 u_2) = \pi_L^{\mathscr{P}}(u_2 \star_3 u_1) \end{cases}$$

$$\begin{cases} \omega_L^{\mathscr{P}}(u_1 \star_1 u_2) = \omega_L^{\mathscr{P}}(u_2 \star_1 u_1) \\ \omega_L^{\mathscr{P}}(u_1 \star_2 u_2) = \omega_L^{\mathscr{P}}(u_2 \star_2 u_1) \\ \omega_L^{\mathscr{P}}(u_1 \star_3 u_2) = \omega_L^{\mathscr{P}}(u_2 \star_3 u_1) \end{cases}$$

$\forall \ u_1, u_2 \in S$.

Theorem 4.1. *The intersection of a family of* **PNSBS** *is a* **PNSBS**.

Theorem 4.2. *If L and M are any two* **PNSBS** *of S_1 and S_2 respectively, then $L \times M$ is a* **PNSBS** *of S.*

Theorem 4.3. *Let L be the* **PNSBS** *of S and V be the strongest Pythagorean relation of S. Then L is a* **PNSBS** *of S if and only if V is a* **PNSBS** *of $S \times S$.*

Theorem 4.4. *Let $(S_1, \oplus_1, \oplus_2, \oplus_3)$ and $(S_2, \odot_1, \odot_2, \odot_3)$ be any two bisemirings.*
(i) The homomorphic image of a **PNSBS** *of S_1 is a* **PNSBS** *of S_2.*
(ii) The homomorphic preimage of a **PNSBS** *of S_2 is a* **PNSBS** *of S_1.*

5. Conclusions

The main goal of this work is to present a Pythagorean normal subbisemiring of bisemiring. We proposed image and preimage of Pythagorean subbisemiring of bisemiring. So in future, we should consider the Pythagorean spherical and cubic subbisemiring of bisemiring.

6. Acknowledgements

The author is obliged the thankful to the reviewer for the numerous and significant suggestions that raised the consistency of the ideas presented in this paper.

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