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EXISTENCE OF GENERALIZED SUPER QUASI-CONSTANT CURVATURE

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Abstract. In this paper we define and study a new kind of quasi-constant curvature called generalized super-quasi-constant curvature. The existence of a generalized super quasi-constant curvature has also been verified by an example. We prove that a super quasi-umbilical hypersurface of a constant curvature manifold is a generalized super quasi-constant curvature manifold. Furthermore, it is proved that manifold of generalized super quasi-constant curvature is a super quasi-Einstein manifold. We have also introduced a sufficient condition to be a Super quasi-Einstein manifold. Lastly, we construct an example super quasi-Einstein manifold in general form.

1. Introduction

An Einstein manifold is a Riemannian or pseudo-Riemannian manifold whose Ricci tensor Ric of type $(0,2)$ is non-zero and proportional to the metric tensor. Einstein manifolds form a natural subclass of various classes of Riemannian or semi-Riemannian manifolds by a curvature condition imposed on their Ricci tensor [1]. Also, the Einstein manifold plays a very important role in Riemannian geometry as well as in general theory of relativity.

In 2000, M.C. Chaki and R.K. Maity introduced and studied the quasi-Einstein manifold [4]. A non flat n-dimensional Riemannian manifold (M^n, g) $(n > 2)$ is said to be a quasi-Einstein manifold if its Ricci tensor Ric of type $(0,2)$ is non-zero and satisfies the following condition

$$Ric(X,Y) = ag(X,Y) + bA(X)A(Y), \tag{1.1}$$

where a, b are scalars and A is a non-zero 1-form such that

$$g(X,\rho) = A(X), \quad g(\rho,\rho) = A(\rho) = 1, \tag{1.2}$$

for all vector field X. ρ being a unit vector field called the generator of the manifold. Also, the 1-form A is called the associated 1-form. From (1.1), it follows that every Einstein manifold is a subclass of a quasi-Einstein manifold. This manifold is denoted by $(QE)_n$.

Quasi Einstein manifolds arose during the study of exact solutions of the Einstein field equations as well as during considerations of semi-Euclidean spaces. Several authors have

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studied Einstein's field equations. For example, in [11], Naschie turned the tables on the theory of elementary particles and showed the expectation number of elementary particles of the standard model using Einstein's unified field equation. He also discussed possible connections between $G\ddot{o}del's$ classical solution of Einstein's field equations and E-infinity in [10]. Also quasi-Einstein manifolds have some importance in the general theory of relativity. For instance, the Robertson-Walker spacetimes are quasi Einstein manifolds. Further, quasi Einstein manifold can be taken as a model of the perfect fluid spacetime in general relativity [5]. Perfect fluid spacetimes in n-dimensions subjected to the restriction $D_m C^m_{jkl} = 0$, where C is the Weyl conformal curvature tensor, recently investigated in [8] by Mantica, Molinari and De (see also [9]).

A non flat Riemannian manifold (M^n, g) $(n \geq 2)$ is said to be a generalized quasi-Einstein manifold [2] if its Ricci tensor Ric of type $(0, 2)$ is non-zero and satisfies the following condition

$$Ric(X, Y) = ag(X, Y) + bA(X)A(Y) + cB(X)B(Y), \tag{1.3}$$

where a, b, c are scalars and A, B are two non-zero 1-forms associated with two orthogonal unit vectors ρ and σ are defined by

$$g(X, \rho) = A(X), \quad g(X, \sigma) = B(X), \quad g(\rho, \rho) = 1, \quad g(\sigma, \sigma) = 1. \tag{1.4}$$

An n-dimensional generalized quasi-Einstein manifold is denoted by $G(QE)_n$.

In [3], Chaki introduced the concept of a super quasi-Einstein manifold as a generalization of the quasi-Einstein manifold. According to him, a super quasi-Einstein manifold $(M^n, g)(n > 2)$ is a non-flat Riemannian or semi-Riemannian manifold whose Ricci tensor Ric of type $(0, 2)$ is non-zero and satisfies the following condition

$$\begin{aligned} Ric(X, Y) = {} & ag(X, Y) + bA(X)A(Y) \\ & + c[A(X)B(Y) + A(Y)B(X)] + dE(X, Y). \end{aligned} \tag{1.5}$$

A non flat Riemannian manifold (M^n, g) $(n \geq 3)$ is said to be a nearly quasi-Einstein manifold [7] if its Ricci tensor Ric of type $(0, 2)$ is non-zero and satisfies the following condition

$$Ric(X, Y) = ag(X, Y) + bE(X, Y), \tag{1.6}$$

where a, b are scalars and E is a non-zero symmetric tensor of type $(0, 2)$. A nearly quasi-Einstein manifold is denoted by $N(QE)_n$. Several authors have been studied on this manifold [13, 14] and many others.

Further, we know that if the Riemannian curvature tensor R of type $(0, 4)$ has the form

$$\overline{K}(X, Y, Z, W) = k[g(Y, Z)g(X, W) - g(X, Z)g(Y, W)] \tag{1.7}$$

then the manifold is said to be of constant curvature k. The generalization of this manifold is the manifold of quasi-constant curvature and in this case the curvature tensor has the following form

$$\begin{aligned} \overline{K}(X, Y, Z, W) = {} & f_1[g(Y, Z)g(X, W) - g(X, Z)g(Y, W)] + f_2[g(Y, Z)A(X)A(W) \\ & - g(Y, W)A(X)A(Z) + g(X, W)A(Y)A(Z) - g(X, Z)A(Y)A(W)], \end{aligned} \tag{1.8}$$

where $g(K(X, Y)Z, W) = \overline{K}(X, Y, Z, W)$, K is the curvature tensor of type $(1, 3)$ and a, b are scalar function of which $b \neq 0$ and A is non-zero 1-form which is earlier defined.

It can be easily seen that if the curvature tensor $\overline{K}$ of the form (1.8), then the manifold is conformally flat. Hence a Riemannian or semi-Riemannian manifold is said to be quasi-constant curvature if the curvature tensor $\overline{K}$ satisfied the relation (1.8). Such a manifold is denoted by $(QC)_n$.

In 2004, U.C. De and G.C. Ghosh [6] introduced the concept of generalized quasi-constant curvature. A Riemannian manifold is said to be a manifold of generalized quasi-constant curvature if the curvature tensor satisfies the following condition

$$\begin{aligned}\overline{K}(X,Y,Z,W) &= f_1[g(Y,Z)g(X,W) - g(X,Z)g(Y,W)] + f_2[g(X,W)A(Y)A(Z) \\ &- g(Y,W)A(X)A(Z) + g(Y,Z)A(X)A(W) - g(X,Z)A(Y)A(W)] \\ &+ f_3[g(X,W)B(Y)B(Z) - g(Y,W)B(X)B(Z) \\ &+ g(Y,Z)B(X)B(W) - g(X,Z)B(Y)B(W)].\end{aligned} \tag{1.9}$$

A non-flat Riemannian or semi-Riemannian manifold (M^n, g) $(n \geq 3)$ shall be called a manifold of super quasi-constant curvature if its curvature tensor $\overline{K}$ of type $(0,4)$ satisfies the following condition

$$\begin{aligned}\overline{K}(X,Y,Z,W) &= f_1[g(Y,Z)g(X,W) - g(X,Z)g(Y,W)] + f_2[g(X,W)A(Y)A(Z) \\ &- g(Y,W)A(X)A(Z) + g(Y,Z)A(X)A(W) - g(X,Z)A(Y)A(W)] \\ &+ f_3[g(X,W)\{A(Y)B(Z) + B(Y)A(Z)\} - g(Y,W)\{A(X)B(Z) \\ &+ B(X)A(Z)\} + g(Y,Z)\{A(X)B(W) + B(X)A(W)\} \\ &- g(X,Z)\{A(Y)B(W) + B(Y)A(W)\}] + f_4[E(Y,Z)g(X,W) \\ &- E(X,Z)g(Y,W) + E(X,W)g(Y,Z) - E(Y,W)g(X,Z)],\end{aligned} \tag{1.10}$$

where f_1, f_2, f_3, f_4 are scalars of which $f_1 \neq 0$, $f_2 \neq 0$, $f_3 \neq 0$, $f_4 \neq 0$ and A, B are two non-zero 1-forms defined earlier, ρ, σ being two unit vector fields such that $g(\rho, \sigma) = 0$. Such an n-dimensional manifold shall be denoted by $S(QC)_n$. If in (1.10), $f_3 = f_4 = 0$ then the manifold reduces to a manifold of quasi-constant curvature. Such a generalization of quasi-constant curvature motivates us to study the generalized super-quasi-constant curvature.

Definition 1.1. A non-flat Riemannian manifold is said to be a manifold of generalized super quasi-constant curvature if the curvature tensor $\overline{K}$ of type $(0,4)$ has the following form

$$\begin{aligned}\overline{K}(X,Y,Z,W) = {} & f_1[g(Y,Z)g(X,W) - g(X,Z)g(Y,W)] + f_2[g(X,W)A(Y)A(Z) \\ & - g(Y,W)A(X)A(Z) + g(Y,Z)A(X)A(W) - g(X,Z)A(Y)A(W)] \\ & + f_3[g(X,W)\{A(Y)B(Z) + B(Y)A(Z)\} - g(Y,W)\{A(X)B(Z) \\ & + B(X)A(Z)\} + g(Y,Z)\{A(X)B(W) + B(X)A(W)\} \\ & - g(X,Z)\{A(Y)B(W) + B(Y)A(W)\}] + f_4[E(Y,Z)g(X,W) \\ & - E(X,Z)g(Y,W) + E(X,W)g(Y,Z) - E(Y,W)g(X,Z)] \\ & + f_5[E(Y,Z)A(X)A(W) - E(X,Z)A(Y)A(W) \\ & + E(X,W)A(Y)A(Z) - E(Y,W)A(X)A(Z)] \\ & + f_6[E(X,W)\{A(Y)B(Z) + B(Y)A(Z)\} - E(Y,W)\{A(X)B(Z) \\ & + B(X)A(Z)\} + E(Y,Z)\{A(X)B(W) + B(X)A(W)\} \\ & - E(X,Z)\{A(Y)B(W) + B(Y)A(W)\}] + f_7[A(X)A(W)B(Y)B(Z) \\ & - A(Y)A(W)B(X)B(Z) + A(Y)A(Z)B(X)B(W) \\ & - A(X)A(Z)B(Y)B(W)] + f_8[E(Y,Z)E(X,W) \\ & - E(X,Z)E(Y,W)],\end{aligned} \tag{1.11}$$

where f_1, f_2, f_3,.......,f_8 are non-zero scalars and A, B are two non-zero 1-forms defined in (1.4), ρ, σ being two unit vector fields such that $g(\rho, \sigma) = 0$ and E is a symmetric tensor of type $(0,2)$ such that $E(X,\rho) = 0$, for all X.

2. Existence of generalized super quasi-constant curvature

In this section, we have studied the hypersurface of a Riemannian manifold and the existence of the generalized super quasi-constant curvature defined by (1.11) has been proved in two different methods. Let $(M_n, \overline{g})$ be a hypersurface of (M_{n+1}, g), for $\overline{g}$ being induced metric on M_{n+1}. If $\overline{H}$, R and U denote the second fundamental tensor, a tensor of type $(1,1)$ and a unit normal vector field respectively then

$$\overline{g}(R_U(Y), Z) = g(\overline{H}(Y,Z), U). \tag{2.1}$$

Let $\overline{H}$ be a symmetric tensor of type $(0,2)$ associated with R such that

$$\overline{g}(R_U(Y), Z) = \overline{H}_U(Y,Z). \tag{2.2}$$

Now, we define a super quasi-umbilical hypersurface.

Definition 2.1. A hypersurface is said to be a super quasi-umbilical hypersurface if the second fundamental tensor $\overline{H}$ has the form

$$\begin{aligned}\overline{H}_U(Y,Z) = {} & \beta_1 g(X,Y) + \beta_2 A(X)A(Y) \\ & + \beta_3[A(X)B(Y) + A(Y)B(X)] + \beta_4 E(X,Y),\end{aligned} \tag{2.3}$$

where β_i, $1 \leq i \leq 4$ are scalars and A, B are non-zero 1-forms and E is the symmetric tensor of type $(0,2)$.

Proposition 2.1. *The hypersurface categories are determined by some conditions on β_i.*

(i) *If $\beta_i = 0$ for $1 \leq i \leq 4$ then hypersurface is called geodesics.*
(ii) *If $\beta_i = 0$ for $2 \leq i \leq 4$ then hypersurface is called umbilical.*
(iii) *If $\beta_i = 0$ for $i \neq 2$ or $i \neq 3$ then hypersurface is called cylindrical.*

(iv) *If* $\beta_i = 0$ *for* $3 \leq i \leq 4$ *then hypersurface is said to be quasi-umbilical.*
(v) *A hypersurface is known as generalized quasi-umbilical if* $\beta_4 = 0$.
(vi) *A hypersurface is said to be nearly quasi-umbilical if* $\beta_i = 0$ *for* $2 \leq i \leq 3$.

Let the hypersurface be a super quasi-umbilical hypersurface, then (2.1), (2.2) and (2.3) together give

$$\begin{aligned}\overline{H}(X,Y) &= \beta_1 g(X,Y)U + \beta_2 A(X)A(Y)U \\ &\quad + \beta_3[A(X)B(Y) + A(Y)B(X)]U + \beta_4 E(X,Y)U.\end{aligned} \tag{2.4}$$

The Gauss equation for the hypersurface is given by [12]

$$\overline{K}(X,Y,Z,W) = R(X,Y,Z,W) + g(\overline{H}(X,W),\overline{H}(Y,Z)) - g(\overline{H}(Y,W),\overline{H}(X,Z)), \tag{2.5}$$

where $\overline{K}$ is the curvature tensor of the hypersurface.

Assuming the manifold has constant curvature, we use (1.7) and (2.4) in (2.5), we get

$$\begin{aligned}\overline{K}(X,Y,Z,W) &= k[g(Y,Z)g(X,W) - g(X,Z)g(Y,W)] + \beta_1^2[g(Y,Z)g(X,W) \\ &\quad - g(X,Z)g(Y,W)] + \beta_1\beta_2[g(X,W)A(Y)A(Z) - g(Y,W)A(X)A(Z) \\ &\quad + g(Y,Z)A(X)A(W) - g(X,Z)A(Y)A(W)] \\ &\quad + \beta_1\beta_3[g(X,W)\{A(Y)B(Z) + B(Y)A(Z)\} - g(Y,W)\{A(X)B(Z) \\ &\quad + B(X)A(Z)\} + g(Y,Z)\{A(X)B(W) + B(X)A(W)\} \\ &\quad - g(X,Z)\{A(Y)B(W) + B(Y)A(W)\}] + \beta_1\beta_4[E(Y,Z)g(X,W) \\ &\quad - E(X,Z)g(Y,W) + E(X,W)g(Y,Z) - E(Y,W)g(X,Z)] \\ &\quad + \beta_2\beta_4[E(Y,Z)A(X)A(W) - E(X,Z)A(Y)A(W) \\ &\quad + E(X,W)A(Y)A(Z) - E(Y,W)A(X)A(Z)] \\ &\quad + \beta_3\beta_4[E(X,W)\{A(Y)B(Z) + B(Y)A(Z)\} \\ &\quad - E(Y,W)\{A(X)B(Z) + B(X)A(Z)\} \\ &\quad + E(Y,Z)\{A(X)B(W) + B(X)A(W)\} \\ &\quad - E(X,Z)\{A(Y)B(W) + B(Y)A(W)\}] \\ &\quad + \beta_3^2 + \beta_2\beta_4[A(X)A(W)B(Y)B(Z) - A(Y)A(W)B(X)B(Z) \\ &\quad + A(Y)A(Z)B(X)B(W) - A(X)A(Z)B(Y)B(W)] \\ &\quad + \beta_4^2[E(Y,Z)E(X,W) - E(X,Z)E(Y,W)],\end{aligned} \tag{2.6}$$

On simplifying, one gets

$$\begin{aligned}
\overline{K}(X,Y,Z,W) &= f_1[g(Y,Z)g(X,W) - g(X,Z)g(Y,W)] + f_2[g(X,W)A(Y)A(Z)\\
&- g(Y,W)A(X)A(Z) + g(Y,Z)A(X)A(W) - g(X,Z)A(Y)A(W)]\\
&+ f_3[g(X,W)\{A(Y)B(Z) + B(Y)A(Z)\} - g(Y,W)\{A(X)B(Z)\\
&+ B(X)A(Z)\} + g(Y,Z)\{A(X)B(W) + B(X)A(W)\}\\
&- g(X,Z)\{A(Y)B(W) + B(Y)A(W)\}] + f_4[E(Y,Z)g(X,W)\\
&- E(X,Z)g(Y,W) + E(X,W)g(Y,Z) - E(Y,W)g(X,Z)]\\
&+ f_5[E(Y,Z)A(X)A(W) - E(X,Z)A(Y)A(W)\\
&+ E(X,W)A(Y)A(Z) - E(Y,W)A(X)A(Z)]\\
&+ f_6[E(X,W)\{A(Y)B(Z) + B(Y)A(Z)\}\\
&- E(Y,W)\{A(X)B(Z) + B(X)A(Z)\}\\
&+ E(Y,Z)\{A(X)B(W) + B(X)A(W)\}\\
&- E(X,Z)\{A(Y)B(W) + B(Y)A(W)\}]\\
&+ f_7[A(X)A(W)B(Y)B(Z) - A(Y)A(W)B(X)B(Z)\\
&+ A(Y)A(Z)B(X)B(W) - A(X)A(Z)B(Y)B(W)]\\
&+ f_8[E(Y,Z)E(X,W) - E(X,Z)E(Y,W)],
\end{aligned} \tag{2.7}$$

where $f_1 = (k + \beta_1^2)$, $f_2 = \beta_1\beta_2$, $f_3 = \beta_1\beta_3$, $f_4 = \beta_1\beta_4$, $f_5 = \beta_2\beta_4$, $f_6 = \beta_3\beta_4$,
$f_7 = {\beta_3}^2 + \beta_2\beta_4$, $f_8 = {\beta_4}^2$.
Thus we can state the following theorem:

Theorem 2.2. *A super quasi-umbilical hypersurface of a manifold of constant curvature is a manifold of generalized super quasi-constant curvature.*

Putting $X = W = e_i$ in (2.7), where $\{e_i\}$ is an orthonormal basis of the tangent space at each point on the manifold and taking summation over i, $1 \leq i \leq n$, we get

$$\begin{aligned}
Ric(Y,Z) &= f_1[ng(Y,Z) - g(Y,Z)] + f_2[nA(Y)A(Z) - A(Y)A(Z)\\
&+ g(Y,Z) - A(Y)A(Z)] + f_3[n\{A(Y)B(Z) + A(Z)B(Y)\}\\
&- A(Y)B(Z) - A(Z)B(Y) - A(Y)B(Z) - A(Z)B(Y)]\\
&+ f_4[nE(Y,Z) - E(Y,Z) - E(Y,Z)] + f_5[E(Y,Z)\\
&- A(Y)A(Z) + A(Y)A(Z)] + f_6[-A(Y)B(Z) - A(Z)B(Y)\\
&- A(Y)B(Z) - A(Z)B(Y)] + f_7[B(Y)B(Z) + A(Y)A(Z)]\\
&- f_8E(Y,Z)
\end{aligned} \tag{2.8}$$

On simplifying, one gets

$$\begin{aligned}
Ric(Y,Z) &= f_1(n-1)g(Y,Z) + f_2[(n-2)A(Y)A(Z) + g(Y,Z)]\\
&+ f_3(n-2)[A(Y)B(Z) + A(Z)B(Y)]\\
&+ f_4(n-2)E(Y,Z) + f_5E(Y,Z)\\
&- 2f_6[A(Y)B(Z) + A(Z)B(Y)] + f_7[B(Y)B(Z)\\
&+ A(Y)A(Z)] - f_8E(Y,Z)
\end{aligned} \tag{2.9}$$

i.e.,

$$\begin{aligned} Ric(Y,Z) &= ag(Y,Z) + bA(Y)A(Z) \\ &\quad + c[A(Y)B(Z) + A(Z)B(Y)] + dE(Y,Z), \end{aligned} \tag{2.10}$$

where $a = f_1(n-1) + f_2, \quad b = f_2(n-2) + f_7, \quad c = f_3(n-2) - 2f_6$,
$dE(Y,Z) = f_4(n-2) + f_5 - f_8$.
Thus, we can state the following theorem:

Theorem 2.3. *A manifold of generalized super quasi-constant curvature is a super quasi-Einstein manifold.*

Also, we can conclude

Theorem 2.4. *A super quasi-umbilical hypersurface of a manifold of constant curvature is a super quasi-Einstein manifold.*

3. Example of generalized super quasi-constant curvature

We define a Riemannian metric g in 4-dimensional space $\mathbb{R}^4$ by the relation

$$ds^2 = g_{ij}dx^i dx^j = (1+2p)[(dx^1)^2 + (dx^2)^2 + (dx^3)^2 + (dx^4)^2], \tag{3.1}$$

where x^1, x^2, x^3, x^4 are non-zero finite and $p = e^{x^1}k^{-2}$. Then the covariant and contravariant components of the metric tensor are

$$g_{11} = g_{22} = g_{33} = g_{44} = (1+2p), \quad g_{ij} = 0 \quad \forall \quad i \neq j \tag{3.2}$$

and

$$g^{11} = g^{22} = g^{33} = g^{44} = \frac{1}{1+2p}, \quad g^{ij} = 0 \quad \forall \quad i \neq j. \tag{3.3}$$

The only non-vanishing components of the Christoffel symbols are

$$\begin{Bmatrix} 1 \\ 11 \end{Bmatrix} = \begin{Bmatrix} 2 \\ 12 \end{Bmatrix} = \begin{Bmatrix} 3 \\ 13 \end{Bmatrix} = \begin{Bmatrix} 4 \\ 14 \end{Bmatrix} = \frac{p}{1+2p} \tag{3.4}$$

and

$$\begin{Bmatrix} 1 \\ 22 \end{Bmatrix} = \begin{Bmatrix} 1 \\ 33 \end{Bmatrix} = \begin{Bmatrix} 1 \\ 44 \end{Bmatrix} = \frac{-p}{1+2p}. \tag{3.5}$$

The non-zero derivatives of (3.4), we have

$$\frac{\partial}{\partial x^1}\begin{Bmatrix} 1 \\ 11 \end{Bmatrix} = \frac{\partial}{\partial x^1}\begin{Bmatrix} 2 \\ 12 \end{Bmatrix} = \frac{\partial}{\partial x^1}\begin{Bmatrix} 3 \\ 13 \end{Bmatrix} = \frac{\partial}{\partial x^1}\begin{Bmatrix} 4 \\ 14 \end{Bmatrix} = \frac{p}{(1+2p)^2} \tag{3.6}$$

and

$$\frac{\partial}{\partial x^1}\begin{Bmatrix} 1 \\ 22 \end{Bmatrix} = \frac{\partial}{\partial x^1}\begin{Bmatrix} 1 \\ 33 \end{Bmatrix} = \frac{\partial}{\partial x^1}\begin{Bmatrix} 1 \\ 44 \end{Bmatrix} = \frac{-p}{(1+2p)^2}. \tag{3.7}$$

For the Riemannian curvature tensor,

$$R^l_{ijk} = \underbrace{\begin{vmatrix} \frac{\partial}{\partial x^j} & \frac{\partial}{\partial x^k} \\ \begin{Bmatrix} l \\ ij \end{Bmatrix} & \begin{Bmatrix} l \\ ik \end{Bmatrix} \end{vmatrix}}_{\text{I}} + \underbrace{\begin{vmatrix} \begin{Bmatrix} m \\ ik \end{Bmatrix} & \begin{Bmatrix} m \\ ij \end{Bmatrix} \\ \begin{Bmatrix} l \\ mk \end{Bmatrix} & \begin{Bmatrix} l \\ mj \end{Bmatrix} \end{vmatrix}}_{\text{II}}.$$

The non-zero components of (I) are:

$$R^1_{221} = -\frac{\partial}{\partial x^1}\begin{Bmatrix} 1 \\ 22 \end{Bmatrix} = \frac{p}{(1+2p)^2},$$

$$R^1_{331} = -\frac{\partial}{\partial x^1}\begin{Bmatrix}1\\33\end{Bmatrix} = \frac{p}{(1+2p)^2},$$
$$R^1_{414} = -\frac{\partial}{\partial x^1}\begin{Bmatrix}1\\44\end{Bmatrix} = \frac{p}{(1+2p)^2}$$

and the non-zero components of (II) are:

$$R^2_{332} = \begin{Bmatrix}m\\32\end{Bmatrix}\begin{Bmatrix}2\\m3\end{Bmatrix} - \begin{Bmatrix}m\\33\end{Bmatrix}\begin{Bmatrix}2\\m2\end{Bmatrix} = -\begin{Bmatrix}1\\33\end{Bmatrix}\begin{Bmatrix}2\\12\end{Bmatrix} = \frac{p^2}{(1+2p)^2},$$
$$R^2_{442} = \begin{Bmatrix}m\\42\end{Bmatrix}\begin{Bmatrix}2\\m4\end{Bmatrix} - \begin{Bmatrix}m\\44\end{Bmatrix}\begin{Bmatrix}2\\m2\end{Bmatrix} = -\begin{Bmatrix}1\\44\end{Bmatrix}\begin{Bmatrix}2\\12\end{Bmatrix} = \frac{p^2}{(1+2p)^2},$$
$$R^3_{443} = \begin{Bmatrix}m\\43\end{Bmatrix}\begin{Bmatrix}3\\m4\end{Bmatrix} - \begin{Bmatrix}m\\44\end{Bmatrix}\begin{Bmatrix}3\\m3\end{Bmatrix} = -\begin{Bmatrix}1\\44\end{Bmatrix}\begin{Bmatrix}3\\13\end{Bmatrix} = \frac{p^2}{(1+2p)^2}.$$

Adding components corresponding (I) and (II), we have

$$R^1_{221} = R^1_{331} = R^1_{441} = \frac{p}{(1+2p)^2} \tag{3.8}$$

and

$$R^2_{332} = R^2_{442} = R^3_{443} = \frac{p^2}{(1+2p)^2}. \tag{3.9}$$

In view of $R_{hijk} = g_{hl}R^l_{ijk}$ and (3.8), (3.9) we can show that

$$R_{1221} = R_{1331} = R_{1441} = \frac{p}{1+2p},$$
$$R_{2332} = R_{2442} = R_{3443} = \frac{p^2}{1+2p}.$$

Let us consider the associated scalars f_i, $1 \le i \le 8$ and the associated tensor E are defined by

$f_1 = -\dfrac{p}{(1+2p)^3}, \quad f_2 = -2f_3, \quad ,f_6 = f_5, \quad f_8 = 3f_5, f_4, \quad f_7$ is arbitrary

and

$E_{11} = -(1+2p), \quad E_{22} = (1+2p), \quad E_{12} = \sqrt{1+2p}, \quad E_{21} = -\sqrt{1+2p}$

the 1-form

$A_1 = B_1 = \sqrt{1+2p}, \quad A_2 = B_2 = 1.$

For the non-zero components of the curvature tensor R, the (1.11) reduces to

$$\begin{aligned} R_{1221} = {} & f_1[g_{12}g_{21} - g_{11}g_{22}] + f_2[g_{12}A_1A_2 + g_{21}A_1A_2 - g_{22}A_1A_1 - g_{11}A_2A_2] \\ & + f_3[g_{12}(A_1B_2 + A_2B_1) + g_{21}(A_1B_2 + A_2B_1) - 2g_{22}A_1B_1 - 2g_{11}A_2B_2] \\ & + f_4[E_{12}g_{21} + E_{21}g_{12} - E_{22}g_{11} - E_{11}g_{22}] \\ & + f_5[E_{12}A_1A_2 + E_{21}A_1A_2 - E_{22}A_1A_1 - E_{11}A_2A_2] \\ & + f_6[E_{12}(A_1B_2 + A_2B_1) + E_{21}(A_1B_2 + A_2B_1) - 2E_{22}A_1B_1 - 2E_{11}A_2B_2] \\ & + f_7[2A_1A_2B_1B_2 - A_1A_1B_2B_2 - A_2A_2B_1B_1] \\ & + f_8[E_{12}E_{21} - E_{11}E_{22}]. \end{aligned} \tag{3.10}$$

$$\begin{aligned}
R.H.S\ of\ (3.10) &= f_1[-(1+2p)^2] + 2f_3[(1+2p)^2 + (1+2p)] \\
&- 2f_3[(1+2p)^2 + (1+2p)] + f_4[(1+2p)^2 - (1+2p)^2] \\
&+ f_5[-(1+2p)^2 + (1+2p)] + 2f_5[-(1+2p)^2 + (1+2p)] \\
&+ f_7[2(1+2p) - 2(1+2p)] - 3f_5[(1+2p)^2 - (1+2p)] \\
&= -f_1(1+2p)^2 \\
&= \frac{p}{(1+2p)^3} * (1+2p)^2 \\
&= \frac{p}{(1+2p)} \\
&= L.H.S\ of\ (3.10)
\end{aligned}$$

Hence it satisfies the condition of generalized super quasi-constant curvature.
Thus we can state the following theorem:

Theorem 3.1. *Let $(\mathbb{R}^4, g)$ be a manifold endowed with the metric given by*

$$ds^2 = g_{ij}dx^i dx^j = (1+2p)[(dx^1)^2 + (dx^2)^2 + (dx^3)^2 + (dx^4)^2]$$

where x^1, x^2, x^3 and x^4 are non-zero finite, then $(\mathbb{R}^4, g)$ is a generalized super quasi-constant curvature.

4. Sufficient condition for Super quasi-Einstein manifolds

In this section, we introduce a sufficient condition for the existence of super quasi-Einstein manifold.

Theorem 4.1. *The Ricci tensor Ric will be of the form in order for a Riemannian manifold to be a super quasi-Einstein manifold*

$$\begin{aligned}
Ric(Y,Z)Ric(X,W) &= f_1[Ric(Y,W)g(X,Z) + Ric(X,Z)g(Y,W)] \\
&+ f_2[g(Y,Z)g(X,W) - g(X,Z)g(Y,W)] \\
&+ f_3[Ric(X,W)E(Y,Z)]
\end{aligned} \tag{4.1}$$

for a symmetric tensor of type $(0,2)$.

Proof. Putting $X = W = \rho$ in (4.1), we have

$$\begin{aligned}
Ric(Y,Z)Ric(\rho,\rho) &= f_1[Ric(Y,\rho)g(\rho,Z) + Ric(\rho,Z)g(Y,\rho)] \\
&+ f_2[g(Y,Z)g(\rho,\rho) - g(\rho,Z)g(Y,\rho)] \\
&+ f_3[Ric(\rho,\rho)E(Y,Z)],
\end{aligned} \tag{4.2}$$

since we know that $Ric(Y,\rho) = g(QY,\rho)$, one gets

$$\begin{aligned}
Ric(Y,Z)Ric(\rho,\rho) &= f_1[g(QY,\rho)g(\rho,Z) + g(QZ,\rho)g(Y,\rho)] \\
&+ f_2[g(Y,Z)g(\rho,\rho) - g(\rho,Z)g(Y,\rho)] \\
&+ f_3[Ric(\rho,\rho)E(Y,Z)] \\
&= f_1[A(QY)A(Z) + A(QZ)A(Y)] \\
&+ f_2[g(Y,Z)g(\rho,\rho) - g(Z,\rho)g(Y,\rho)] \\
&+ f_3[Ric(\rho,\rho)E(Y,Z)].
\end{aligned} \tag{4.3}$$

If we take $A(QY) = B(Y)$, then

$$\begin{aligned} Ric(Y,Z)Ric(\rho,\rho) &= f_1[B(Y)A(Z) + B(Z)A(Y)] \\ &\quad + f_2[g(Y,Z)g(\rho,\rho) - g(Z,\rho)g(Y,\rho)] \\ &\quad + f_3[Ric(\rho,\rho)E(Y,Z)]. \end{aligned} \tag{4.4}$$

On simplifying, one gets

$$\begin{aligned} Ric(Y,Z) &= \frac{f_1}{Ric(\rho,\rho)}[B(Y)A(Z) + B(Z)A(Y)] + \frac{f_2 g(\rho,\rho)}{Ric(\rho,\rho)}g(Y,Z) \\ &\quad - \frac{f_2}{Ric(\rho,\rho)}A(Y)A(Z) + \frac{f_3}{Ric(\rho,\rho)}E(Y,Z). \end{aligned} \tag{4.5}$$

Hence we get

$$\begin{aligned} Ric(Y,Z) &= ag(Y,Z) + bA(Y)A(Z) \\ &\quad + c[A(Y)B(Z) + A(Z)B(Y)] + dE(Y,Z), \end{aligned} \tag{4.6}$$

where $a = \dfrac{f_2 g(\rho,\rho)}{Ric(\rho,\rho)}, \quad b = -\dfrac{f_2}{Ric(\rho,\rho)}, \quad c = \dfrac{f_1}{Ric(\rho,\rho)}, \quad d = f_3$

Hence, a Riemannian manifold satisfying (4.1) is a super quasi-Einstein manifold. □

We will also construct an example of a super quasi-Einstein manifold in general form in the same sequence:

5. EXAMPLE OF SUPER QUASI-EINSTEIN MANIFOLD

Example 5.1. We define a Riemannian metric g in 4-dimensional space $\mathbb{R}^4$ by the relation

$$ds^2 = g_{ij}dx^i dx^j = (1+2p)[(dx^1)^2 + (dx^2)^2 + (dx^3)^2 + (dx^4)^2], \tag{5.1}$$

where x^1, x^2 ,x^3, x^4 are non-zero finite and $p = e^{x^1}k^{-2}$. Then the covariant and contravariant components of the metric tensor are

$$g_{11} = g_{22} = g_{33} = g_{44} = (1+2p), \quad g_{ij} = 0 \quad \forall \quad i \neq j \tag{5.2}$$

and

$$g^{11} = g^{22} = g^{33} = g^{44} = \frac{1}{1+2p}, \quad g^{ij} = 0 \quad \forall \quad i \neq j. \tag{5.3}$$

The only non-vanishing components of the Christoffel symbols are

$$\begin{Bmatrix} 1 \\ 11 \end{Bmatrix} = \begin{Bmatrix} 2 \\ 12 \end{Bmatrix} = \begin{Bmatrix} 3 \\ 13 \end{Bmatrix} = \begin{Bmatrix} 4 \\ 14 \end{Bmatrix} = \frac{p}{1+2p} \tag{5.4}$$

$$\begin{Bmatrix} 1 \\ 22 \end{Bmatrix} = \begin{Bmatrix} 1 \\ 33 \end{Bmatrix} = \begin{Bmatrix} 1 \\ 44 \end{Bmatrix} = \frac{-p}{1+2p}. \tag{5.5}$$

The non-zero derivatives of (5.4), we have

$$\frac{\partial}{\partial x^1}\begin{Bmatrix} 1 \\ 11 \end{Bmatrix} = \frac{\partial}{\partial x^1}\begin{Bmatrix} 2 \\ 12 \end{Bmatrix} = \frac{\partial}{\partial x^1}\begin{Bmatrix} 3 \\ 13 \end{Bmatrix} = \frac{\partial}{\partial x^1}\begin{Bmatrix} 4 \\ 14 \end{Bmatrix} = \frac{p}{(1+2p)^2} \tag{5.6}$$

$$\frac{\partial}{\partial x^1}\begin{Bmatrix} 1 \\ 22 \end{Bmatrix} = \frac{\partial}{\partial x^1}\begin{Bmatrix} 1 \\ 33 \end{Bmatrix} = \frac{\partial}{\partial x^1}\begin{Bmatrix} 1 \\ 44 \end{Bmatrix} = \frac{-p}{(1+2p)^2}. \tag{5.7}$$

For the Riemannian curvature tensor,

$$R^l_{ijk} = \underbrace{\begin{vmatrix} \frac{\partial}{\partial x^j} & \frac{\partial}{\partial x^k} \\ \left\{ {l \atop ij} \right\} & \left\{ {l \atop ik} \right\} \end{vmatrix}}_{=I} + \underbrace{\begin{vmatrix} \left\{ {m \atop ik} \right\} & \left\{ {m \atop ij} \right\} \\ \left\{ {l \atop mk} \right\} & \left\{ {l \atop mj} \right\} \end{vmatrix}}_{=II}.$$

The non-zero components of (I) are:

$$R^1_{221} = -\frac{\partial}{\partial x^1}\left\{ {1 \atop 22} \right\} = \frac{p}{(1+2p)^2},$$
$$R^1_{331} = -\frac{\partial}{\partial x^1}\left\{ {1 \atop 33} \right\} = \frac{p}{(1+2p)^2},$$
$$R^1_{441} = -\frac{\partial}{\partial x^1}\left\{ {1 \atop 44} \right\} = \frac{p}{(1+2p)^2}$$

and the non-zero components of (II) are:

$$R^2_{332} = \left\{ {m \atop 32} \right\}\left\{ {2 \atop m3} \right\} - \left\{ {m \atop 33} \right\}\left\{ {2 \atop m2} \right\} = -\left\{ {1 \atop 33} \right\}\left\{ {2 \atop 12} \right\} = \frac{p^2}{(1+2p)^2},$$
$$R^2_{442} = \left\{ {m \atop 42} \right\}\left\{ {2 \atop m4} \right\} - \left\{ {m \atop 44} \right\}\left\{ {2 \atop m2} \right\} = -\left\{ {1 \atop 44} \right\}\left\{ {2 \atop 12} \right\} = \frac{p^2}{(1+2p)^2},$$
$$R^3_{443} = \left\{ {m \atop 43} \right\}\left\{ {3 \atop m4} \right\} - \left\{ {m \atop 44} \right\}\left\{ {3 \atop m3} \right\} = -\left\{ {1 \atop 44} \right\}\left\{ {3 \atop 13} \right\} = \frac{p^2}{(1+2p)^2}.$$

Adding components corresponding (I) and (II), we have

$$R^1_{221} = R^1_{331} = R^1_{441} = \frac{p}{(1+2p)^2} \tag{5.8}$$

and

$$R^2_{332} = R^2_{442} = R^3_{443} = \frac{p^2}{(1+2p)^2}. \tag{5.9}$$

In view of $R_{hijk} = g_{hl}R^l_{ijk}$ and (5.8), (5.9) we can show that

$$R_{1221} = R_{1331} = R_{1441} = \frac{p}{1+2p},$$

$$R_{2332} = R_{2442} = R_{3443} = \frac{p^2}{1+2p}$$

and the Ricci tensor

$$R_{11} = g^{jh}R_{1j1h} = g^{22}R_{1212} + g^{33}R_{1313} + g^{44}R_{1414} = \frac{3p}{(1+2p)^2},$$
$$R_{22} = g^{jh}R_{2j2h} = g^{11}R_{2121} + g^{33}R_{2323} + g^{44}R_{2424} = \frac{p}{(1+2p)},$$
$$R_{33} = g^{jh}R_{3j3h} = g^{11}R_{3131} + g^{22}R_{3232} + g^{44}R_{3434} = \frac{p}{(1+2p)},$$
$$R_{44} = g^{jh}R_{4j4h} = g^{11}R_{4141} + g^{22}R_{4242} + g^{33}R_{4343} = \frac{p}{(1+2p)}.$$

Let us consider the associated scalars a, b, c, d and the associated tensor E are defined by

$$a = -\frac{p}{(1+2p)^3}, \quad b = 2p, \quad c = \frac{2\sqrt{p}}{(1+2p)}, \quad d = \frac{-2}{(1+2p)^2}$$

and

$$E_{ij} = \begin{cases} p, & \text{if i=j=1} \\ -p, & \text{if i=j=3} \\ 0, & \text{otherwise} \end{cases}$$

the 1-form

$$A_i(x) = \begin{cases} \frac{1}{1+2p}, & \text{if i=1} \\ 0, & \text{otherwise} \end{cases} \quad and \quad B_i(x) = \begin{cases} \sqrt{p}, & \text{if i=1} \\ -\sqrt{p}, & \text{if i=2} \\ 0, & \text{otherwise} \end{cases}$$

where generators are unit vector fields, then from (1.5), we have

$$R_{11} = ag_{11} + bA_1A_1 + 2cA_1B_1 + dE_{11}, \tag{5.10}$$

$$R_{22} = ag_{22} + bA_2A_2 + 2cA_2B_2 + dE_{22}, \tag{5.11}$$

$$R_{33} = ag_{33} + bA_3A_3 + 2cA_3B_3 + dE_{33}, \tag{5.12}$$

$$R_{44} = ag_{44} + bA_4A_4 + 2cA_4B_4 + dE_{44}, \tag{5.13}$$

$$\begin{aligned} R.H.S.\ of\ (5.10) &= ag_{11} + bA_1A_1 + 2cA_1B_1 + dE_{11} \\ &= -\frac{p}{(1+2p)^2} + \frac{2p}{(1+2p)^2} + \frac{4p}{(1+2p)^2} - \frac{2p}{(1+2p)^2} \\ &= \frac{3p}{(1+2p)^2} \\ &= L.H.S.\ of\ (5.10) \end{aligned}$$

By similar argument it can be shown that (5.11) to (5.13) are also true.
Hence $(\mathbb{R}^4, g)$ is a $S(QE)_4$.

Example 5.2. Let $(\mathbb{R}^4, g)$ be a Lorentzian manifold endowed with the metric given by

$$ds^2 = g_{ij}dx^i dx^j = -(1+2p)(dx^1)^2 + (1+2p)[(dx^2)^2 + (dx^3)^2 + (dx^4)^2]$$

where x^1, x^2, x^3 and x^4 are non-zero finite, then $(\mathbb{R}^4, g)$ is a $S(QE)_4$.

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