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L-FUZZIFYING PROXIMITY, L-FUZZIFYING UNIFORM SPACE AND L-FUZZIFYING STRONG UNIFORM SPACE

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ABSTRACT. In this paper the concept of proximity in L-fuzzifying topology is established and some of its properties are discussed. Furthermore we introduce and study the concepts of$L$-fuzzifying uniform space and L-fuzzifying strong uniform space.

1. PRELIMINARIES

In 1993, M. Ying [11] introduced and studied the uniformity in $[0, 1]-$fuzzifying topology as a fuzzy concept, i.e., as a fuzzy subset of $P(X \times X)$ for an ordinary set X. In 2003, H. F. Kheder [5], introduced and studied concepts of proximity and strong uniformity in fuzzifying topology as fuzzy concepts. In this paper we introduce and study the concept of proximity, uniformity and strong uniformity in L-fuzzifying topology. In section 2, we extend the concept of fuzzifying proximity due to (Kheder, et al (2003)[5]) into L-fuzzifying setting. Some of basic properties of this extenstion are studied.

Section 3, is devoted to extend and study the concept of uniformity in the sense of (Ying (1993)[10])in L–fuzzifying topology. Finally, the notion of fuzzifying strong uniform space (Kheder, et al (2003)[5]) is generalized by introducing the concept of L-fuzzifying strong uniform spaces. Some results concerning this concept are obtained. In the present paper L is assumed to be a completely residuated lattice such that the following conditions are satisfied:

(1) L is totally ordered as a poset.(i.e. for each $a, b \in L$, $a < b$, or $b < a$.)

(2) L satisfies that '$\wedge$' is disributive over arbitrary joins.

Definition 1.2. [9]. A structure $(L,\vee, \wedge,*, \rightarrow, \perp, \top)$ is called a complete residuated lattice iff

(1) $(L,\vee, \wedge, \perp, \top)$ is a complete lattice whose greatest and least element are $\top, \perp$ respectively,

(2) $(L,*, \top)$ is a commutative monoid, i.e.,

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(a) $*$ is a commutative and associative binary operation on L, and
(b) $\forall\, a \in L, a * \top = \top *\ a = a$,
(3)(a) $*$ is isotone,
(b) $\rightarrow$ is a binary operation on L which is antitone in the first and isotone in the second variable,
(c) $\rightarrow$ is couple with $*$ as: $a * b \leq c$ iff $a \leq b \rightarrow c \quad \forall\, a, b, c \in L$. The basic operations on the family L^X of all L-sets on a non-empty set X was defined as follows:

Definition 1.3. [1]. A complete lattice L is called completely distributive if the following law is satisfied:
$\forall \{A_j \,|j \in J\} \subseteq P(L)$, where $P(L)$ is the power subset of L we have,
$\bigwedge\limits_{j\in J} \bigvee A_j = \bigvee\limits_{f \in \prod\limits_{j\in J} A_j} (\bigwedge\limits_{j\in J} f(j))$.

Definition 1.4.(Csa′sza′r (1978)[2]). A binary relation δ on $P(X) \times P(X)$ is called a proximity on a set X if it satisfies the following conditions:
(P1) If $(A, B) \in \delta$, then $A \neq \phi$ and $B \neq \phi$ and $\delta(\phi, X) = 0$,
(P2) If $A \cap B \neq \phi$, then $(A, B) \in \delta$
(P3) If $(A_1 \cup A_2, C) \in \delta$, then $(A_1, C) \in \delta$ or $(A_2, C) \in \delta$
(P4) If $(A, B) \in \delta$, then $(B, A) \in \delta$
(P4) If $(A, B) \notin \delta$, then there exists D such that $(A, D) \notin \delta$ and $(X - D, B) \notin \delta$
The pair (X, δ) is said to be a proximity space.

The following concepts are given in (Kheder, et. al. (2003)[5]).

Definition 1.5. Let X be a set and let $\delta \in I^{(P(X)\times P(X))}$, i.e., $\delta : P(X) \times P(X) \rightarrow [0, 1]$. Assume that for any $A, B, C \in P(X)$ the following axioms are satisfied:
$(FP1)\ \models \neg(X, \phi) \in \delta$,
$(FP2)\ \models\ (A, B) \in \delta \leftrightarrow (B, A) \in \delta$,
$(FP3)\ \models\ (A, B \cup C) \in \delta \leftrightarrow (A, B) \in \delta \vee (A, C) \in \delta$,
$(FP4)$ for every $A, B \subseteq X$, there exists $C \subseteq X$ such that
$\models\ ((A, C) \in \delta \vee (B, X - C) \in \delta) \rightarrow (A, B) \in \delta$,
$(FP5)\ \models \{x\} \equiv \{y\} \leftrightarrow (\{x\}, \{y\}) \in \delta$. Then δ is called a fuzzifying proximity on X and (X, δ) is called a fuzzifying proximity space.

Theorem 1.1. Let (X, δ) be a fuzzifying proximity space. Then we have
(1) $\models (A, B) \in \delta \wedge B \subseteq C \rightarrow (A, C) \in \delta$,
(2) $\models\ (A \cap B) \neq \phi \rightarrow (A, B) \in \delta$,
(3) $\models \neg\delta(A, \phi)$.

Proposition 1.1. For every $\alpha \in (0, 1],\ \delta_\alpha$ is a proximity on X, where δ_α is the α-level of δ, i.e., $\delta_\alpha = \{(A, B) : \delta(A, B) \geq \alpha\}$.

Definition 1.6. Let (X, δ) be a fuzzifying proximity space. For each $\alpha \in (0, 1]$, we define the interior operation induced by δ_α, denoted by
$int_{\delta_\alpha} : P(X) \rightarrow P(X)$, as follows: $int_{\delta_\alpha}(A) = \bigcup\limits_{B\in P(X), (B, X-A)\notin\delta_\alpha} B \quad \forall A \in P(X)$.

Proposition 1.2. For every $\alpha \in (0, 1]$, the family $\tau_{\delta_\alpha} = \{A : A \subseteq X$ and $int_{\delta_\alpha}(A) = A\}$ is a topology on X.

Theorem 1.2. Let (X, δ) be a fuzzifying proximity space. The mapping $\tau_\delta : P(X) \to [0,1]$ defined by: $\tau_\delta(A) = \bigvee_{\alpha\in(0,1)\ ,A\in\tau_{\delta_\alpha}} \alpha$ is a fuzzifying topology and is called the fuzzifying topology induced by the fuzzifying proximity δ.

Definition 1.7. (Csa′sza′r (1978)[2]). A uniform structure U on a set X is a family of subsets of
$X \times X$, called entourage, which satisfies the following properties:
$(U1)$ If $u \in U$, then $\triangle \subseteq u$, where $\triangle$ is the diagonal:
$\triangle = \{(x,x)\,|x \in X\}$
$(U2)$ If $v \subseteq u$, and $v \in U$ then $u \in U$,
$(U3)$ for every $u, v \in U,\ u \cap v \in U$,
$(U4)$ If $u \in U$, then $u^{-1} \in U$, where $u^{-1} = \{(x,y)\,|(y,x) \in u\}$.
$(U3)$ for every $u \in U$, there exists $v \subseteq U$ such that $v \circ v \subseteq u$, where $v \circ v \subseteq u$, where $v \circ u$ is defined by:
$v \circ u = \{(x,y)|\,\exists z \in X$ such that $(x,z) \in u$ and $(z,y) \in u\}$, $\forall x, y \in X$. The pair (X, U) is said to be a uniform space.

The following results are given in [Ying (1992)[11]).

Definition 1.8. Let X be a set and $\mathcal{U} \in I^{P(X\times X)}$. If for any $U, V \subseteq X \times X$,
$(U1) \models (U \in \mathcal{U}) \to (\triangle \subseteq U)$,
$(U2) \models (U \in \mathcal{U}) \to (U^{-1} \in \mathcal{U})$,
$(U3) \models (U \in \mathcal{U}) \to (\exists V)(V \in \mathcal{U}) \wedge (V \circ V \subseteq U)$,
$(U4) \models (U \in \mathcal{U}) \wedge (V \in \mathcal{U}) \to (U \cap V \subseteq \mathcal{U})$,
$(U5) \models (U \in \mathcal{U}) \wedge (U \subseteq V) \to (V \in \mathcal{U})$. Then, $\mathcal{U}$ is called Fuzzifying uniformity and $(X, \mathcal{U})$ is called fuzzifying uniform space.

Lemma 1.1. Let $(X, \mathcal{U})$ be a fuzzifying uniform space and $\Im \in I^{P(X)}$ defined by:
$T \in \Im := (\forall x)(x \in T) \to (\exists U)((U \in \mathcal{U}) \wedge (U[x] \subseteq T)))$, $T \subseteq X$ i.e.,
$\Im(T) := \bigwedge_{x\in T} \bigvee_{U[x]\subseteq T} \mathcal{U}(U)$, $T \subseteq X$. where $U[x] = \{y \in X : (x,y) \in U\}$. Then $\Im$ is a fuzzifying topology on X and called the fuzzifying (uniform) topology of $\mathcal{U}$.

The following concepts are given in (Kheder, et. al. (2003)[5]).

Definition 1.9. Let X be a set and let $\mathcal{U} : P(X \times X) \to I$. Assume that $\mathcal{U}$ is normal, i.e. $\exists\, U \subseteq X \times X$ s.t. $\mathcal{U}[U] = 1$. If for any $U, V \subseteq X \times X$,
$(FU1) \models (U \in \mathcal{U}) \to (\triangle \subseteq U)$,
$(FU2) \models (U \in \mathcal{U}) \to (U^{-1} \in \mathcal{U})$,
$(FU3)^*$ There exists $H \sqsubset P(X \times X)\ s.t.\ \models (U \in \mathcal{U}) \to (\exists V)(V \in H) \wedge (V \in \mathcal{U}) \wedge (V \circ V \subseteq U)$,where $\sqsubset$ stands for "a finite subset of",
$(FU4) \models (U \in \mathcal{U}) \wedge (V \in \mathcal{U}) \to (U \cap V \subseteq \mathcal{U})$,
$(FU5) \models (U \in \mathcal{U}) \wedge (U \subseteq V) \to (V \in \mathcal{U})$. Then, $\mathcal{U}$ is called a strong fuzzifying uniformity and $(X, \mathcal{U})$ is called a strong fuzzifying uniform space.

Theorem 1.3. Let $(X,\mathcal{U})$ be a strong fuzzifying uniform space. Then for each $\alpha \in (0,1)$, the α level of $\mathcal{U}$ denoted by $\mathcal{U}_\alpha$ is a classical uniformity on X. where $\mathcal{U}_\alpha = \{U \in P(X \times X)\ s.t. \mathcal{U}(U) \geq \alpha\}$.

Theorem 1.4. Let $(X,\mathcal{U})$ be a strong fuzzifying uniform space. The fuzzy set $\tau_\mathcal{U} \in (\mathcal{F}(P(X))$, defined by: $\tau_\mathcal{U}(A) = \bigvee_{\alpha\in(0,1], A\in\tau_{\mathcal{U}_\alpha}} \alpha$, is a fuzzifying topology. It is called the fuzzifying topology induced by the strong fuzzifying uniformity $\mathcal{U}$

Theorem 1.5 Let $\delta_{\mathcal{U}_\alpha}$ be the proximity induced by the uniformity $\mathcal{U}_\alpha$. Then the mapping $\delta_\mathcal{U} : P(X \times X) \to [0,1]$, defined by $\delta_\mathcal{U}(A,B) = \bigvee_{\alpha\in(0,1],(A,B)\in\delta_{\mathcal{U}_\alpha}} \alpha$, is a fuzzifying proximity. It is called the fuzzifying proximity induced by the strong fuzzifying uniformity $\mathcal{U}$.

2. L-fuzzifying proximity space

Definition 2.1. The binary crisp predicate $CE \in \{\bot,\top\}^{P(X)\times P(X)}$, called crisp equality, is given as follows:

$$CE(A,B) = \begin{cases} \top & \text{if } A = B \\ \bot & \text{if } A \neq B \end{cases}$$

Definition 2.2. Let X be a set and let $\delta \in L^{P(X)\times P(X)}$, i.e.,
$\delta : P(X) \times P(X) \to L$. Assume that for every $A, B, C \in P(X)$, the following axioms are satisfied:
$(LFP1)$ $\delta(X, \phi) = \bot$,
$(LFP2)$ $\delta(B, A) = \delta(A, B)$,
$(LFP3)$ $\delta(A, B \cup C) = \delta(A, B) \vee \delta(A, C)$,
$(LFP4)$ For every $A, B \in P(X)$, $\exists C \in P(X)$
$s.t.\ \delta(A, B) \geq \delta(A, C) \vee \delta(B, X - C)$,
$(LFP5)$ $\delta(\{x\}, \{y\}) = CE(\{x\}, \{y\})$. Then δ is called an L-fuzzifying proximity on X and (X, δ) is called an L-fuzzifying proximity space.

Definition 2.3. The binary crisp predicate $\subseteq \in \{\bot,\top\}^{P(X)\times P(X)}$,called crisp inclusion, is defined as follows:

$$\subseteq (A,B) = \begin{cases} \top & \text{if } A \subseteq B, \\ \bot & \text{if } A \nsubseteq B. \end{cases}$$

Definition 2.4. The binary crisp predicate $\cap \in \{\bot,\top\}^{P(X)\times P(X)}$, called crisp intersection, is defined as follows:

$$\cap(A,B) = \begin{cases} \top & \text{if } A \cap B \neq \phi \\ \bot & \text{if } A \cap B = \phi \end{cases}$$

Lemma 2.1. If $\subseteq (B,C) = \top$, then $\delta(A,B) \leq \delta(A,C)\ \ \forall A \in P(X)$.

Proof. $\delta(A,C) = \delta(A, B \cup C) = \delta(A,B) \vee \delta(A,C) \geq \delta(A,B)$.

Theorem 2.1. Let (X, δ) be an L-fuzzifying proximity space. For every $A, B, C \in P(X)$,then we have
(1) $\delta(A, C) \geq \delta(A, B) \wedge \subseteq (B, C)$,
(2) $\delta(B, A) \geq \cap(A, B)$,
(3) $\delta(A, \phi) \rightarrow \perp = \top$.

Proof. (1) If $\subseteq (B, C) = \perp$, then $\delta(A, C) \geq \delta(A, B) \wedge \perp$
and if $\subseteq (B, C) = \top$, from Lemma 2.1 we have
$\delta(A, C) \geq \delta(A, B) \wedge \top$. Then $\delta(A, C) \geq \delta(A, B) \wedge \subseteq (B, C)$.

(2) If $\cap(A, B) = \perp$, the result hold. Let $\cap(A, B) = \top$, i.e., $\exists x \in A \cap B$.
From $(LFP5)$, $\delta(\{x\}, \{x\}) = \top$. Applying Lemma 2.1 and $(LFP2)$,
$\delta(A, B) \geq \delta(A, \{x\}) = \delta(\{x\}, A) \geq \delta(\{x\}, \{x\}) = \top$. Hence $\delta(A, B) \geq \cap(A, B)$.

(3) From Lemma 2.1, $\delta(A, \phi) \leq \delta(X, \phi)$.
Then $\delta(A, \phi) \rightarrow \perp \geq \delta(X, \phi) \rightarrow \perp = \perp \rightarrow \perp = \top$.

Theorem 2.2. For every $\alpha \in L$-$\{\perp\}$, δ_α is a proximity on X, where δ_α is the α-cut of an L-fuzzifying proximity δ, i.e., $\delta_\alpha = \{(A, B) : \delta(A, B) \geq \alpha\}$.

Proof. Let $\alpha \in L - \{\perp\}$.

$(P1)$ From $(LFP1)$ we have $\delta(X, \phi) = \perp$. Then $\delta(X, \phi) < \alpha$. So, $(X, \phi) \notin \delta_\alpha$.

$(P2)$ Suppose $(A, B) \in \delta_\alpha$. Then $\delta(A, B) \geq \alpha$.
From $(LFP2)$, $\delta(B, A) = \delta(A, B) \geq \alpha$. Hence $(B, A) \in \delta_\alpha$.
$(P3)$ Let $(A, B \cup C) \in \delta_\alpha$ then $\delta(A, B \cup C) \geq \alpha$.
From $(LFP3)$ we have $\delta(A, B) \geq \alpha$ or $\delta(A, C) \geq \alpha$
and hence $(A, B) \in \delta_\alpha$ or $(A, C) \in \delta_\alpha$.

$(P4)$ Let $(A, B) \notin \delta_\alpha$.since L is totally ordered
then we have $\delta(A, B) < \alpha$. From $(LFP4)$ there exists $C \in P(X)$ such that
$\delta(A, B) \geq \delta(A, C) \vee \delta(B, X - C)$. Then $\delta(A, C) \vee \delta(B, X - C) < \alpha$ which implies that $\delta(A, C) < \alpha$ and $\delta(B, X - C) < \alpha$ which implies that $(A, C) \notin \delta_\alpha$ and $(B, X - C) \notin \delta_\alpha$.

$(P5)$ Suppose $x = y$. Then $CE(\{x\}, \{y\}) = \top$ so that from $(LFP5)$,
$\delta(\{x\}, \{y\}) = \top \geq \alpha$. Hence $(\{x\}, \{y\}) \in \delta_\alpha$.

Definition 2.5. Let (X, δ) be an L- fuzzifying proximity space. For each $\alpha \in L$-$\{\perp\}$, the interior operation induced by δ_α, denoted by $int_{\delta_\alpha} : P(X) \rightarrow P(X)$, is defined as follows: $int_{\delta_\alpha}(A) = \bigcup_{B \in P(X), (B, X-A) \notin \delta_\alpha} B \quad \forall A \in P(X)$.

Theorem 2.3. For every $\alpha \in L$-$\{\perp\}$, the family $\tau_{\delta_\alpha} = \{A : A \subseteq X \text{ and } int_{\delta_\alpha}(A) = A\}$ is a classical topology on X.

Proof. Let $\alpha \in L$-$\{\perp\}$. Then:

(1)*since* $int_{\delta_\alpha}(X) = \bigcup_{B\in P(X),(B,\phi)\notin\delta_\alpha} B = X$ and $int_{\delta_\alpha}(\phi) = \bigcup_{B\in P(X),(B,X)\notin\delta_\alpha} B$ $= \phi$, then $X,\ \phi \in \tau_{\delta_\alpha}$.

(2) Let $A,\ C \in \tau_{\delta_\alpha}$ $s.t.$ $int_{\delta_\alpha}(A) = A$ and $int_{\delta_\alpha}(C) = C$.

Then $int_{\delta_\alpha}(A\cap C) = \bigcup_{B\in P(X),(B,X-(A\cap C))\notin\delta_\alpha} B$

$$= \bigcup_{B\in P(X),(B,(X-A)\cup(X-C))\notin\delta_\alpha} B = \left(\bigcup_{B\in P(X),(B,X-A)\notin\delta_\alpha} B\right)\cap\left(\bigcup_{B\in P(X),(B,X-C)\notin\delta_\alpha} B\right)$$

$= A\cap C$. So, $A\cap C \in \tau_{\ \delta_\alpha}$.

(3) Let $\{A_\lambda : \lambda\in\Lambda\} \subseteq \tau_{\ \delta_\alpha}$. Now $\bigcup_{\lambda\in\Lambda} A_\lambda = \bigcup_{\lambda\in\Lambda} int_{\ \delta_\alpha}(A_\lambda) \subseteq int_{\ \delta_\alpha}(\bigcup_{\lambda\in\Lambda} A_\lambda)$,

because $int_{\ \delta_\alpha}$ is monotone (indeed, If $A\subseteq C$, then $int_{\ \delta_\alpha}(A) = \bigcup_{B\in P(X),(B,X-A)\notin\delta_\alpha} B$

$\subseteq \bigcup_{B\in P(X),(B,X-C)\notin\delta_\alpha} B = int_{\ \delta_\alpha}(C)$.). Also, $int_{\ \delta_\alpha}(\bigcup_{\lambda\in\Lambda} A_\lambda) \subseteq \bigcup_{\lambda\in\Lambda} A_\lambda$
because $int_{\ \delta_\alpha}(A) = \bigcup_{B\in P(X),(B,X-A)\notin\delta_\alpha} B \subseteq \bigcup_{B\ \in P(X),\ \cap(B,\ X-A)=\bot} B$
$= \bigcup_{B\in P(X),B\ \subseteq A} B = A$ for any $A\in P(X)$

Then $int_{\ \delta_\alpha}(\bigcup_{\lambda\in\Lambda} A_\lambda) = \bigcup_{\lambda\in\Lambda} A_\lambda$. Hence $\bigcup_{\lambda\in\Lambda} A_\lambda \in \tau_{\ \delta_\alpha}$.

Theorem 2.4. Let (X,δ) be an L- fuzzifying proximity space and let X satisfies the completely distributive law. The mapping $\tau_\delta : P(X)\to L$ defined by: $\tau_\delta(A) = \bigvee_{\alpha\in L-\{\bot\},A\in\tau_{\ \delta_\alpha}} \alpha$ is an L- fuzzifying topology and is called the L-fuzzifying topology induced by the L-fuzzifying proximity δ.

Proof. (1) $\tau_\delta(X) = \bigvee_{\alpha\in L-\{\bot\},\ X\in\tau_{\ \delta_\alpha}} \alpha = \top,\ \tau_\delta(\phi) = \bigvee_{\alpha\in L-\{\bot\},\phi\in\tau_{\ \delta_\alpha}} \alpha = \top.$

(2) $\tau_\delta(A\cap B) = \bigvee_{\alpha\in L-\{\bot\},A\cap B\in\tau_{\ \delta_\alpha}} \alpha \geq \bigvee_{\alpha\in L-\{\bot\},A\in\tau_{\ \delta_\alpha}\wedge B\in\tau_{\ \delta_\alpha}} \alpha$

$$= \left(\bigvee_{\alpha\in L-\{\bot\},A\in\tau_{\ \delta_\alpha}} \alpha\right)\wedge\left(\bigvee_{\alpha\in L-\{\bot\},B\in\tau_{\ \delta_\alpha}} \alpha\right) = \tau_\delta(A)\wedge\tau_\delta(B).$$

(3) Let $\{A_\lambda : \lambda \in \Lambda\} \subseteq P(X)$. Then we have, $\tau_\delta(\bigcup_{\lambda\in\Lambda} A_\lambda) = \bigvee_{\alpha\in L-\{\perp\}, \bigcup_{\lambda\in\Lambda} A_\lambda \in \tau_{\delta_\alpha}} \alpha$

$\geq \bigvee_{\alpha\in L-\{\perp\}, A_\lambda\in\tau_{\delta_\alpha}, \lambda\in\Lambda} \alpha = \bigwedge_{\lambda\in\Lambda} \bigvee_{\alpha\in L-\{\perp\}, A_\lambda\in\tau_{\delta_\alpha}} \alpha = \bigwedge_{\lambda\in\Lambda} \tau_\delta(A_\lambda)$.

3. L-fuzzifying uniformity and L-fuzzifying strong uniformity

Definition 3.1. Let X be a nonempty set and let $\mathcal{U} \in L^{P(X\times X)}$. Assume that the following statments are satisfied:

$(LFU0)$ There exists $U \in P(X \times X)$ s.t. $\mathcal{U}(U) = \top$,

$(LFU1)$ For any $U \in P(X \times X)$, $\mathcal{U}(U) > 0$, $[[\triangle, U[[= \top$,

$(LFU2)$ For any $U \in P(X \times X)$, $\mathcal{U}(U) = \mathcal{U}(U^{-1})$,

$(LFU3)$ For any $U \in P(X \times X)$, $\bigvee_{V\subseteq X\times X} \mathcal{U}(V) \wedge \subseteq (V \circ V, U) \geq \mathcal{U}(U)$,

$(LFU4)$ For any $U, V \in P(X \times X)$, $\mathcal{U}(U \cap V) \geq \mathcal{U}(U) \wedge \mathcal{U}(V)$,

$(LFU5)$ For any $U, V \in P(X \times X)$, $\mathcal{U}(V) \geq \mathcal{U}(U) \wedge \subseteq (U, V)$. Then $\mathcal{U}$ is called an L-fuzzifying uniformity and $(X, \mathcal{U})$ is called an L-fuzzifying uniform space.

Theorem 3.1. Let $(X, \mathcal{U})$ be an L-fuzzifying uniform space . Then $\tau \in L^{P(X)}$ defined by $\tau(A) = \bigwedge_{x\in A} \bigvee_{U[x]\subseteq A} \mathcal{U}(U)$ $\forall A \in P(X)$. is an L-fuzzifying topology on X and is called the L-fuzzifying topology on X induced by $\mathcal{U}$.

Proof. It is clear from $(LFU0)$ that $\tau(X) = \top$. Let $A_1, A_2 \subseteq X$. From $(LFU4)$ we have,

$$\tau(A_1) \wedge \tau(A_2) = (\bigwedge_{x_1\in A_1} \bigvee_{U_1[x_1]\subseteq A_1} \mathcal{U}(U_1)) \wedge (\bigwedge_{x_2\in A_2} \bigvee_{U_2[x_2]\subseteq A_2} \mathcal{U}(U_2))$$

$$= \bigwedge_{x_1\in A_1, x_2\in A_2} \bigvee_{U_1[x_1]\subseteq A_1,\, U_2[x_2]\subseteq A_2} (\mathcal{U}(U_1) \wedge \mathcal{U}(U_2))$$

$$\leq \bigwedge_{x_1\in A_1, x_2\in A_2} \bigvee_{U_1[x_1]\subseteq A_1,\, U_2[x_2]\subseteq A_2} \mathcal{U}(U_1 \cap U_2)$$

$$\leq \bigwedge_{x\in A_1\cap A_2} \bigvee_{U_1\cap\, U_2[x]\subseteq A_1\cap A_2} \mathcal{U}(U_1 \cap U_2)$$

$\leq \bigwedge_{x\in A_1\cap A_2} \bigvee_{U[x]\subseteq A_1\cap A_2} \mathcal{U}(U) = \tau(A_1 \cap A_2)$.Finally, for any $A_i \subseteq X$ $(i \in I)$,

we have $\tau(\bigcup_{i\in I} A_i) = \bigwedge_{x\in \bigcup_{i\in I} A_i} \bigvee_{U[x]\subseteq \bigcup_{i\in I} A_i} \mathcal{U}(U) = \bigwedge_{i\in I} (\bigwedge_{x\in A_i} \bigvee_{U[x]\subseteq \bigcup_{i\in I} A_i} \mathcal{U}(U))$

$\geq \bigwedge_{i\in I} (\bigwedge_{x\in A_i} \bigvee_{U[x]\subseteq A_i} \mathcal{U}(U)) = \bigwedge_{i\in I} \tau(A_i)$.

In [5], the authers introduced a counterexample in $[0, 1]$-fuzzifying setting to illustrate that there exists some α-cut of the $[0, 1]$-fuzzifying uniformity in the sense of M. S. Ying [11], which not a uniformity. In the following we introduce the concept of an L-fuzzifying strong uniform space as a generalization of the concept of fuzzifying strong uniform space [5].

Definition 3.3. Let X be a nonempty set and let $\mathcal{U} \in L^{P(X\times X)}$. If the following statments are satisfied:

$(LFSU0)$ There exists $U \in P(X \times X)$ s.t. $\mathcal{U}(U) = \top$

$(LFSU1)$ For any $U \in P(X \times X)$, $[[\triangle, U[[= \top$,

$(LFSU2)$ For any $U \in P(X \times X)$, $\mathcal{U}(U) \leq \mathcal{U}(U^{-1})$,

$(LFSU3)$ For any $U \in P(X \times X)$, $\exists\, V \in P(X \times X)$ s.t. $V \circ V \subseteq U$ and $\mathcal{U}(V) \geq \mathcal{U}(U)$,

$(LFSU4)$ For every $U, V \in P(X \times X)$, $\mathcal{U}(U \cap V) \geq \mathcal{U}(U) \wedge \mathcal{U}(V)$, and

$(LFSU5)$ For every $U, V \in P(X\times X)$, $\mathcal{U}(V) \geq \mathcal{U}(U) \wedge \subseteq (U, V)$, then $\mathcal{U}$ is called an L-fuzzifying strong uniform and $(X, \mathcal{U})$ is called an L-fuzzifying strong uniformity space.

Remark 3.1. If $L = [0, 1]$, the condition $(LFSU3)$ implies the condition $(FU3)^*$ in Definition 1.9.

Theorem 3.2. Let $(X, \mathcal{U})$ be an L-fuzzifying strong uniformity space. Then for each $\alpha \in L - \{\perp\}$, then the α-cut of $\mathcal{U}$ denoted by $\mathcal{U}_\alpha$ is a uniformity .

Proof. Let $\alpha \in L - \{\perp\}$.

$(U0)$ From $(LFSU0)$ $\exists U \in P(X \times X)$ s.t. $\mathcal{U}(U) = \top \geq \alpha$ so that $\mathcal{U}_\alpha \neq \phi$.

$(U1)$ Let $U \in \mathcal{U}_\alpha$. So from condition $(LFSU1)$, $[[\triangle, U[[= \top$ so that $\triangle \subseteq U$.

$(U2)$ Let $U \in \mathcal{U}_\alpha$. So from $(LFSU2)$, $\mathcal{U}(U^{-1}) \geq \mathcal{U}(U) \geq \alpha$.
Then $U^{-1} \in \mathcal{U}_\alpha$.

$(U3)$ Let $U \in \mathcal{U}_\alpha$. Then from $(LFSU3)$, $\exists\, V \in P(X \times X)\ s.t.$
$V \circ V \subseteq U$ and $\mathcal{U}(V) \geq \mathcal{U}(U) \geq \alpha$. Hence $V \in \mathcal{U}_\alpha$.

$(U4)$ Let $U, V \in \mathcal{U}_\alpha$. Then from $(LFSU4)$, $\mathcal{U}(U \cap V) \geq \mathcal{U}(U) \wedge \mathcal{U}(V) \geq \alpha$ so that $U \cap V \in \mathcal{U}_\alpha$.

$(U5)$ *Let* $U \in \mathcal{U}_\alpha$ and $U \subseteq V$. Then from $(LFSU5)$,
$\mathcal{U}(V) \geq \mathcal{U}(U) \wedge \subseteq (U, V) = \mathcal{U}(U) \geq \alpha$. Hence $V \in \mathcal{U}_\alpha$.

Theorem 3.3. Let $(X, \mathcal{U})$ be an L-fuzzifying strong uniformity space and let X satisfies the completely distributive law. The L-fuzzy set $\tau_{\mathcal{U}} \in L^{P(X)}$, defined by: $\tau_{\mathcal{U}}(A) = \bigvee_{\alpha \in L-\{\perp\},\ A \in \tau_{\mathcal{U}_\alpha}} \alpha$ is an L-fuzzifying topology. It is called the L-fuzzifying topology induced by the L-fuzzifying strong uniformity $\mathcal{U}$.

Proof. (1) Since $X, \phi \in \tau_{\mathcal{U}_\top}$, then we have that $\tau_{\mathcal{U}}(X) = \bigvee_{\alpha \in L-\{\perp\},\ X \in \tau_{\mathcal{U}_\alpha}} \alpha = \top.$ and $\tau_{\mathcal{U}}(\phi) = \bigvee_{\alpha \in L-\{\perp\},\ \phi \in \tau_{\mathcal{U}_\alpha}} \alpha = \top.$

$$(2)\ \tau_{\mathcal{U}}(A \cap B) = \bigvee_{\alpha \in L-\{\perp\},\ A\cap B \in \tau_{\mathcal{U}_\alpha}} \alpha \geq \bigvee_{(\alpha_1 \wedge \alpha_2) \in L-\{\perp\},\ A \in \tau_{\mathcal{U}_{\alpha_1}}, B \in \tau_{\mathcal{U}_{\alpha_2}}} (\alpha_1 \wedge \alpha_2)$$

$$= \bigvee_{\alpha\in L-\{\perp\},\ A\in\tau_{\mathcal{U}_{\alpha_1}}} \alpha_1 \wedge \bigvee_{\alpha\in L-\{\perp\},\ B\in\tau_{\mathcal{U}_{\alpha_2}}} \alpha_2 = \tau_{\mathcal{U}}(A) \wedge \tau_{\mathcal{U}}(B).$$

(3) $\tau_{\mathcal{U}}(\bigcup_{\lambda\in\Lambda} A_\lambda) = \bigvee_{\alpha\in L-\{\perp\},\ \bigcup_{\lambda\in\Lambda} A_\lambda\in\tau_{\mathcal{U}_\alpha}} \alpha \geq \bigvee_{\alpha\in L-\{\perp\},\ A_\lambda\in\tau_{\mathcal{U}_\alpha},\lambda\in\Lambda} \alpha = \bigvee_{f\in\prod_{\lambda\in\Lambda} M_\lambda} \bigwedge_{\lambda\in\Lambda} f(\lambda)$

$$= \bigwedge_{\lambda\in\Lambda} \bigvee M_\lambda = \bigwedge_{\lambda\in\Lambda} \bigvee_{\alpha\in M_\lambda} \alpha = \bigwedge_{\lambda\in\Lambda} \bigvee_{A_\lambda\in\tau_{\mathcal{U}_\alpha}} \alpha = \bigwedge_{\lambda\in\Lambda} \tau_{\mathcal{U}}(A_\lambda).$$

Where $M_\lambda = \{\alpha \in L-\{\perp\} :\ A_\lambda \in \tau_{\mathcal{U}_\alpha}\ \forall\ \lambda \in \Lambda\}$.

Theorem 3.4. Let $\delta_{\mathcal{U}_\alpha}$ be the proximity induced by the uniformity $\mathcal{U}_\alpha$ $\alpha \in L-\{\perp\}$. Then the mapping $\delta_{\mathcal{U}} \in L^{P(X)\times P(X)}$ defined by $\delta_{\mathcal{U}}(A,B) = \bigvee_{\alpha\in L-\{\perp\},\ (A,B)\in\delta_{\mathcal{U}_\alpha}} \alpha$ is an L-fuzzifying proximity. It is called the L-fuzzifying proximity induced by the L-fuzzifying strong uniformity $\mathcal{U}$.

Proof. $(LFP1)$ $\delta_{\mathcal{U}}(X,\phi) = \bigvee_{\alpha\in L-\{\perp\},\ (X,\phi)\in\delta_{\mathcal{U}_\alpha}} \alpha = \perp.$

$(LFP2)$ $\delta_{\mathcal{U}}(A,B) = \bigvee_{\alpha\in L-\{\perp\},\ (A,B)\in\delta_{\mathcal{U}_\alpha}} \alpha = \bigvee_{\alpha\in L-\{\perp\},\ (B,A)\in\delta_{\mathcal{U}_\alpha}} \alpha = \delta_{\mathcal{U}}(B,A).$

$(LFP3)$ $\delta_{\mathcal{U}}(A,B\cup C) = \bigvee_{\alpha\in L-\{\perp\},\ (A,B\cup C)\in\delta_{\mathcal{U}_\alpha}} \alpha$

$$= \left(\bigvee_{\alpha\in L-\{\perp\},\ (A,B)\in\delta_{\mathcal{U}_\alpha}} \alpha\right) \vee \left(\bigvee_{\alpha\in L-\{\perp\},\ (A,C)\in\delta_{\mathcal{U}_\alpha}} \alpha\right) = \delta_{\mathcal{U}}(A,B) \vee \delta_{\mathcal{U}}(A,C).$$

$(LFP4)$ $(A,B) \notin \delta_{\mathcal{U}_\alpha} \Rightarrow \exists C \in P(X)\ s.t.\ (A,C) \notin \delta_{\mathcal{U}_\alpha}$

and $(B, X-C) \notin \delta_{\mathcal{U}_\alpha}$. Therefore $\delta_{\mathcal{U}}(A,B) = \bigvee_{\alpha\in L-\{\perp\},\ (A,B)\in\delta_{\mathcal{U}_\alpha}} \alpha$

$$\geq \bigvee_{\alpha\in L-\{\perp\},\ (A,C)\in\delta_{\mathcal{U}_\alpha}\ \text{or}\ \alpha\in L-\{\perp\},\ (B,X-C)\in\delta_{\mathcal{U}_\alpha}} \alpha$$

$= (\bigvee_{\alpha\in L-\{\perp\},\ (A,C)\in\delta_{\mathcal{U}_\alpha}} \alpha) \vee (\bigvee_{\alpha\in L-\{\perp\},\ (B,X-C)\in\delta_{\mathcal{U}_\alpha}} \alpha) = \delta_{\mathcal{U}}(A,C) \vee \delta_{\mathcal{U}}(B,X-C).$

$(LFP5)$ Frist suppose that $CE(\{x\},\{y\}) = \top$. Then $\{x\} = \{y\}$. So,

$(\{x\},\{y\}) \in \delta_{\mathcal{U}_\alpha}$, for any $\alpha \in L-\{\bot\}$, Therefore

$$\delta_{\mathcal{U}}(\{x\},\{y\}) = \bigvee_{\alpha\in L-\{\bot\},\ (\{x\},\{y\})\in\delta_{\mathcal{U}_\alpha}} \alpha = \top.$$ Second if $CE(\{x\},\{y\}) = \bot$,

then $x \notin \{y\}$. So, $(\{x\},\{y\}) \notin \delta_{\mathcal{U}_\alpha}$, for any $\alpha \in L-\{\bot\}$.

Hence $\delta_{\mathcal{U}}(\{x\},\{y\}) = \bigvee_{\alpha\in L-\{\bot\},\ (\{x\},\{y\})\notin\delta_{\mathcal{U}_\alpha}} \alpha = \bot$.

4. Conclusions

in this paper, the notion of fuzzifying strong uniform space (Kheder, et al (2003)[5]) is generalized by introducing the concept of L-fuzzifying strong uniform spaces. Some results concerning this concept are obtained. In the present paper L is assumed to be a completely residuated lattice such that the following conditions are satisfied:
(1) L is totally ordered as a poset.(i.e. for each $a, b \in L$, $a < b$, or $b < a$.)
(2) L satisfies that '$\wedge$' is disributive over arbitrary joins.

In the future, we will study topological notions defined by means of regular open sets when these are planted into the framework of Ying's fuzzifying topological spaces (in Lukasiewicz fuzzy logic). We used fuzzy logic to introduce almost separation axioms (almost Hausdorff)-, (almost-regular)-and (almost-normal). we gave the relations of these axioms with each other as well as the relations with other fuzzifying separation axioms.

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