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ON SOME RELATIVE WEAKLY HYPERIDEALS AND RELATIVE PRIME BI-HYPERIDEALS IN ORDERED HYPERSEMIGROUPS AND IN INVOLUTION ORDERED HYPERSEMIGROUPS

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ABSTRACT. The aim of the present paper is to define and bring together the fundamental definitions such as relative hyperideals, relative bi-hyperideals, relative quasi-hyperideals, relative prime hyperideals, relative weakly prime hyperideals, relative semiprime hyperideals, relative prime and relative semiprime bi-hyperideals, and hyper relative regularity of dynamic algebraic character to develop the theory of hypersemigroups, and obtain the results relating to and connecting these hyperideal-theoretic definitions of this vast theory to the larger framework of the algebraic area of ordered hypersemigroups as well as of involution ordered hypersemigroups.

1. INTRODUCTION

The concepts of ideal and prime ideal was created by Dedekind [28], [29], [34] as a generalization of the concept of ideal numbers as a special subset of a ring defined by Kummer [28]. Thereafter, these notions were extended by Hilbert and Noether [32]. Noether and Artin gave classical definitions of basic notions such as one sided ideal, ideal and other algebraic notions [24], [31]. The concept of prime ideals in rings was defined and studied by McCoy [52] . The notion of a bi-ideal in semigroup was introduced by Good and Hughes [62]. Then, the notion of bi-ideal and generalized bi-ideal in semigroups was introduced and studied by Lajos [63], [64]. Thereafter, the notion of a quasi-ideal was introduced by Steinfeld [54], [55] in rings and semigroups. The theory of ordered semigroups was introduced and studied by Conard [59]. The theory of an ordered quasi-ideal in ordered semigroups was introduced by Kehayopulu et al. [43], [47]. They investigated theory of ordered semigroups relating to ordered ideals from the theory of semigroups relating to ideals. For regular ordered semigroups, left regular ordered semigroups and intra-regular ordered semigroups, one can refer [23], [42], [44], [45], [46], [47]. The notion of prime and weakly prime ideal in semigroups was given by Szasz [26], and Petrich [35] further studied this notion by extending and generalizing these notions in semigroups. Furthermore, Kehayopulu [40], [41], [45] introduced prime, weakly prime ideals in ordered semigroups

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(partially ordered semigroups) by generalizing the analogous concepts of ring theory that was introduced and studied by McCoy [52] and Steinfeld [53].

Wallace [1] introduced the notion of relative ideals (T-ideals) on semigroup S. This concept of T-ideal (or relative ideal) was generalized by Hrmov [61] when he introduced the generalized version of T-ideal (or relative ideal) in terms of (B_1, B_2)-ideal in a semigroup S $(T, B_1, B_2 \subseteq S)$. Walt [3] introduced the notions of prime and semiprime bi-ideals for an associative ring with unity. Roux [27] used the notions defined by Walt [3] and further studied the structure of a ring containing prime and semiprime bi-ideals. Recently, N. M. Khan and M. F. Ali [50] defined and studied relative ideals, relative bi-ideals, relative regularity, relative intra-regularity relative prime ideals, relative prime and relative semiprime bi-ideals, relative weakly prime ideals and relative quasi-ideals and obtained nice results connecting these notions in ordered semigroups by introducing, extending and generalizing the analogous notions in semigroups and rings that were introduced by Wallace [1], Roux [27], Kehayopulu [40], [41], [48], Saritha [60], Hrmov [61].

The notion of involution semigroups was introduced by Foulis [20]. Thereafter, rings with involution was studied by Baxter [67], and Drazin [39] investigated regular semigroups with involution. Herstein [30] studied ring with involution and Wu [18] studied intra-regular ordered semigroups with involution.

Hyperstructure theory was introduced by Marty [25]. Algebraic hyperstructures are a natural generalization of classical algebraic structures. In a classical algebraic structure, the composition of two elements is an element, while in an algebraic hyperstructure, the composition of two elements is a set. Chvalina [33] initiated the work on ordered semihypergroups. For more on ordered semihypergroups and ordered hyperstructures, one can refer [21], [22]. Basar et al. [14] introduced involution in ordered hypersemigroups. For detailed related work on ordered semigroups, hypersemigroups, ordered hypersemigroups, ordered Γ-semihypergroups and Γ-semigroups, one can refer the monographs, books and papers [2], [5], [6], [7], [8], [9], [10], [11], [12], [13], [15], [16], [17], [23], [36], [37], [38], [45], [49], [51], [56], [57], [58], [65], [66].

2. Basic Definitions and Preliminaries

A hyperstructure H is a nonvoid set equipped with an hyperoperation

$$\circ : H \times H \to \mathbb{P}^*(H) \mid (x, y) \to (x \circ y)$$

on H and an induced operation of H defined as follows:

$$* : \mathbb{P}^*(H) \times \mathbb{P}^*(H) \to \mathbb{P}^*(H) \mid (X, Y) \to X * Y$$

on $\mathbb{P}^{*(}H)$ such that

$$X * Y = \bigcup_{(x,y) \in X \times Y} (x \circ y)$$

for any $X, Y \in \mathbb{P}^*(H)$. A hypergroupoid H is denoted by $(H, \circ)$ since the operation " $*$ " depends on the hyperoperation " $\circ$ ". Obviously, we have $X \subseteq Y \Rightarrow X * D \subseteq Y * D, D * X \subseteq D * Y$ for any $X, Y, D \in \mathbb{P}^*(H)$ and $H * H \subseteq H$. For a subset X of an hypergroupoid H, we define by $(X]$ the subset of H as follows:

$$(X] = \{s \in H \mid s \leq x \text{ for some x} \in \text{X}\}.$$

If " $\leq$ " is an order relation on a hypersemigroup H, we define the order relation " $\preceq$ " on $\mathbb{P}^*(H)$ as follows:

$$\preceq := \{(X,Y) \mid \forall x \in X \; \exists \, y \in Y \text{ such that x} \leq \text{y}\}.$$

Therefore, for $X, Y \in \mathbb{P}^*(H)$, we denote $X \preceq Y$ if for every $x \in X$, there exists $y \in Y$ such that $x \leq y$. This is clearly, reflexive and transitive relation on $\mathbb{P}^*(H)$.
A hyperstructure $(H, \circ)$ is called a semihypergroup if for all $x, y, z \in H$, $(x \circ y) \circ z = x \circ (y \circ z)$, i. e.,

$$\bigcup_{m \in x \circ y} m \circ z = \bigcup_{n \in y \circ z} x \circ n.$$

A nonempty subset A of a semihypergroup $(H, \circ)$ is called a subsemihypergroup of H if $A * A \subseteq A$. An semihypergroup $(H, \circ)$ equipped with a partial order " $\leq$ " on H that is compatible with semihypergroup operation " $\preceq$ " such that for all $x, y, z \in H$,

$$x \leq y \Rightarrow z \circ x \preceq z \circ y \text{ and x} \circ \text{z} \preceq \text{y} \circ \text{z},$$

ia called an ordered semihypergroup. Throughout this paper, H will denote a semihypergroup(hypersemigroup) unless otherwise specified. Let $(H, \circ, \leq)$ be an ordered semihypergroup, $S \subseteq H$ and let X, Y be nonempty subsets of S, then we easily have the following:

(i) If $x \in X * Y$, then $x \in x^{'} \circ y$ for some $x^{'} \in X$, $y \in Y$.
(ii) If $x \in X$, $y \in Y$, then $x \circ y \subseteq X * Y$.
(iii) $X \subseteq (X]_S$;
(iv) If $X \subseteq Y$, then $(X]_S \subseteq (Y]_S$;
(v) $(X]_S * (Y]_S \subseteq (X * Y]_S$;
(vi) $((X]_S * (Y]_S]_S = (X * Y]_S$;
(vii) For every left (resp. right) S-hyperideal I of S, $(I]_S = I$.

Note that if I is a S-hyperideal(or relative hyperideal) of H, then the condition $I = (I]_S$ is equivalent to the condition: for any $a \in I$ and $b \in S$, $b \leq a \Rightarrow b \in I$. Let $S \subseteq H$. Then the intersection of all left S-hyperideals of H containing its nonempty subset I is a S-hyperideal and is of the form: $L(I) = (I \cup S * I]_S$. In a similar fashion, the intersection of all right S-hyperideals of H containing its nonempty subset I is a S-hyperideal and is of the form: $L(I) = (I \cup I * S]_S$.

Definition 2.1. [4] *Suppose that $(H, \circ, \leq)$ is an ordered hypersemigroup and $S \subseteq H$. Then, a nonempty subset I of H is called a right (resp. left) S-hyperideal(or relative hyperideal) of H if*

(i) $I * S \subseteq I$(resp. $S * I \subseteq I$); and
(ii) if $x \in I$ and $S \ni y \leq x$, then $y \in I$, i. e., if $(I]_S = I$.

A subset of S which is both a right and left S-hyperideal of S is called an S-hyperideal of H. It is to be noted that $I*S \subseteq I(resp. S*I \subseteq I)\, if\, and\, only\, if\, x \circ g \subseteq I(resp. g \circ x \subseteq I)$ for every $x \in I$ and every $s \in S$. Obviously, every right(left) S-hyperideal of an ordered hypersemigroup S is a sub-hypersemigroup of H.

Definition 2.2. [14] *An ordered semihypergroup $(H, \circ, \leq)$ with a unary operation $\star : H \longrightarrow H$ is called an ordered semihypergroup with involution if*

(i) $(x^\star)^\star = x$; and
(ii) $(x \circ y)^\star = y^\star \circ x^\star$

for all $x, y \in H$. The unary operation $\star$ is called an involution. Furthermore, if for all $a, b \in H$ with $a \leq b \Rightarrow a^\star \leq b^\star$, then we call $\star$ an order preserving involution.

Definition 2.3. *Suppose that $(H, \circ, \leq)$ is an ordered hypersemigroup with involution and $P, S \subseteq H$. Then an S-relative hyperideal P of H is called a relative prime hyperideal of H if $A, B \subseteq S$, $A * B \subseteq P$ implies $A^\star \subseteq P$ or $B^\star \subseteq P$. Equivalently: $x, y \in S$, $x \circ y \subseteq P$, then $x^\star \in P$ or $y^\star \in P$.*

Definition 2.4. *Suppose that $(H, \circ, \leq)$ is an ordered hypersemigroup with involution and $P, S \subseteq H$. Then a relative S-hyperideal P of H is called relative weakly prime hyperideal of H if for relative S-hyperideals A, B of H such that $A * B \subseteq P \Rightarrow A^\star \subseteq P$ or $B^\star \subseteq P$.*

Definition 2.5. *Suppose that $(H, \circ, \leq)$ is an ordered hypersemigroup with involution and $P, S \subseteq H$. Then a relative S-hyperideal P of H is called a relative S-semiprime hyperideal of H if for any subset A of S, $A * A \subseteq P$ implies $A^\star \subseteq P$. Equivalently: $x \in S$, $x \circ x \subseteq P$, then $x^\star \in P$.*

Definition 2.6. [4] *Let $(H, \circ, \leq)$ be an ordered hypersemigroup and let S, T be any nonempty subsets of H. Then, T is said to be a S-bi-hyperideal(or relative bi-hyperideal) of H if*

- *(i) $T * S * T \subseteq T$; and*
- *ii) for all $t \in T$, $S \ni g \leq t \Rightarrow g \in T$.*

Definition 2.7. [4] *Let $(H, \circ, \leq)$ be an ordered hypersemigroup and $S \subseteq H$. Then, H is called left S-regular(resp. right S-regular) if $g_1 \leq g_2 \circ g_1^2 (resp. g_1 \leq g_1^2 \circ g_2)$ for all $g_1, g_2 \in S$.*
Equivalently:

- *(i) $g \in (S * g^2]_S$(resp., $g \in (g^2 * S]_S$) for all $g \in S$; and*
- *(ii) $S \subseteq (S * A^2]_S$(resp. $S \subseteq (A^2 * S]_S$) for all $A \subseteq S$.*

Definition 2.8. *Suppose that $(H, \circ, \leq)$ is an ordered hypersemigroup and $S_1, S_2 \subseteq H$. Then H is called (S_1, S_2)-regular (or relative regular) if for every $s \in S$, there exist $k \in S_1 \cup S_2$ such that $s \preceq s \circ k \circ s$. Equivalently: for all $B \subseteq S, B \subseteq (S * B^2 * S]_S$.*
Equvalently:

- *(i) $s \in (s * S * s]_S$ for all $s \in S = S_1 \cup S_2$; and*
- *(ii) $B \subseteq (B * S * B]_S$ for all $B \subseteq S$.*

Definition 2.9. *Suppose that $(H, \circ, \leq)$ is an ordered hypersemigroup and S_1, S_2 are nonempty subsets of H. A nonempty subset S of H is called an (S_1, S_2)-hyperideal or a relative hyperideal of H if $S_1 * S \subseteq S$, $S * S_2 \subseteq S$ and $S_1 \cup S_2 \ni a \leq b$ for some $b \in S \Rightarrow a \in S$. If $S_1 = \varnothing$ or $S_2 = \varnothing$, then the (S_1, S_2)-hyperideal coincides with one sided relative hyperideal of H. We represent the set of all (S_1, S_2)-hyperideals of S by $I(S_1, S_2)$.*

Definition 2.10. *Suppose that $(H, \circ, \leq)$ is an ordered hypersemigroup and let S_1, S_2 be any nonempty subsets of H. Then a nonempty subset P of H is said to be an (S_1, S_2)-prime hyperideal of H if*

- *(i) P is an (S_1, S_2)-hyperideal of H; and*
- *(ii) Let $X, Y \subseteq S_1 \cup S_2$ with $X * Y \subseteq P$, then we have, either $X \subseteq P$ or $Y \subseteq P$.*

Definition 2.11. *Suppose that $(H, \circ, \leq)$ is an ordered hypersemigroup and let S_1, S_2 be any nonempty subsets of H. Then a nonempty subset P of H is said to be an (S_1, S_2)-weakly prime hyperideal of H if*

- *(i) P is an (S_1, S_2)-hyperideal of H; and* *(ii)* *Let X, Y be arbitrary (S_1, S_2)-hyperideals of H and $X, Y \subseteq S_1 \cup S_2$ with $X * Y \subseteq P$, then we have, either $X \subseteq P$ or $Y \subseteq P$.*

Definition 2.12. *Suppose that $(H, \circ, \leq)$ is an ordered hypersemigroup and let S_1, S_2 be any nonempty subsets of H. Then a nonempty subset P of H is said to be an (S_1, S_2)-semiprime hyperideal of H if*

(i) *$P \in I(S_1, S_2)$; and*
(ii) *For $S \subseteq S_1 \cup S_2$ with $S * S \subseteq P$, we have $X \subseteq P$.*

Definition 2.13. *Suppose that $(H, \circ, \leq)$ is an ordered hypersemigroup and S_1, S_2 are nonempty subsets of H. Then, $S (\neq \varnothing)$ is called an (S_1, S_2)-bi-hyperideal or a relative bi-hyperideal of H if*

(i) *$S * (S_1 \cup S_2) * S = S * S_1 * S \cup S * S_2 * S \subseteq S$; and*
(ii) *for all $s \in S$, $S_1 \cup S_2 \ni k \leq s \Rightarrow k \in S$.*

We denote the set of all relative bi-hyperideals of H by $\mathcal{B}(S_1, S_2)$.

Definition 2.14. *Suppose that $(H, \circ, \leq)$ is an ordered hypersemigroup and let $S_1, S_2 \subseteq H$. A nonempty subset Q of H is called an (S_1, S_2)-quasi hyperideal of H if*

(i) *$(Q * S_2]_S (S_1 * Q]_S \subseteq Q$, where $S = S_1 \cup S_2$; and*
(ii) *$m \in Q, S \ni n \leq m \Rightarrow n \in Q$.*

We denote an (S_1, S_2)-bi-hyperideal $B_R(s)$ and (S_1, S_2)-quasi hyperideal $Q_R(s)$ of H generated by an element s of H as follows: $B_R(s) = (s \cup s^2 \cup s * S * s]_S$ and $Q_R(s) = (s \cup ((s * S_2]_S \cap (S_1 * s]_S)]_S$ respectively, where $S = S_1 \cup S_2$.

Definition 2.15. *Suppose that $(H, \circ, \leq)$ is an ordered hypersemigroup and S_1, S_2 are nonempty subsets of H. Then $S (\neq \varnothing)$ is called an (S_1, S_2)-prime bi-hyperideal of H if*

(i) *$S \in \mathcal{B}(S_1, S_2)$; and*
(ii) *$s_1 * (S_1 \cup S_2) * s_2 = s_1 * S_1 * s_2 \cup s_1 * S_2 * s_2 \subseteq S \Rightarrow$ either $s_1 \in S$ or $s_2 \in S$.*

Equivalently: $X, Y \subseteq P = S_1 \cup S_2$ with $X * P * Y \subseteq S$ implies either $X \subseteq S$ or $Y \subseteq S$.

Definition 2.16. *Suppose that $(H, \circ, \leq)$ is an ordered hypersemigroup and S_1, S_2 are nonempty subsets of H. Then $S (\neq \varnothing)$ is called an (S_1, S_2)-semiprime bi-hyperideal of H if*

(i) *$S \in \mathcal{B}(S_1, S_2)$; and*
(ii) *$s * (S_1 \cup S_2) * s = s * S_1 * s \cup s * S_2 * s \subseteq S \Rightarrow s \in S$.*

Equivalently: $X \subseteq P = S_1 \cup S_2$ with $X * P * X \subseteq S \Rightarrow X \subseteq S$.

Suppose that H is an ordered hypersemigroup and $P \in \mathcal{B}(S_1, S_2)$, where $S_1, S_2 \subseteq H$. Then we have the following notation:

$$L(P) = \{p \in P \mid S_1 * p \subseteq P\},$$

and

$$M(P) = \{s \in L(P) \mid s * S_2 \subseteq L(P)\}.$$

3. Relative Weakly Prime Hyperideals in Involution ordered Hypersemigroups

In this section, we extend results proved in [45], [46] for intra-regular ordered semigroups, in [18], [19] for intra-regular ordered $\star$-semigroups and in [14], [37] for ordered Γ-semigroups and ordered involution hypersemigroups. We start this section with the following lemma similar to Lemma 1 of [46] which may easily be proved.

Lemma 3.1. *Suppose that $(H, \circ, \leq, \star)$ is an ordered hypersemigroup with involution and $S \subseteq H$. Then the following assertions are true.*

(i) $A \subseteq (A]_S$ *for any* $A \subseteq S$.
(ii) $(A]_S \subseteq (B]_S$ *for any* $A \subseteq B \subseteq S$.
(iii) $(A]_S * (B]_S \subseteq (A * B]_S$ *for all* $A, B \subseteq S$.
(iv) $((A]_S]_S \subseteq (A]_S$ *for all* $A \subseteq S$.
(v) *For any right (left, two-sided) relative hyperideal* I *of* S, $(I]_S = I$.
(vi) *If* I *and* J *are relative hyperideals of* H, *then* $(I * J]_S$ *and* $(I \cap J)_S$ *are also relative hyperideals of* H.
(vii) *For any* $s \in S$, $(S * s * S]_S$ *is a relative hyperideal of* H.

Lemma 3.2. *Suppose that $(H, \circ, \leq, \star)$ is an involution ordered hypersemigroup such that the involution $\star$ admits order and $S \subseteq H$. Then:*

(i) $(b * S * a]_S^\star = (a^\star * S * b^\star]_S$ *for any* $a, b \in S$.
(ii) $(S * a * S]_S^\star = (S * a^\star * S]_S$ *for any* $a \in S$.
(iii) $I^\star$ *is a relative hyperideal of* H *for any relative hyperideal* I *of* H.

Proof. *(i) Suppose that $x \in (b * S * a]_S^\star$. Since $x^\star \in (b * S * a]_S$, $\{x^\star\} \preceq b \circ s \preceq a$ for $s \in S$. Then, $\{x\} \preceq (b \circ s \circ a)^\star = a^\star \circ s^\star \circ b^\star \subseteq a^\star * S * b^\star$ as $\star$ is an order preserving involution. Therefore, $x \in (a^\star * S * b^\star]$ and thus, we receive $(b * S * a]^\star \subseteq (a^\star * S * b^\star]_S$. Furthermore, if $x \in (a^\star * S * b^\star]_S$, then we have $\{x\} \preceq a^\star \circ s \circ b^\star$ for some $s \in S$. Therefore, $\{x^\star\} \preceq b \circ s^\star \circ a \subseteq b * S * a$, since $a^\star \circ s \circ b^\star = (b \circ s^\star \circ a)^\star$. It follows that $x^\star \in (b * S * a]$ and $x \in (b * S * a]^\star$. Therefore, $(a^\star * S * b^\star]_S \subseteq (b * S * a]_S^\star$. Hence, $(b * S * a]_S^\star = (a^\star * S * b^\star]_S$.*
(ii) The proof is similar to (i).
(iii) Suppose that I is a relative hyperideal of H. Since, $S * I \subseteq I$, we obtain $(S * I)^\star \subseteq (I)^\star$. Therefore, $I^\star * S^* \subseteq I^\star$. Since $\star$ is an involution, we have $(s^\star)^\star = s$ for every $s \in S$, and thus, $S^\star = S$. Therefore, $I^\star * S \subseteq I^\star$. Similarly, $I * S \subseteq I$, we obtain $S * I^\star \subseteq I^\star$. Suppose that $a \in I^\star$, and $b \leq a$, then $b^\star \leq a^\star$. Since $a^\star \in I$ and I is a relative hyperideal of H. Therefore, $b^\star \in I$, and so $b \in I^\star$. Hence, $I^\star$ is a relative hyperideal of H. □

Theorem 3.3. *Suppose that $(H, \circ, \leq, \star)$ is an ordered hypersemigroup such that H admits an order preserving involution $\star$. A hyperideal of H is a relative prime hyperideal of H if and only if it is both relative weakly prime and relative semiprime. Furthermore, if H is commutative, then the relative prime and relative weakly prime hyperideals coincide.*
Proof. *Let I be a relative prime hyperideal of H. Then it is obviously relative weakly prime and relative semiprime hyperideal of H.*
Conversely, let P be a hyperideal of H which is relative weakly prime and relative semiprime. Suppose that $a \circ b \subseteq P$, we need to prove that $a^\star \in P$ or $b^\star \in P$. By Lemma 3.1, we have

$$(b * S * a]_S * (b * S * a]_S \subseteq (S * a * b * S]_S \subseteq (S * P * S]_S \subseteq (P]_S = P.$$

Thus, P is relative semiprime and it follows that $(b * S * a]_S^\star \subseteq P$. Now, we have the following:

$$\begin{aligned}(S * a^\star * S]_S * (S * b^\star * S]_S &\subseteq (S * a^\star * S * S * b^\star * S]_S \\ &\subseteq (S * (a^\star * S * b^\star) * S]_S \\ &= (S * ((S * b^\star)^\star * a)^\star * S]_S \\ &= (S * (b * S * a)^\star * S]_S \\ &\subseteq (S * (b * S * a]^\star * S]_S \\ &\subseteq (S * P * S]_S \\ &\subseteq P.\end{aligned}$$

We recall that $(S * a^\star * S]_S$ and $(S * b^\star * S]_S$ are relative hyperideals of H, and P is relative weakly prime hyperideal of H. Therefore, $(S*a^\star*S]_S^\star \subseteq P$ or $(S*b^\star*S]_S^\star \subseteq P$. Hence, by Lemma 3.2, we have $(S * a * S]_S \subseteq P$ or $(S * b * S]_S \subseteq P$. Now, to prove that P is relative prime, we simply need to prove that if $(S * a * S]_S \subseteq P$ then $a^\star \in P$. The other statement can be shown similarly. If $(S * a * S]_S \subseteq P$ then, we have

$$\begin{aligned}I(a) * I(a) * I(a) &= (a \cup S * a \cup a * S \cup S * a * S]_S^3 \\ &\subseteq ((a \cup S * a \cup a * S \cup S * a * S)^3]_S \\ &\subseteq (S * (a \cup S * a \cup a * S \cup S * a * S) * S]_S \\ &\subseteq (S * a * S]_S \subseteq P.\end{aligned}$$

Thus, by Lemma 3.2, we have

$$(I(a) * (I(a) * I(a))]_S = (I(a)]_S * (I(a) * I(a)]_S \subseteq ((I(a))^3]_S \subseteq (P]_S = P.$$

We know that P is relative weakly prime and $I(a)$, $(I(a) * I(a)]_S$ are relative hyperideals of H. This implies that $(I(a))^\star \subseteq P$ or $(I(a) * I(a)]_S^\star \subseteq P$. Let $(I(a))^\star \subseteq P$. Therefore, $a^\star \in (I(a))^\star \subseteq P$. Again, let $(I(a) * I(a)]_S^\star \subseteq P$. Therefore, $a^\star \circ a^\star \subseteq (I(a) * I(a))^\star \subseteq (I(a) * I(a)]_S^\star \subseteq P$ since, $a \circ a \subseteq I(a) * I(a)$ and therefore, $a = (a^\star)^\star \in P$ since, P is relative semiprime. Now, P is a relative hyperideal of H shows that $a \circ a \subseteq P$, therefore, $a^\star \in P$ as P is relative semiprime. Now, we prove the last statement. Suppose that P is a relative hyperideal of H. If P is relative prime then, obviously P is relative weakly prime. Conversely, suppose that P is relative weakly prime. Let $a \circ b \subseteq P$. Since, H is commutative, we obtain

$$\begin{aligned}I(a) * I(b) &= (a \cup S * a \cup a * S \cup S * a * S]_S * (b \cup S * b \cup b * S \cup S * b * S]_S \\ &\subseteq ((a \cup S * a \cup a * S \cup S * a * S]_S * (b \cup S * b \cup b * S \cup S * b * S)]_S \\ &\subseteq (a \circ b \cup S * a \circ b]_S.\end{aligned}$$

We recall that $(a \circ b \cup S * a \circ b]_S \subseteq (P]_S = P$. Therefore, $I(a) * I(b) \subseteq P$, and so, we obtain $(I(a))^\star \in P$ or $(I(b))^\star \in P$ since, P is relative weakly prime. Hence, $a^\star \in P$ or $b^\star \in P$. It follows that P is a relative prime hyperideal of H. □

Proposition 3.4. *Suppose that $(H, \circ, \leq, \star)$ is an ordered hypersemigroup with order preserving involution $\star$ and $S \subseteq H$. Then the following statements are equivalent:*

(i) $(A^\star * A^\star]_S = A$ *for any relative hyperideal A of S.*

(ii) $A^\star \cap B^\star = (A * B]_S$ *for any relative hyperideals A, B of H.*

(iii) $I(a)\cap I(b)=((I(a))^{\star}*(I(b))^{\star}]_S$ *for any* $a,b\in S$.

(iv) $I(a)=(I(a^{\star})*I(a^{\star})]_S$ *for any* $a\in S$.

(v) $a\in(S*a^{\star}*S*a^{\star}*S]_S$ *for any* $a\in S$.

Proof. $(i)\Rightarrow(ii)$. *Since* $A^{\star}$, $B^{\star}$ *are relative hyperideals of* H, *by the assumption and Lemma 3.1, we obtain the following:*

$$(A*B]_S\subseteq(A*S]_S\subseteq(A]_S=((A^{\star}*A^{\star}]_S]_S=(A^{\star}*A^{\star}]_S\subseteq(A^{\star}]_S=A^{\star}.$$

Similarly, we have

$$(A*B]_S\subseteq(S*B]_S\subseteq(B]_S=((B^{\star}*B^{\star}]_S]_S=(B^{\star}*B^{\star}]_S\subseteq(B^{\star}]_S=B^{\star}.$$

Therefore, $(A*B]_S\subseteq A^{\star}\cap B^{\star}$. *Moreover,* $A^{\star}\cap B^{\star}$ *is a relative hyperideal of* H *shows that*

$$\begin{aligned}A^{\star}\cap B^{\star}&=((A^{\star}\cap B^{\star})^{\star}*(A^{\star}\cap B^{\star})^{\star}]_S\\&=((A\cap B)*(A\cap B)]_S\\&\subseteq(A*B]_S.\end{aligned}$$

Thus, we obtain $(A*B]_S\subseteq A^{\star}\cap B^{\star}$ *and* $A^{\star}\cap B^{\star}\subseteq(A*B]_S$. *Hence,* $A^{\star}\cap B^{\star}=(A*B]_S$.
$(ii)\Rightarrow(iii)$. *By Lemma 3.2, we have* $(I(a))^{\star}$ *and* $(I(b))^{\star}$ *are relative hyperideals of* H. *Hence follows the result.*
$(iii)\Rightarrow(iv)$. *Since,* $I(a)=((I(a))^{\star}*(I(a))^{\star}]_S$ *by the assumption, we just need to show that* $(I(a))^{\star}=I(a^{\star})$. *Clearly,* $a^{\star}\in(I(a))^{\star}$. *Therefore,* $I(a^{\star})\subseteq(I(a))^{\star}$ *since* $(I(a))^{\star}$ *is a relative hyperideal of* H. *Now, suppose that* $x\in(I(a))^{\star}$. *We have*

$$x^{\star}\in I(a)=(a\cup a*S\cup S*a\cup S*a*S]_S.$$

This shows that $x^{\star}\leq a$ *or* $x^{\star}\preceq a\circ v$ *or* $x^{\star}\preceq v\circ a$ *or* $x^{\star}\preceq v\circ a\circ w$ *for some* $v,w\in S$. *Therefore,* $x\leq a^{\star}$ *or* $x\preceq v^{\star}\circ a^{\star}\subseteq S*a^{\star}$ *or* $x\preceq a^{\star}\circ v^{\star}\subseteq a^{\star}*S$ *or* $x^{\star}\preceq w^{\star}\circ a^{\star}\circ v^{\star}\subseteq S*a^{\star}*S$ *for some* $v^{\star},w^{\star}\in S$, *and thus,* $x\in(a^{\star}]_S$ *or* $x\in(S*a^{\star}]_S$ *or* $x\in(a^{\star}*S]_S$ *or* $x\in(S*a^{\star}*S]_S$. *Therefore,* $x\in(a^{\star}]_S\cup(S*a^{\star}]_S\cup(a^{\star}*S]_S\cup(S*a^{\star}*S]_S\subseteq(a^{\star}\cup S*a^{\star}\cup a^{\star}*S\cup S*a^{\star}*S]_S=I(a^{\star})$. *This implies* $(I(a))^{\star}\subseteq I(a^{\star})$. *Hence,* $(I(a))^{\star}=I(a^{\star})$.
$(iv)\Rightarrow(v)$. *For this, we show (1).* $I(a)=((I(a^{\star})^6*I(a)]_S$, *and (2).* $((I(a^{\star}))^6*I(a)]_S\subseteq(S*a^{\star}*S*a^{\star}*S]_S$. *This will imply that* $a\in I(a)\subseteq(S*a^{\star}*S*a^{\star}*S]_S$.
(1) By Lemma 3.1, and our assumption, we obtain

$$\begin{aligned}I(a)&=(I(a^{\star})*I(a^{\star})]_S\\&=((I(a)*I(a)]_S*(I(a)*I(a)]_S]_S\\&\subseteq((I(a)*I(a)*I(a)*I(a)]_S]_S\\&=(I(a)*I(a)*I(a)*I(a)]_S.\end{aligned}$$

Moreover,

$$\begin{aligned}(I(a)*I(a)*I(a)*I(a)]_S&=((I(a^{\star})*I(a^{\star})]_S*(I(a^{\star})*I(a^{\star})]_S*(I(a^{\star})*I(a^{\star})]_S*(I(a)]_S\\&\subseteq((I(a^{\star}))^6*I(a)]_S\\&\subseteq(S*I(a)]*I(a)\subseteq(I(a)]_S\\&=I(a)\end{aligned}$$

such that $I(a) \subseteq ((I(a^\star))^6 * I(a)]_S \subseteq I(a)$. Thus, $I(a) = ((I(a^\star))^6 * I(a)]_S$.
(2). As $(I(a))^3 \subseteq (S * a *]_S$ by Theorem 3.3, we obtain the following:

$$\begin{aligned} (I(a))^5 &= (I(a))^3 * I(a) * I(a) \\ &\subseteq (S * a * S]_S * (a \cup a * S \cup S * a \cup S * a * S]_S * (S]_S \\ &\subseteq (S * a * S * (a \cup a * S \cup S * a \cup S * a * S) * S]_S \end{aligned}$$

Obviously, $S * (a \cup a * S \cup S * a \cup S * a * S) * S \subseteq S * a * S$, and so

$(S*a*S*(a\cup a*S\cup S*a\cup S*a*S)*S]_S \subseteq (S*a*S*S*a*S]_S \subseteq (S*a*S*a*S]_S$.

Therefore, $(I(a))^5 \subseteq (S * a * S * a * S]_S$ and so, $(I(a^\star))^5 \subseteq (S * a^\star * S * a^\star * S]_S$. We now have

$$\begin{aligned} ((I(a^\star))^6 * I(a)]_S &\subseteq ((S * a^\star * S * a^\star * S]_S * I(a^\star) * I(a)]_S \\ &\subseteq ((S * a^\star * S * a^\star * S]_S * (S]_S]_S \\ &\subseteq (S * a^\star * S * a^\star * S * S]_S \\ &\subseteq (S * a^\star * S * a^\star * S]_S \end{aligned}$$

Therefore, $((I(a^\star))^6 * I(a)]_S \subseteq (S * a^\star * S * a^\star * S]_S$.
$(v) \Rightarrow (i)$. Let $x \in (A^\star * A^\star]_S$. Then $x \preceq y \circ z$ for some $y, z \in A^\star$. By the assumption, if $y \in (S * y^\star * S * y^\star * S]_S$, then $y \preceq u_1 \circ y^\star \circ u_2 \circ y^\star \circ u_3$ for some $u_i \in S$, $i = 1, 2, 3$. In a similar fashion, $z \preceq v_1 \circ z^\star \circ v_2 \circ z^\star \circ v_3$ for some $v_i \in S$, $i = 1, 2, 3$. Therefore, we have

$$y \circ z \preceq u_1 \circ y^\star \circ u_2 \circ y^\star \preceq u_3 \circ v_1 \circ z^\star \circ v_2 \circ z^\star \circ v_3 \subseteq S * y^\star * S \subseteq S * A * S \subseteq A.$$

So, $x \in (A]_S$ since $x \preceq y \circ z$, and thus, $(A^\star * A^\star]_S \subseteq (A]_S = A$. If $x \in A$, then we obtain $x \preceq w_1 \circ x^\star \circ w_2 \circ x^\star \circ w_3$ for some $w_i \in S$, $i = 1, 2, 3$ since $x \in (S * x^\star * S * x^\star * S]_S$. It is now obvious that $w_1 \circ x^\star \circ w_2 \subseteq A^\star$, and $x^\star \circ w_3 \subseteq A^\star$ as $A^\star$ is a relative ordered hyperideal of H by Lemma 3.2. Therefore, we have

$$x \preceq w_1 \circ x^\star \circ w_2 \circ x^\star \circ w_3 \subseteq A^\star * A^\star,$$

and so, we have $A \subseteq (A * A^\star]_S$. Hence, $A = (A^\star * A^\star]_S$. □

Theorem 3.5. *Suppose that $(H, \circ, \leq, \star)$ is an ordered hypersemigroup having order preserving involution $\star$. Then, the relative hyperideals of H are relative weakly prime if and only if $A^\star = (A * A]_S$ for any relative hyperideal A of H and any two relative hyperideals of H are comparable under the inclusion relation.*

Proof. *Let the relative hyperideals of H be relative weakly prime. Suppose that A, B are any relative hyperideals of H. As $B^\star$ is a relative hyperideal of H and $(A * B^\star]_S$ is relative weakly prime. Thus $A * B^\star \subseteq (A * B^\star]_S$ shows that $A^\star \subseteq (A * B^\star]_S$ or $B \subseteq (A * B^\star]_S$. If $A^\star \subseteq (A * B^\star]_S$, then $A^\star \subseteq (S * B^\star]_S \subseteq (B^\star]_S = B^\star$ and so $(A^\star)^\star \subseteq (B^\star)^\star$. This means $A \subseteq B$. If $B \subseteq (A * B^\star]_S$, then $B \subseteq (A * S]_S \subseteq (A]_S = A$. It now follows that A and B are comparable. We claim $A^\star = (A * A]_S$. As $(A * A]_S$ is weakly prime and $A * A \subseteq (A * A]_S$, we obtain $A^\star \subseteq (A * A]_S$. Also, suppose that $x \in (A * A]_S$. Then, $x \preceq a_1 \circ a_2 \subseteq A * A$ for some $a_1, a_2 \in A$. As $A^\star \subseteq (A * A]_S$, we obtain $a_1^\star \preceq u_1 \circ v_1 \subseteq A * A$ and $a_2^\star \preceq u_2 \circ v_2 \subseteq A * A$ for some $u_1, u_2, v_1, v_2 \in A$. Thus, $a_1 \preceq (u_1 \circ v_1)^\star$ and $a_2 \preceq (u_2 \circ v_2)^\star$. This shows that $x \preceq a_1 \circ a_2 \preceq (u_1 \circ v_1)^\star \circ (v_1 \circ v_2)^\star \subseteq (A * A)^\star * (A * A)^\star = A^\star * A^\star * A^\star * A^\star \subseteq A^\star$ since $A^\star$ is a relative hyperideal of H. It follows that $x \in (A^\star]_S = A^\star$. So, $(A * A]_S \subseteq A^\star$.*
Conversely, assume that A, B and P are relative hyperideals of H such that $A * B \subseteq P$. As $A^\star = (A * A]_S$, we obtain $A^\star \cap B^\star = (A * B]_S$ by Proposition 3.4. As A and B

are comparable, two cases arise. If $A \subseteq B$, then $A^\star \subseteq B^\star$, and so, $A^\star = A^\star \cap B^\star = (A * B]_S \subseteq (P]_S = P$ by Proposition 3.4. Also, if $B \subseteq A$, then $B^\star \subseteq A^\star$, and so, $B^\star = A^\star \cap B^\star = (A * B]_S \subseteq (P]_S = P$. Hence, P is a relative weakly prime hyperideal of H. □

Proposition 3.6. *Suppose that $(H, \circ, \leq, \star)$ is an involution ordered hypersemigroup and $S \subseteq H$. Then, H is relative intra-regular if and only if the relative hyperideals of H are relative semiprime hyperideal of H.*

Proof. *Let I be a relative hyperideal of H having $s \circ s \subseteq I$ for some $s \in S$. As H is hyper intra-regular, we obtain $s^\star \in (S * s \circ s * S]_S \subseteq (S * I * S]_S \subseteq (I]_S = I$ and therefore, I is semiprime.*

Conversely, let $s \in S$. It is now obvious that $(S * s^\star \circ s^\star * S]_S$ is a relative hyperideal of H. Therefore, $(s * s^\star \circ s^\star * S]_S$ is semiprime by the assumption. This shows that $s \circ s = (s^\star \circ s^\star)^\star \subseteq (S * s^\star \circ s^\star * S]_S$ since $(s^\star \circ s^\star) \circ (s^\star \circ s^\star) \subseteq S * s^\star \circ s^\star * S \subseteq (S * s^\star \circ s^\star * S]_S$. So, $s^\star \in (S * s^\star \circ s^\star * S]_S$ and thus, $s^\star \circ s^\star \subseteq (S * s^\star \circ s^\star * S]_S$. Hence, $s \in (S * s^\star \circ s^\star * S]_S$ and it follows that H is relative intra-regular. □

Proposition 3.7. *Suppose that $(H, \circ, \leq, \star)$ is an ordered hypersemigroup with involution and $S \subseteq H$. If H is relative intra-regular, then $(S * x \circ y * S]_S = (S * x^\star \circ y^\star * S]_S$ for some $x, y \in S$.*

Proof. *Suppose that $x, y \in S$. As H is relative intra-regular, it follows that $x \circ y \subseteq (S * (x \circ y)^\star \circ (x \circ y)^\star * S]_S = (S * y^\star \circ x^\star \circ y^\star \circ x^\star * S]_S \subseteq (S * x^\star \circ y^\star * S]_S$. Therefore, $x \circ y \preceq u_1 \circ x^\star \circ y^\star \circ u_2$ for some $u_1, u_2 \in S$. Therefore $u_3 \circ x \circ y \circ u_4 \preceq u_3 \circ u_1 \circ x^\star \circ y^\star \circ u_2 \circ u_4 \subseteq S * x^\star \circ y^\star * S \subseteq (S * x^\star \circ y^\star * S]_S$ for any $u_3, u_4 \in S$. This shows that $S * x \circ y * S \subseteq (S * x^\star \circ y^\star *]_S$, therefore, $(S * x \circ y * S]_S \subseteq ((S * x^\star \circ y^\star * S]_S]_S = (S * x^\star \circ * S]_S$ by Lemma 3.1. We obtain $(S * x^\star \circ y^\star * S]_S \subseteq (S * x \circ y * S]_S$. Hence, $(S * x \circ y * S]_S = (S * x^\star \circ y^\star * S]_S$.* □

Proposition 3.8. *Suppose that $(H, \circ, \leq, \star)$ is an ordered hypersemigroup with order preserving involution $\star$ and $S \subseteq H$. If the relative hyperideals of H are semiprime, then the following are true:*

(i) $I(s) = (S * s * S]_S$ for any $s \in S$; and
(ii) $I(x \circ y) = I(x) \cap I(y)$ for any $x, y \in S$.

Proof. *(i) Suppose $s \in S$. Since $(S * s * S]_S$ is a relative hyperideal of H and so, it is relative semiprime. Since $(s \circ s) \circ (s \circ s) = (s \circ s)^2 = s^4 \subseteq (S * s * S]_S$ gives $s^\star \circ s^\star = (s \circ s)^\star \subseteq (S * s * S]_S$. In a similar fashion, $s \in (S * s * S]_S$ so that $I(s) \subseteq (S * s * S]_S$. Moreover, $(S * s * S]_S \subseteq (s \cup s * S \cup S * s \cup S * s * S]_S = I(s)$. Hence, $I(x) = (S * x * S]_S$.*

(ii) As $x \circ y \subseteq I(x) * S \subseteq I(x)$, we obtain $I(x \circ y) \subseteq I(x)$. Also, $I(x \circ y) \subseteq I(y)$ since $x \circ y \subseteq S * I(y) \subseteq I(y)$. So, $I(x \circ y) \subseteq I(x) \cap I(y)$. If $z \in I(x) \cap I(y)$, then $z \in (S * x * S]_S \cap (S * y * S]_S$ by (i), and so $z \preceq u_1 \circ x \circ u_2$ and $z \preceq v_1 \circ y \circ v_2$ for some $u_1, u_2, v_1, v_2 \in S$. Since

$$(y \circ v_2 \circ u_1 \circ x)^2 = (y \circ v_2 \circ u_1 \circ x) \circ (y \circ v_2 \circ u_1 \circ x) \subseteq (S * x \circ y * S]_S = I(x \circ y)$$

and that $I(x \circ y)$ is relative semiprime. So, $(y \circ v_2 \circ u_1 \circ x)^\star \subseteq I(x \circ y)$. So, $z^\star \circ z^\star \preceq (u_1 \circ x \circ u_2)^\star \circ (v_1 \circ y \circ v_2)^\star = u_2^\star \circ (y \circ v_2 \circ u_1 \circ x)^\star \circ v_1^\star \subseteq I(x \circ y)$, and so, $z^\star \circ z^\star \subseteq (I(x \circ y)]_S = I(x \circ y)$. This implies that $z \in I(x \circ y)$, then $I(x) \cap I(y) \subseteq I(x \circ y)$.
□

Theorem 3.9. *Suppose that $(H, \circ, \leq, \star)$ is an involution ordered hypersemigroup such that the involution admits the order and $S \subseteq H$. Then the relative hyperideals of H are* *relative prime if and only if H is relative intra-regular and any two relative hyperideals of H are comparable under the inclusion relation.*

Proof. *If the relative hyperideals of $H$are relative prime, then they are relative weakly prime and thus, they are comparable by Theorem 3.5. Suppose that $s \in S$. Since, $(S * s^\star \circ s^\star * S]_S$ is a relative hyperideal by Lemma 3.1, and therefore, it is relative prime. So, $(s \circ s) \circ (s \circ s) = s^4 \subseteq (S * s^\star \circ s^\star * S]_S$ since $(s^\star)^4 \circ (s^\star)^4 \subseteq (S * s^\star \circ s^\star * S]_S$. In a similar fashion, we have $(s^\star \circ s^\star) = (s^\star)^2 \subseteq (S * s^\star \circ s^\star * S]_S$ and $s \in (S * s^\star \circ s^\star * S]_S$. It follows that H is relative intra-regular.*

Conversely, assume that H is relative intra-regular and any two relative hyperideals of H are comparable under the inclusion relation $\subseteq$. Suppose that T is any relative hyperideal of H and $a \circ b \subseteq T$, where $a, b \in S$. We claim that $a^\star \in T$ or $b^\star \in T$. By Proposition 3.6, $I(a)$ is relative semiprime. Thus, $a \circ a \subseteq I(a)$ implies $a^\star \in I(a)$. One can similarly show that $b^\star \in I(b)$. By the assumption, we obtain $I(a) \subseteq I(b)$ or $I(b) \subseteq I(a)$. If $I(a) \subseteq I(b)$, then $a^\star \in I(a) = I(a) \cap I(b) = I(a \circ b) \subseteq T$ by Proposition 3.8. If $I(b) \subseteq I(a)$, then we obtain

$$b^\star \in I(b) = I(a) \cap I(b) = I(a \circ b) \subseteq T.$$

4. Relative Prime, Weakly Prime and Semiprime bi-hyperideals of Ordered Hypersemigroups

We generalize the following results proved in [27], [60] for an associative ring without unity, and in [50] for ordered semigroups. First, we state the follwing Theorem which will be needed to prove some results in the sequel of the present paper.

Theorem 4.1. [4] *Suppose $(S, \circ, \leq)$ is an ordered hypersemigroup and $G \subseteq S$. Then, for any G-hyperideal of S, the following assertions are equivalent:*

(i) I is G-weakly prime.
(ii) If $x, y \in G$, such that $(x * G * y]_G \subseteq I$, then either $x \in I$ or $y \in I$.
(iii) If $x, y \in G$ such that $I_R(x) * I_R(y) \subseteq I$, then either $x \in I$ or $y \in I$.
(iv) if X and Y are left G-hyperideals of S such that $X * Y \subseteq I$, then either $X \subseteq I$ or $Y \subseteq I$.
(v) if X and Y are right G-hyperideals of S such that $X * Y \subseteq I$, then either $X \subseteq I$ or $Y \subseteq I$.
(vi) If X is a right G-hyperideal and Y is a left G-hyperideal of S such that $X*Y \subseteq I$, then either $X \subseteq I$ or $Y \subseteq I$.

Theorem 4.2. *Let $(H, \circ, \leq)$ be an ordered hypersemigroup and S be a sub-hypersemigroup of H. An S-hyperideal of H is S-weakly semiprime if and only if one of the four equivalent assertions are true in H.*

(i) For every $s \in S$, we have $(s * S * s]_S \subseteq P \Rightarrow s \in P$.
(ii) For $s \in S$, we have $(I_R(s))^2 \subseteq P \Rightarrow s \in P$.
(iii) For right S-hyperideal A of H, we have $A^2 \subseteq P \Rightarrow A \subseteq P$.
(iv) For left S-hyperideal A of H, we have $A^2 \subseteq P \Rightarrow A \subseteq P$.

Proposition 4.3. *Suppose that $(H, \circ, \leq)$ is an ordered hypersemigroup and S_1, S_2 are subhypersemigroups of H and $S_2 * S_1 \subseteq S_1 \cup S_2$. Suppose that $P \in I(S_1, S_2)$ and $P \in \mathcal{B}(S_1, S_2)$. Then, the (S_1, S_2)-bi-hyperideal P of H is (S_1, S_2)-hyperprime if and only if $R * L \subseteq P$ such that $R \in I(\varnothing, S_2)$, $L \in I(S_1, \varnothing)$ and $R, L \subseteq S_1 \cup S_2$ implies either $R \subseteq P$ or $L \subseteq P$.*

Proof. $\Rightarrow$ *Suppose that P is a (S_1, S_2)-prime bi-hyperideal of ordered hypersemigroup*

$(H,\circ,\leq)$, and $R*L\subseteq P$. Let $R\nsubseteq P$. For all $a\in L$ and $b\in R\setminus P$, we obtain

$$\begin{aligned} b*(S_1\cup S_2)*a &= b*S_1*a\cup b*S_2*a \\ &\subseteq R*S_1*L\cup R*S_2*L \\ &\subseteq R*L\cup R*L \\ &\subseteq R*L\subseteq P. \end{aligned}$$

Since, P is an (S_1,S_2) prime-bi-hyperideal of H and $b\notin P$, we receive $a\in P$ for all $a\in L$. So, $L\subseteq P$.
$\Leftarrow$ Suppose that $R*L\subseteq P\Rightarrow R\subseteq P$ or $L\subseteq P$ for any $R\in I(\varnothing,S_2)$ and $L\in I(S_1,\varnothing)$. Suppose that $s_1,s_2\in S=S_1\cup S_2$ with $s_1*S*s_2\subseteq P$. Therefore,

$$(s_1*S_2]_S*(S_1*s_2]_S\subseteq(s_1*S_2*S_1*s_2]_S\subseteq(s_1*S*s_2]_S\subseteq(P]_S.$$

As, $(s_1*S_2]_S\subseteq I(\varnothing,S_2)$ and $(S_1*s_2]_S\subseteq I(S_1,\varnothing)$, we obtain $(s_1*S_2]_S\subseteq P$ or $(S_1*s_2]_S\subseteq P$. Since, $s_1,s_2\in S$, the proof now proceeds as follows:
Case (i). Let $s_2\in S_1,s_1\in S_2$. Take $(s_1*S_2]_S\subseteq P$. Therefore, $s_1^2\subseteq P$. Thus, $S_1*(s_1)$ and $S_2*(s_1)$ are $(S_1,\varnothing)$-hyperideal of H and $(\varnothing,S_2)$-hyperideal of H. We now have the following:

$$\begin{aligned} S_1*(s_1)*S_2*(s_1) &= (s_1\cup S_1*s_1]_S*(s\cup s_1*S_2]_S \\ &\subseteq ((s_1\cup S_1*s_1)*(s_1\cup s_1*S_2)]_S \\ &= (s_1^2\cup s_1^2*S_2\cup S_1*s_1^2\cup S_1*s_1^2*S_2]_S \\ &\subseteq (P\cup P*S_2\cup S_1*P\cup S_1*P*S_2]_S \\ &\subseteq (P\cup P\cup P\cup P]_S\subseteq(P]_S=P. \end{aligned}$$

By the assumption, we have either $S_1*(s_1)\subseteq P$ or $S_2*(s_1)\subseteq P$. Thus, $s_1\in P$. Let $(S_1*s_2]_S\subseteq P$. Then, $s_2^2\subseteq P$. Similarly, we receive $s_2\in P$. Therefore, P is a (S_1,S_2)-prime bi-ideal of H.
Case (ii). Let $s_1\in S_1$ and $s_2\in S_2$. Thus, obviously, $s_1*s_2\subseteq P$. As, $S_2*(s_1)$ and $S_1*(s_2)$ are $I(\varnothing,S_2)$ and $I(S_1,\varnothing)$ hyperideals of H, respectively, we obtain the following:

$$\begin{aligned} S_2*(s_1)*S_1*(s_2) &= (s_1\cup s_1*S_2]_S*(s_2\cup S_1*s_2]_S \\ &\subseteq ((s_1\cup s_1*S_2)*(s_2\cup S_1*s_2]_S \\ &= (s_1*s_2\cup s_1*S_1*s_2\cup s_1*S_2*s_2\cup s_1*S_2*S_1*s_2]_S \\ &\subseteq (s_1*s_2*\cup s_1*S*s_2\cup s_1*S*s_2\cup s_1*S*s_2]_S \\ &\subseteq (P\cup P\cup P\cup P]_S\subseteq(P]_S=P. \end{aligned}$$

By the hypothesis, we obtain either $S_2*(s_1)\subseteq P$ or $S_1*(s_2)\subseteq P$. Therefore, either $s_1\in P$ or $s_2\in P$. Hence, P is an (S_1,S_2)-prime bi-hyperideal of H.
Case (iii). Let $s_1,s_2\in S_1$ or $s_1,s_2\in S_2$. Thus, by combining the preceding cases, one can prove that either $s_1\in P$ or $s_2\in P$. Hence, P is an (S_1,S_2)-prime bi-hyperideal of H.

Proposition 4.4. *Suppose that $(H,\circ,\leq)$ is an ordered hypersemigroup and S_1, S_2 are sub-hypersemigroups of H such that $S_1*S_2\subseteq S=S_1\cup S_2$ and $S_2*S_1\subseteq S$. Then, an (S_1,S_2)-prime bi-hyperideal of H is either $(\varnothing,S_2)$-prime hyperideal of H or $(S_1,\varnothing)$-prime-hyperideal of H.*
Proof. *Suppose that P is an (S_1,S_2)-prime bi-hyperideal of H. We only require to prove*

that $P \in I(\varnothing, S_2)$ or $P \in I(S_1, \varnothing)$. Obviously,

$$\begin{aligned}(P * S_2]_S * (S_1 * P]_S &\subseteq (P * S_2 * S_1 * P]_S \\ &\subseteq (P * S * P]_S \\ &\subseteq (P]_S = P.\end{aligned}$$

As, $(P * S_2]_S \in I(\varnothing, S_2)$, $(S_1 * P]_S \in I(S_1, \varnothing)$ and $(P * S_2]_S$, $(S_1 * P]_S \subseteq S$. By Proposition 4.3, we obtain either $(P * S_2]_S \subseteq P$ or $(S_1 * P]_S \subseteq P$. Therefore, either $P * S_2 \subseteq P$ or $S_1 * P \subseteq P$. Let $s \in S = S_1 \cup S_2$ and $p \in P$ with $q \leq p$. As, $P \in \mathcal{B}(S_1, S_2)$, we obtain $q \in P$.

Lemma 4.5. *Suppose that H is an ordered hypersemigroup and S_1, S_2 are sub-hypersemigroups of H. Let $P \in \mathcal{B}(S_1, S_2)$. Then, $L(P) \in I(S_1, \varnothing)$.*

Proof. *Suppose that $p \in L(P)$ and $s \in S_1$. Thus, $s \circ p \subseteq S_1 * p \subseteq P$, and $S_1 * (s \circ p) \subseteq S_1 * S_2 * p \subseteq S_1 * p \subseteq P \Rightarrow s \circ p \subseteq L(T)$. Consider, $m \in L(P) \subseteq P$ with $S_1 \ni s_1 \leq m$. So, $s_1 \in P$ since $P \in \mathcal{B}(S_1, S_2)$. Since, $s_1 \leq m \Rightarrow s_2 \circ s_1 \preceq s_2 \circ m$ for all $s_2 \in S_1$. Therefore,*

$$s_2 \circ s_1 \preceq s_2 \circ m \subseteq S_1 * m \subseteq (S_1 * m]_S \subseteq (P] = P.$$

It follows that $s_2 \circ s_1 \subseteq P$ for all $s_2 \in S_1$. So, $S_1 * s_1 \subseteq P \Rightarrow s_1 \in L(P)$. Hence, $L(P) \in I(S_1, \varnothing)$.

Proposition 4.6. *Suppose that $(H, \circ, \leq)$ is an ordered hypersemigroup and S_1, S_2 are sub-hypersemigroups of H. Let $P \in \mathcal{B}(S_1, S_2)$. Then, $M(P)$ is the unique largest (S_1, S_2)-hyperideal of H contained in P.*

Proof. *Obviously, $M(P) \subseteq L(P) \subseteq P$. Consider $x \in M(P), x_1 \in S_1$ and $x_2 \in S_2$. Thus, $x \in P, x \in L(P)$, $S_1 * x \subseteq P$ and $x * S_2 \subseteq L(P)$. Obviously, $x_1 \circ x \subseteq S_1 * x \subseteq S_1 \Rightarrow x_1 \circ x \subseteq P$. Moreover, $S_1 * (x_1 \circ x) \subseteq S_1 * S_1 * x \subseteq S_1 * x \subseteq P \Rightarrow x_1 \circ x \subseteq L(P)$. Furthermore, $x \circ x_2 \subseteq x * S_2 \subseteq L(P) \Rightarrow x \circ x_2 \subseteq L(P)$. Next. we prove that $x_1 \circ x \subseteq M(P)$ and $x \circ x_2 \subseteq M(P)$. Since, $(x \circ x_2) * S_2 \subseteq x * S_2 * S_2 \subseteq x * S_2 \subseteq L(P)$, we obtain $x \circ x_2 \subseteq M(P)$. Again, $(x_1 \circ x) * S_2 \subseteq S_1 * x * S_2 \subseteq S_1 * L(P) \subseteq L(T) \Rightarrow x \circ x_2 \subseteq M(P)$. Suppose that $y \in M(P)$, $S_1 \cup S_2 = S \ni k \leq y$. Then, $k \in L(P)$ since $M(P) \subseteq L(P)$ and $L(P) \in I(S_1, \varnothing)$. As, $k \leq y$ and $k \in S_1$ or $k \in S_2$, we obtain $k \circ h \preceq y \circ h$ for all $h \in S_2$. Then, $k \circ h \preceq y \circ h \preceq y * S_2 \subseteq L(P) \Rightarrow k \circ h \subseteq L(P)$ for all $h \in S_2$. So, $k * S_2 \subseteq L(P) \Rightarrow k \in M(P)$. Hence, $M(P) \in I(S_1, S_2)$. Let I be any $I(S_1, S_2)$-hyperideal of H and $I \subseteq P$. For $i \in I$, $i \in P$ and $S_1 * i \subseteq I \subseteq P \Rightarrow I \subseteq L(P)$. Furthermore, for $i \in L(P)$, since $i * S_2 \subseteq I \subseteq L(P)$, we obtain $i \in M(P)$. Hence, $I \subseteq M(P)$.*

Proposition 4.7. *Suppose that $(H, \circ, \leq)$ is an ordered hypersemigroup and H is a sub-hypersemigroup of H. Let P be a S-prime bi-hyperideal of H. Then, $M(P)$ is a weakly prime S-hyperideal of H.*

Proof. *Suppose that P is S-prime bi-hyperideal of H. As, P is S-prime bi-hyperideal of H, $M(P) \in I(S, S)$. Now, we need to show that $M(P)$ is S-weakly prime. Consider $x, y \in S$ with $I_R(x) * I_R(y) \subseteq M(P)$. Therefore, by Theorem 4.1, either $I_R(x) \subseteq P$ or $I_R(y) \subseteq P$. Since $M(P)$ is the unique largest S-hyperideal in P, we obtain $I_R(x) \subseteq M(P)$ or $I_R(y) \subseteq M(P)$. So, we have either $x \in M(P)$ or $y \in M(P)$. Hence, by Theorem 4.1, $M(P)$ is S-weakly prime S-hyperideal of H.*

Proposition 4.8. *Suppose that $(H, \circ, \leq)$ is an ordered hypersemigroup and S_1, S_2 are nonempty subsets of H. Suppose that P is an (S_1, S_2)-semiprime bi-hyperideal of H and $B \in I(S_1, S_2)$. Then, $B^2 \subseteq P$ implies that $B \subseteq P$ for each $B \in I(S_1, S_2)$.*

Proof. *Suppose that P is (S_1, S_2)-semiprime bi-hyperideal of H with $B^2 \subseteq P$. On the contrary, assume that $B \nsubseteq P$, then there exists $b \in B$ such that $b \notin P$. Since, $B \in I(S_1, S_2)$, we obtain*

$$b * (S_1 \cup S_2) * b \subseteq B * (S_1 \cup S_2) * B = B * S_1 * B \cup B * S_2 * B \subseteq B^2 \cup B^2 = B^2 \subseteq P.$$

As, P is (S_1, S_2)-semiprime, we obtain $b \in P$ which is a contradiction. Hence, $B \subseteq P$.

Proposition 4.9. *Suppose that $(H, \circ, \leq)$ is an ordered hypersemigroup and S is a sub-hypersemigroup of H. Let P be a S-bi-hyperideal of H. Then, $M(P)$ is a S-weakly semiprime hyperideal of H.*

Proof. *Suppose that P is S-bi-hyperideal of H. By Proposition 4.6, we receive $M(P)$ is S-hyperideal of H. We need to show that $M(P)$ is S-weakly semiprime. Consider $s \in S$ with $(I_R(h))^2 \subseteq M(P)$. By Theorem 4.2, $I_R(h) \subseteq P$ as $(I_R(h))^2 \subseteq P$. Since, $M(P)$ is the unique largest S-hyperideal of P, we obtain $I_R(h) \subseteq M(P)$ which implies that $h \in M(P)$. Hence, by Theorem 4.2, $M(P)$ is an S-weakly semiprime hyperideal of H.*

Proposition 4.10. *Suppose that $(H, \circ, \leq)$ is an ordered hypersemigroup and S_1, S_2 are sub-hypersemigroups of H with $S_2 * S_1 \subseteq S_1 \cup S_2$. Then, each (S_1, S_2)-semiprime bi-hyperideal of H is an (S_1, S_2)-quasi-hyperideal of H.*

Proof. *Suppose that P is (S_1, S_2)-semiprime bi-hyperideal of H. Suppose $t \in P * S_2 \cap S_1 * P$. Therefore, $t \in P * S_2$ and $t \in S_1 * P$. Now,*

$$\begin{aligned} t * (S_1 \cup S_2) * t &\subseteq (P * S_2) * (S_1 \cup S_2) * (S_1 * P) \\ &= (P * S_2 * S_1^2 \cup P * S_2^2 * S_1) \\ &\subseteq P * S_2 * S_1 * P \cup P * S_2 * S_1 * P \\ &\subseteq P * (S_1 \cup S_2) * P \subseteq P. \end{aligned}$$

As, P is an (S_1, S_2)-semiprime bi-hyperideal of H, we obtain $t \in P$. Hence, $(P * S_2 \cap S_1 * P) \subseteq P$. Moreover, $h \in P$, $S_1 \cup S_2 \ni t \leq h$. Then, since $P \in \mathcal{B}(S_1, S_2)$, $t \in P$. Hence, P is an (S_1, S_2)-quasi-hyperideal of H.

Proposition 4.11. *Suppose that $(H, \circ, \leq)$ is an ordered hypersemigroup and $S_1, S_2 \subseteq H$ with $S_2 * S_1, S_1 * S_2 \subseteq S = S_1 \cup S_2$. Then, H is (S_1, S_2)-regular if and only if each (S_1, S_2)-bi-hyperideal of H is (S_1, S_2)-hypersemiprime.*

Proof. *$\Rightarrow$ Suppose that H is an (S_1, S_2)-regular ordered hypersemigroup and $P \in \mathcal{B}(S_1, S_2)$. Let $s * S * S \subseteq P$ for $s \in S$. Therefore, since H is (S_1, S_2)-regular, we have $k \in S$ with $s \preceq s \circ k \circ a$. But, $s \circ k \circ s \subseteq s * S * S \subseteq P$. Since, $S \ni s \preceq s \circ k \circ s$ and $P \in \mathcal{B}(S_1, S_2)$, $s \in P$. Hence, P is (S_1, S_2)-semiprime.*

$\Leftarrow$ Suppose that all (S_1, S_2)-bi-hyperideal of H is (S_1, S_2)-semiprime. Suppose that $s \in S$. Obviously, $A = (s * S * s] \in \mathcal{B}(S_1, S_2)$. Thus, we have either $s \in S_1$ or $s \in S_2$. Suppose that $s \in S_1$. We prove that $A * S * A = A * S_1 * S \cup A * S_2 * A \subseteq A$. Then,

$$\begin{aligned} A * S_1 * A &= (s * S * s]_S * S_1 * (s * S * s]_S \\ &\subseteq ((s * S_1 * s \cup s * S * s) * S_1 * (s * S_1 * s \cup s * S_2 * s)]_S \\ &= (s * S_1 * s * S_1 * s * S_1 * s \cup s * S_1 * s * S_1 * s * S_2 * s \cup s * \\ &\quad S_2 * s * S_1 * s * S_1 * s \cup s * S_2 * s * S_1 * s * S_2 * s]_S \\ &\subseteq (s * S_1^3 * s \cup s * S_1^2 * S_2 * s \cup s * S_2 * S_1^2 * s \cup s * S_2 * S_1 * S_2 * s]_S \\ &\subseteq (s * S_1 * s \cup s * S * s \cup s * S * s \cup s * S * s]_S \\ &\subseteq (s * S * s]_S = A. \end{aligned}$$

Obviously, $((s * S * s]_S]_S = (s * S * s]_S$*. By the assumption,* $(s * S * s]_S$ *is* (S_1, S_2)*-semiprime for* $s \in S$*. As,* $s * S * s \subseteq (s * S * s]_S \Rightarrow s \in (s * S * s]_S \Rightarrow s \preceq s \circ k \circ s$ *for some* $k \in S$*. Hence,* H *is* (S_1, S_2)*-regular.*

Proposition 4.12. *Suppose that* $(H, \circ, \leq)$ *is a commutative ordered hypersemigroup and* S_1, S_2 *are sub-hypersemigroups of* H *with* $S_2 * S_1 \subseteq S_1 \cup S_2$*. Therefore,* H *is* (S_1, S_2)*-regular if and only if each* (S_1, S_2)*-hyperideal of* H *is* (S_1, S_2)*-semiprime.*

Proof. $\Rightarrow$ *Suppose that* H *is* (S_1, S_2)*- regular commutative ordered hypersemigroup and* $P \in I(S_1, S_2)$*. Suppose* $k^2 \subseteq P$ *for some* $k \in S = S_1 \cup S_2$*. Thus, there exists* $h \in S$ *such that* $k \preceq k \circ h \circ k$*. For* $h \in S_1$*, we obtain* $k \preceq k \circ h \circ k = (k \circ h) \circ k = h \circ (k \circ k) = h \circ k^2 \subseteq S_1 * P \subseteq P$*. This implies that* $k \in P$*. Since* $P \in I(S_1, S_2)$*. Let* $h \in S_2$*. Then, we receive*

$$k \preceq k \circ h \circ k = k \circ (h \circ k) = k \circ (k \circ h) = k^2 \circ h \subseteq P * S_2 \subseteq P \Rightarrow k \in P.$$

Hence, P *is* (S_1, S_2)*-semiprime.*

$\Leftarrow$ *Suppose that each* (S_1, S_2)*- hyperideal of* H *is* (S_1, S_2)*-semiprime. Suppose that* $s \in S = S_1 \cup S_2$*. Since,* $(s^2 * S]_S \in I(S_1, S_2)$*. By the assumption,* $(s^2 * S]_S$ *is* (S_1, S_2)*-semiprime. As,* $s^4 \subseteq (s^2 * S]_S \Rightarrow s^2 \subseteq (s * S * s]_S \Rightarrow s \in (s^2 * S]_S$*, therefore, we have* $s \preceq s^2 \circ k$ *for some* $k \in S$*. This implies that* $s \preceq s \circ s \circ k = s \circ k \circ s$ *for some* $k \in S$*. Hence,* H *is* (S_1, S_2)*-regular.*

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