ANNALS OF COMMUNICATIONS IN MATHEMATICS
Volume 4, Number 2 (2021), 141-148
ISSN: 2582-0818

DIFFERENT TYPES OF PRIME BI-IDEALS IN TERNARY SEMIRINGS

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ABSTRACT. To interact three different types of prime bi-ideals are introduce and the relation between them is obtain. Also we have define three different m^b-systems. The bi-ideal P of a ternary semiring $\mathbb{R}$ is 2-prime if and only if $RML \subseteq P$, where R is a right ideal, M is a lateral ideal and L is a left ideal of $\mathbb{R}$, implies $R \subseteq P$ or $M \subseteq P$ or $L \subseteq P$. If B is a bi-ideal of $\mathbb{R}$, then H_B is the unique largest two-sided ideal of $\mathbb{R}$ contained B. Let M be a m_3^bsystem and bi-ideal B of $\mathbb{R}$ with $B \cap M = \phi$. Then there exists a 3-prime P of $\mathbb{R}$ containing B with $P \cap M = \phi$.

1. INTRODUCTION

D. H. Lehmer initiated the concept of ternary algebraic systems called triplexes in 1932 [9]. He investigated certain ternary algebraic systems called triplexes which turn out to be commutative ternary groups. The notion of semiring was introduced by Vandiver in 1934 [6]. In 1962, Hestenes [7] studied the notion of ternary algebra with application to matrices and linear transformation. In 1971, Lister characterized those additive subgroups of rings which are closed under the triple ring product and he called this algebraic system a ternary ring. Non-commutative ternary semirings have been an object of systematic study only quite recently, during the 20th century. Other natural non-commutative objects that arise are matrices. Algebraic structures play with applications in many area such as control engineering, combinatorics, functional analysis, graph theory, Euclidean geometry, optimization theory, discrete event dynamical systems, automata theory. Ternary semiring is a generalization of semirings. The results in ordinary semirings may be extended to n-ary semirings for arbitrary n but the transition from $n = 3$ to arbitrary n entails a great degree of complexity that makes it undesirable for exposition. The ring of integers $\mathbb{Z}$ which plays a role in the ring theory. The subset $\mathbb{Z}^+$ of $\mathbb{Z}$ is an additive semigroup which is closed under the ring product, that is $\mathbb{Z}^+$ is a semiring. Now, if we consider the subset $\mathbb{Z}^-$ of $\mathbb{Z}$, then we see that $\mathbb{Z}^-$ is an additive semigroup which is closed under the triple ring product, that is $\mathbb{Z}^-$ forms a ternary semiring. Arulmozhi et. al was introduced by new type of ideals in semirings and ternary semirings [1, 2, 13, 14, 15]. Let B be any bi-ideal of a semiring S and $L_B = \{x \in B | Sx \subseteq B\}$ and $H_B = \{y \in L_B | yS \subseteq L_B\}$ [16, 17].

2010 *Mathematics Subject Classification.* 16Y60, 16D25.
Key words and phrases. 1-prime; 2-prime; 3-prime; m_1^b-system; m_2^b-system; m_3^b-system.
Received: April 21, 2021. Accepted: July 15, 2021. Published: September 30, 2021.
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2. PRELIMINARIES

Throughout this paper, $\mathbb{R}$ denote a ternary semiring unless otherwise specified.

Definition 2.1. [5] A non-empty set $\mathbb{R}$ together with a binary operation, called addition and a ternary multiplication, denoted by juxtaposition is said to be a ternary semiring if $(\mathbb{R}, +)$ is a commutative semigroup and ternary multiplication satisfies the following:
(i) $(abc)de = a(bcd)e = ab(cde)$,
(ii) $(a + b)cd = acd + bcd$,
(iii) $a(b + c)d = abd + acd$,
(iv) $ab(c + d) = abc + abd$ for all $a, b, c, d, e \in \mathbb{R}$

Definition 2.2. [5] A ternary semiring $\mathbb{R}$ is said to commutative if $abc = acb = cab = cba$ for all $a, b, c \in \mathbb{R}$.

Definition 2.3. [5] An additive subsemigroup I of $\mathbb{R}$ is called a
(i) ternary subsemiring if $i_1i_2i_3 \in I$ for all $i_1, i_2, i_3 \in I$.
(ii) right (lateral, left) ideal if $ir_1r_2 \in I(r_1ir_2 \in I, r_1r_2i \in I)$ for all $r_1, r_2 \in \mathbb{R}$ and $i \in I$.

Lemma 2.1. *Let $a \in \mathbb{R}$. Then the principal*
(i) right ideal generated by "a" is given by $\langle a\rangle_r = \{na + a\mathbb{R}\mathbb{R} | n \in \mathbb{Z}_0^+\}$,
(ii) left ideal generated by "a" is given by $\langle a\rangle_l = \{na + \mathbb{R}\mathbb{R}a | n \in \mathbb{Z}_0^+\}$,
(iii) lateral ideal generated by "a" is given by $\langle a\rangle_{lat} = \{na + \mathbb{R}a\mathbb{R} + \mathbb{R}\mathbb{R}a\mathbb{R}\mathbb{R} | n \in \mathbb{Z}_0^+\}$,
(iv) two sided ideal generated by "a" is given by $\langle a\rangle_t = \{na+\mathbb{R}\mathbb{R}a+a\mathbb{R}\mathbb{R}+\mathbb{R}\mathbb{R}a\mathbb{R}\mathbb{R} | n \in \mathbb{Z}_0^+\}$,
(v) ideal generated by "a" is given by $\langle a\rangle = \{na+a\mathbb{R}\mathbb{R}+\mathbb{R}a\mathbb{R}+\mathbb{R}\mathbb{R}a\mathbb{R}\mathbb{R}+\mathbb{R}\mathbb{R}a | n \in \mathbb{Z}_0^+\}$,
(vi) bi-ideal generated by "a" is given by $\langle a\rangle_b = \{na + ma^3 + a\mathbb{R}a\mathbb{R}a | n, m \in \mathbb{Z}_0^+\}$.

Theorem 2.2. [5] *If P is an ideal of $\mathbb{R}$, then the following are equivalent.*
(i) P is a prime ideal.
(ii) $a\mathbb{R}b\mathbb{R}c \subseteq P, a\mathbb{R}\mathbb{R}b\mathbb{R}\mathbb{R}c \subseteq P, a\mathbb{R}\mathbb{R}b\mathbb{R}c\mathbb{R} \subseteq P, \mathbb{R}a\mathbb{R}b\mathbb{R}\mathbb{R}c \subseteq P$ implies $a \in P$ or $b \in P$ or $c \in P$.
(iii) $\langle a\rangle\langle b\rangle\langle c\rangle \subseteq P$ implies $a \in P$ or $b \in P$ or $c \in P$.

3. TYPES OF PRIME BI-IDEALS

By Theorem 2.2, if we replace the ideal P by bi-ideal P then we find that all the three conditions are different.

Definition 3.1. A bi-ideal P of $\mathbb{R}$ is called a
(i) 1-prime if $\mathscr{B}_1\mathscr{B}_2\mathscr{B}_3 \subseteq P$ implies $\mathscr{B}_1 \subseteq P$ or $\mathscr{B}_2 \subseteq P$ or $\mathscr{B}_3 \subseteq P$ for any bi-ideals $\mathscr{B}_1, \mathscr{B}_2$ and $\mathscr{B}_3$ of $\mathbb{R}$.
(ii) 2-prime if $a\mathbb{R}b\mathbb{R}c \subseteq P$ implies $a \in P$ or $b \in P$ or $c \in P$.
(iii) 3-prime if $\mathscr{I}_1\mathscr{I}_2\mathscr{I}_3 \subseteq P$ implies $\mathscr{I}_1 \subseteq P$ or $\mathscr{I}_2 \subseteq P$ or $\mathscr{I}_3 \subseteq P$ for any ideals $\mathscr{I}_1, \mathscr{I}_2$ and $\mathscr{I}_3$ of $\mathbb{R}$.

Theorem 3.1. *The bi-ideal P of $\mathbb{R}$ is 2-prime if and only if $RML \subseteq P$, with R is a right ideal, M is a lateral ideal and L is a left ideal of $\mathbb{R}$, implies $R \subseteq P$ or $M \subseteq P$ or $L \subseteq P$.*

Proof. Let P be an 2-prime of $\mathbb{R}$ and $RML \subseteq P$. To prove that $R \subseteq P$ or $M \subseteq P$ or $L \subseteq P$. Suppose that $R \nsubseteq P$ and $M \nsubseteq P$ implies that $a \in R$ but $a \notin P$ and $b \in M$ but $b \notin P$. To prove that $L \subseteq P$. For $c \in L$, $a\mathbb{R}b\mathbb{R}c \subseteq RML \subseteq P$. Since P be a 2-prime of $\mathbb{R}$ and $a \notin P$ and $b \notin P$, implies that $c \in P$. Thus, $L \subseteq P$.

Conversely, Suppose that $a\mathbb{R}b\mathbb{R}c \subseteq P$. Now, $(a\mathbb{R}\mathbb{R})(\mathbb{R}b\mathbb{R})(\mathbb{R}\mathbb{R}c) \subseteq a\mathbb{R}b\mathbb{R}c \subseteq P$ implies $a\mathbb{R}\mathbb{R} \subseteq P$ or $\mathbb{R}b\mathbb{R} \subseteq P$ or $\mathbb{R}\mathbb{R}c \subseteq P$. If $a\mathbb{R}\mathbb{R} \subseteq P$, then

$$\begin{aligned}\langle a\rangle_r \cdot \langle b\rangle_{lat} \cdot \langle c\rangle_l &= \Big[\{na|n \in \mathbb{Z}^+\} + a\mathbb{R}\mathbb{R}\Big] \cdot \Big[\{mb|m \in \mathbb{Z}^+\} + [\mathbb{R}b\mathbb{R} + \\ &\quad \mathbb{R}\mathbb{R}b\mathbb{R}\mathbb{R}]\Big] \cdot \Big[\{m^{'}c|m^{'} \in \mathbb{Z}^+\} + \mathbb{R}\mathbb{R}c\Big] \\ &\subseteq \{n^{'}abc|n^{'} \in \mathbb{Z}^+\} + a\mathbb{R}b\mathbb{R}c \\ &\subseteq a\mathbb{R}\mathbb{R} \subseteq P.\end{aligned}$$

Thus, $a \in P$ or $b \in P$ or $c \in P$.
Similarly, if $\mathbb{R}b\mathbb{R} \subseteq P$ then $\langle a\rangle_r \cdot \langle b\rangle_{lat} \cdot \langle c\rangle_l \subseteq [\mathbb{R}b\mathbb{R} \cup \mathbb{R}\mathbb{R}b\mathbb{R}\mathbb{R}] \subseteq P$.
If $\mathbb{R}\mathbb{R}c \subseteq P$, then $\langle a\rangle_r \cdot \langle b\rangle_{lat} \cdot \langle c\rangle_l \subseteq \mathbb{R}\mathbb{R}c \subseteq P$ implies $a \in P$ or $b \in P$ or $c \in P$.

Theorem 3.2. *Every 1-prime is a 2-prime of* $\mathbb{R}$.

Proof. Let P be 1-prime of $\mathbb{R}$. For the bi-ideals $\mathscr{B}_1, \mathscr{B}_2$ and $\mathscr{B}_3$ of $\mathbb{R}$ such that $\mathscr{B}_1\mathscr{B}_2\mathscr{B}_3 \subseteq P$ implies $\mathscr{B}_1 \subseteq P$ or $\mathscr{B}_2 \subseteq P$ or $\mathscr{B}_3 \subseteq P$. To prove P is a 2-prime of $\mathbb{R}$. Let $a\mathbb{R}b\mathbb{R}c \subseteq P$, for $a, b, c \in \mathbb{R}$. Now, $(a\mathbb{R}\mathbb{R}) \cdot (\mathbb{R}b\mathbb{R}) \cdot (\mathbb{R}\mathbb{R}c) \subseteq a\mathbb{R}b\mathbb{R}c \subseteq P$, since $a\mathbb{R}\mathbb{R}$ and $\mathbb{R}b\mathbb{R}$ and and $\mathbb{R}\mathbb{R}c$ are bi-ideals. Hence $a\mathbb{R}\mathbb{R} \subseteq P$ or $\mathbb{R}b\mathbb{R} \subseteq P$ or or $\mathbb{R}\mathbb{R}c \subseteq P$. Suppose that $a\mathbb{R}\mathbb{R} \subseteq P$. Consider $\langle a\rangle_b \cdot \langle a\rangle_b \cdot \langle a\rangle_b \subseteq a\mathbb{R}\mathbb{R} \subseteq P$. Then $a \in P$. Similarly if $\mathbb{R}b\mathbb{R} \subseteq P$ and $\mathbb{R}\mathbb{R}c \subseteq P$, then $b \in P$ and $c \in P$ respectively. Thus P is a 2-prime of $\mathbb{R}$.

Converse of the Theorem 3.2 need not be true as the following Example.

Example 3.2. Consider the set $\mathbb{R} = \{a, b, c, d, e, f, g, h, i, j, k, l, m, n, o, p\}$ with the following compositions:

$+$	a	b	c	d	e	f	g	h	i	j	k	l	m	n	o	p
a	a	b	c	d	e	f	g	h	i	j	k	l	m	n	o	p
b	b	a	f	g	k	c	d	p	o	l	e	j	n	m	i	h
c	c	f	a	n	h	b	m	e	l	o	p	i	g	d	j	k
d	d	g	n	a	i	m	b	l	e	p	o	h	f	c	k	j
e	e	k	h	i	a	p	o	c	d	m	b	n	j	l	g	f
f	f	c	b	m	p	a	n	k	j	i	h	o	d	g	l	e
g	g	d	m	b	o	n	a	j	k	h	i	p	c	f	e	l
h	h	p	e	l	c	k	j	a	n	g	f	d	o	i	m	b
i	i	o	l	e	d	j	k	n	a	f	g	c	p	h	b	m
j	j	l	o	p	m	i	h	g	f	a	n	b	e	k	c	d
k	k	e	p	o	b	h	i	f	g	n	a	m	l	j	d	c
l	l	j	i	h	n	o	p	d	c	b	m	a	k	e	f	g
m	m	n	g	f	j	d	c	o	p	e	l	k	a	b	h	i
n	n	m	d	c	l	g	f	i	h	k	j	e	b	a	p	o
o	o	i	j	k	g	l	e	m	b	c	d	f	h	p	a	n
p	p	h	k	j	f	e	l	b	m	d	c	g	i	o	n	a

$\cdot$	a	b	c	d	e	f	g	h	i	j	k	l	m	n	o	p
a	b_1	b_1	b_1	b_1	b_1	b_1	b_1	b_1	b_1	b_1	b_1	b_1	b_1	b_1	b_1	b_1
b	b_1	b_2	b_3	b_1	b_1	b_6	b_2	b_3	b_1	b_6	b_2	b_3	b_6	b_3	b_2	b_6
c	b_1	b_1	b_1	b_2	b_3	b_1	b_2	b_3	b_6	b_6	b_3	b_6	b_2	b_2	b_6	b_3
d	b_1	b_4	b_5	b_1	b_1	b_9	b_4	b_5	b_1	b_9	b_4	b_5	b_9	b_5	b_4	b_9
e	b_1	b_1	b_1	b_4	b_5	b_1	b_4	b_5	b_9	b_9	b_5	b_9	b_4	b_4	b_9	b_5
f	b_1	b_2	b_3	b_2	b_3	b_6	b_1	b_1	b_6	b_1	b_6	b_2	b_3	b_6	b_3	b_2
g	b_1	b_7	b_8	b_1	b_1	b_{10}	b_7	b_8	b_1	b_{10}	b_7	b_8	b_{10}	b_8	b_7	b_{10}
h	b_1	b_1	b_1	b_7	b_8	b_1	b_7	b_8	b_{10}	b_{10}	b_8	b_{10}	b_7	b_7	b_{10}	b_8
i	b_1	b_4	b_5	b_4	b_5	b_9	b_1	b_1	b_9	b_1	b_9	b_4	b_5	b_9	b_5	b_4
j	b_1	b_7	b_8	b_7	b_8	b_{10}	b_1	b_1	b_{10}	b_1	b_{10}	b_7	b_8	b_{10}	b_8	b_7
k	b_1	b_2	b_3	b_4	b_5	b_6	b_7	b_8	b_9	b_{10}	b_{11}	b_{12}	b_{13}	b_{14}	b_{15}	b_{16}
l	b_1	b_4	b_5	b_7	b_8	b_9	b_2	b_3	b_{10}	b_6	b_{12}	b_{13}	b_{11}	b_{15}	b_{16}	b_{14}
m	b_1	b_7	b_8	b_2	b_3	b_{10}	b_4	b_5	b_6	b_9	b_{13}	b_{11}	b_{12}	b_{16}	b_{14}	b_{15}
n	b_1	b_4	b_5	b_2	b_3	b_9	b_7	b_8	b_6	b_{10}	b_{14}	b_{16}	b_{15}	b_{11}	b_{13}	b_{12}
o	b_1	b_7	b_8	b_4	b_5	b_{10}	b_2	b_3	b_9	b_6	b_{15}	b_{14}	b_{16}	b_{12}	b_{11}	b_{13}
p	b_1	b_2	b_3	b_7	b_8	b_6	b_4	b_5	b_{10}	b_9	b_{16}	b_{15}	b_{14}	b_{13}	b_{12}	b_{11}

$\cdot$	a	b	c	d	e	f	g	h	i	j	k	l	m	n	o	p
b_1	a	a	a	a	a	a	a	a	a	a	a	a	a	a	a	a
b_2	a	b	c	a	a	f	b	c	a	f	b	c	f	c	b	f
b_3	a	a	a	b	c	a	b	c	f	f	c	f	b	b	f	c
b_4	a	d	e	a	a	i	d	e	a	i	d	e	i	e	d	i
b_5	a	a	a	d	e	a	d	e	i	i	e	i	d	d	i	e
b_6	a	b	c	b	c	f	a	a	f	a	f	b	c	f	c	b
b_7	a	g	h	a	a	j	g	h	a	j	g	h	j	h	g	j
b_8	a	a	a	g	h	a	g	h	j	j	h	j	g	g	j	h
b_9	a	d	e	d	e	i	a	a	i	a	i	d	e	i	e	d
b_{10}	a	g	h	g	h	j	a	a	j	a	j	g	h	j	h	g
b_{11}	a	b	c	d	e	f	g	h	i	j	k	l	m	n	o	p
b_{12}	a	d	e	g	h	i	b	c	j	f	l	m	k	o	p	n
b_{13}	a	g	h	b	c	j	d	e	f	i	m	k	l	p	n	o
b_{14}	a	d	e	b	c	i	g	h	f	j	n	p	o	k	m	l
b_{15}	a	g	h	d	e	j	b	c	i	f	o	n	p	l	k	m
b_{16}	a	b	c	g	h	f	d	e	j	i	p	o	n	m	l	k

Clearly $P = \{a, d, e, i\}$ is a 2-prime. Now, $\{a, g\} \cdot \{a, h\} \cdot \{a, f\} = \{a\} \subseteq P$ with $\{a, g\} \nsubseteq P$ and $\{a, h\} \nsubseteq P$ and $\{a, f\} \nsubseteq P$. This implies that P is not a 1-prime.

Theorem 3.3. *Every 2-prime is a 3-prime of* $\mathbb{R}$.

Proof. For ideals $\mathscr{I}_1, \mathscr{I}_2$ and $\mathscr{I}_3$ of $\mathbb{R}$ such that $\mathscr{I}_1 \cdot \mathscr{I}_2 \cdot \mathscr{I}_3 \subseteq P$. If $\mathscr{I}_2 \nsubseteq P$ and $\mathscr{I}_3 \nsubseteq P$, let $b \in \mathscr{I}_2 \setminus P$ and $c \in \mathscr{I}_3 \setminus P$. For any $a \in \mathscr{I}_1$, $a\mathbb{R}b\mathbb{R}c \subseteq \langle a\rangle \cdot \langle b\rangle \cdot \langle c\rangle \subseteq \mathscr{I}_1 \cdot \mathscr{I}_2 \cdot \mathscr{I}_3 \subseteq P$. Hence $a \in P$. Then $\mathscr{I}_1 \subseteq P$. Thus P is a 3-prime of $\mathbb{R}$.

Converse of the above theorem not true as shown by the Example.

Example 3.3. In Example 3.2, Clearly $P = \{a, h\}$ is a 3-prime of $\mathbb{R}$. Now, $g\mathbb{R}b\mathbb{R}c = \{a, h\} \subseteq P$ but $g \notin P$,$b \notin P$ and $c \notin P$. This means that P is not a 2-prime of $\mathbb{R}$.

Definition 3.4. A subset M of $\mathbb{R}$ is called a
(i) m_1^b-system if for any $a, b, c \in M$, there exists $a_1 \in \langle a\rangle_b$, $b_1 \in \langle b\rangle_b$ and $c_1 \in \langle c\rangle_b$ such that $a_1 \cdot b_1 \cdot c_1 \in M$.
(ii) m_2^b-system if for any $a, b, c \in M$, there exists $a_1 \in \langle a\rangle_r$, $b_1 \in \langle b\rangle_{lat}$ and $c_1 \in \langle c\rangle_l$ such that $a_1 \cdot b_1 \cdot c_1 \in M$.
(iii) m_3^b-system if for any $a, b, c \in M$, there exists $a_1 \in \langle a\rangle$, $b_1 \in \langle b\rangle$ and $c_1 \in \langle c\rangle$ such that $a_1 \cdot b_1 \cdot c_1 \in M$.

Theorem 3.4. *If P is a bi-ideal of $\mathbb{R}$, then P is a 1-prime (2-prime, 3-prime) if and only if $\mathbb{R} \setminus P$ is an m_1^b-system (m_2^b-system, m_3^b-system) of $\mathbb{R}$.*

Proof. Let P be an 1-prime of $\mathbb{R}$ and let $a, b, c \in \mathbb{R} \setminus P$. Hence $a, b, c \in \mathbb{R}$ but $a \notin P, b \notin P$ and $c \notin P$. Then $\langle a\rangle_b \cdot \langle b\rangle_b \cdot \langle c\rangle_b \nsubseteq P$ implies there exists $a_1 \in \langle a\rangle_b$, $b_1 \in \langle b\rangle_b$ and $c_1 \in \langle c\rangle_b$ such that $a_1 \cdot b_1 \cdot c_1 \in \mathbb{R} \setminus P$. Thus $\mathbb{R} \setminus P$ is a m_1^b-system.

Conversely, Let $\mathbb{R} \setminus P$ be an m_1^b-system. let $\mathscr{B}_1 \cdot \mathscr{B}_2 \cdot \mathscr{B}_3 \subseteq P$ for the bi-ideals $\mathscr{B}_1, \mathscr{B}_2$ and $\mathscr{B}_3$ of $\mathbb{R}$. To prove that $\mathscr{B}_1 \subseteq P$ or $\mathscr{B}_2 \subseteq P$ or $\mathscr{B}_3 \subseteq P$. Let us arrive at a contradiction. If $\mathscr{B}_1 \nsubseteq P$, $\mathscr{B}_2 \nsubseteq P$ and $\mathscr{B}_3 \nsubseteq P$, let $b_1 \in \mathscr{B}_1 \setminus P$, $b_2 \in \mathscr{B}_2 \setminus P$ and $b_3 \in \mathscr{B}_3 \setminus P$. Thus $b_1, b_2, b_3 \in \mathbb{R} \setminus P$ implies $\langle b_1\rangle_b \cdot \langle b_2\rangle_b \cdot \langle b_3\rangle_b \nsubseteq P$, which is a contradiction. Thus $\mathscr{B}_1 \subseteq P$ or $\mathscr{B}_2 \subseteq P$ or $\mathscr{B}_3 \subseteq P$. Hence P is a 1-prime of $\mathbb{R}$. Similarly we can prove other two cases.

Lemma 3.5. *The following statements are true.*
(i) Every m_1^b-system is an m_2^b-system.
(ii) Every m_2^b-system is an m_3^b-system.

Converse of the above lemma not true by the following Example.

Example 3.5. By the Example 3.2, Clearly, (i) $M = \{b, c, f, g, h, j, k, l, m, n, o, p\}$ is an m_2^b-system, but not m_1^b-system. For $b, f, g \in M$, but there is no $x_1 \in \langle b\rangle_b$ and $y_1 \in \langle f\rangle_b$ and $z_1 \in \langle g\rangle_b$ such that $x_1 \cdot y_1 \cdot z_1 \in M$. Since $\langle b\rangle_b \cdot \langle f\rangle_b \cdot \langle g\rangle_b = \{a, b\} \cdot \{a, f\} \cdot \{a, g\} =$

$\{a\} \notin M$.
(ii) Clearly, $M = \{b, c, d, e, f, g, i, j, k, l, m, n, o, p\}$ is a m_3^b-system, but not a m_2^b-system by $j\mathbb{R}d\mathbb{R}c = \{a, h\}$.

Lemma 3.6. *Every m_2^b-system is an m-system and vice versa.*

Proof. Let M be an m_2^b-system. Let $a, b, c \in M$, then there exist $a_1 \in \langle a\rangle_r, b_1 \in \langle b\rangle_{lat}$ and $c_1 \in \langle c\rangle_l$ such that $a_1 \cdot b_1 \cdot c_1 \in M$. Now,

$$\begin{aligned}
a_1 \cdot b_1 \cdot c_1 &= [n_1a + ar_1r_2] \cdot [n_2b + r_3br_4 + r_5r_6br_7r_8] \cdot [n_3c + r_9r_{10}c] \\
&= \Big[n_1n_2ab + (n_1a)r_3br_4 + (n_1a)r_5r_6br_7r_8 + ar_1r_2a(n_2b) + \\
&\quad ar_1r_2r_3br_4 + ar_1r_2r_5r_6br_7r_8\Big] \cdot [n_3c + r_9r_{10}c] \\
&= \Big[n_1n_2n_3abc + (n_1a)r_3br_4(n_3c) + (n_1a)r_5r_6br_7r_8(n_3c) + ar_1r_2a \\
&\quad (n_2n_3bc) + ar_1r_2r_3br_4(n_3c) + ar_1r_2r_5r_6br_7r_8(n_3c) + n_1n_2abr_9r_{10}c \\
&\quad +(n_1a)r_3br_4r_9r_{10}c + (n_1a)r_5r_6br_7r_8r_9r_{10}c + ar_1r_2a(n_2b)r_9r_{10}c + \\
&\quad ar_1r_2r_3br_4r_9r_{10}c + ar_1r_2r_5r_6br_7r_8r_9r_{10}c\Big] \\
&= n_1n_2n_3abc + ar^{'}br^{''}c + n_1n_2abr^{'''}c \in M.
\end{aligned}$$

Again $a, b, n_1n_2n_3abc + ar^{'}br^{''}c + n_1n_2abr^{'''}c \in M$, then there exist $a_2 \in \langle a\rangle_r, b_2 \in \langle b\rangle_{lat}$ and $c_2 \in \langle n_1n_2n_3abc + ar^{'}br^{''}c + n_1n_2abr^{'''}c\rangle_l$ such that $a_2 \cdot b_2 \cdot c_2 \in M$. Now, $a_2 \cdot b_2 \cdot c_2 = ar_{11}br_{12}c \in a\mathbb{R}b\mathbb{R}c$. Thus, $ar_{11}br_{12}c = a_2 \cdot b_2 \cdot c_2 \in M$. Therefore M is a m-system.

Conversely, let M be an m-system and $a, b, c \in M$. Then there exist $r_1, r_2 \in \mathbb{R}$ such that $ar_1br_2c \in M$. Let $ar_1 = a_1$ and $r_2c = c_1$, then there exists $a_1 \in \langle a\rangle_r,\ b \in \langle b\rangle_{lat}$ and $c_1 \in \langle c\rangle_l$ such that $a_1 \cdot b \cdot c_1 \in M$. Therefore M is a m_2^b-system.

Definition 3.6. Let B be any bi-ideal of $\mathbb{R}$.
(i) let $L_B = \{x \in B|\mathbb{R}\mathbb{R}x \subseteq B\}$ and relative to $H_B = \{y \in L_B|y\mathbb{R}\mathbb{R} \subseteq L_B\}$.
(ii) $R_B = \{x \in B|x\mathbb{R}\mathbb{R} \subseteq B\}$ and relative to $H_B = \{y \in R_B|\mathbb{R}\mathbb{R}y \subseteq R_B\}$.

Lemma 3.7. *If B is a bi-ideal of $\mathbb{R}$, then H_B is the unique largest two-sided ideal of $\mathbb{R}$ contained B.*

Proof. Let B is any bi-ideal of $\mathbb{R}$. To prove that H_B is a two-sided ideal of $\mathbb{R}$. Since $L_B \subseteq B$ and $H_B \subseteq L_B$. Therefore $H_B \subseteq L_B \subseteq B$. Let $x \in H_B$ and $y_1, y_2 \in \mathbb{R}$. Then $x \in H_B \subseteq B$ implies that $x \in B$. Since x is an element of L_B. We have $\mathbb{R}\mathbb{R}x \subseteq B$ and $x\mathbb{R}\mathbb{R} \subseteq L_B$. Then $y_1y_2x \in \mathbb{R}\mathbb{R}x \subseteq B$ implies $y_1y_2x \in B$ and $\mathbb{R}\mathbb{R}y_1y_2x \subseteq \mathbb{R}\mathbb{R}\mathbb{R}\mathbb{R}x \subseteq \mathbb{R}\mathbb{R}x \subseteq B$ implies that $y_1y_2x \in L_B$. Now, $xy_1y_2 \in x\mathbb{R}\mathbb{R} \subseteq L_B$. Hence $xy_1y_2 \in L_B$ and $y_1y_2x \in L_B$. Hence L_B is a two-sided ideal of $\mathbb{R}$. To prove that H_B is a two-sided ideal of $\mathbb{R}$, that is $xy_1y_2 \in H_B$ and $y_1y_2x \in H_B$. Now, $xy_1y_2\mathbb{R}\mathbb{R} \subseteq x\mathbb{R}\mathbb{R}\mathbb{R}\mathbb{R} \subseteq x\mathbb{R}\mathbb{R} \subseteq L_B$. Hence $xy_1y_2\mathbb{R}\mathbb{R} \subseteq L_B$ implies $xy_1y_2 \in H_B$. Now, $y_1y_2x\mathbb{R}\mathbb{R} \subseteq \mathbb{R}\mathbb{R}x\mathbb{R}\mathbb{R} \subseteq \mathbb{R}\mathbb{R}L_B \subseteq L_B$. Since L_B is a left ideal of $\mathbb{R}$, $y_1y_2x \in H_B$. Hence H_B is a two-sided ideal of $\mathbb{R}$. It enough to prove H_B is a largest two sided ideal of $\mathbb{R}$. Let I be any ideal of $\mathbb{R}$ and $I \subseteq B$. Let $i \in I$, then $i \in B$ and $\mathbb{R}\mathbb{R}i \subseteq I \subseteq B$. Hence $\mathbb{R}\mathbb{R}i \subseteq B$ implies $i \in L_B$. Hence $I \subseteq L_B$. Next, $i \in L_B$ and $i\mathbb{R}\mathbb{R} \subseteq I \subseteq L_B$. Therefore $i\mathbb{R}\mathbb{R} \subseteq L_B$. Thus, $i \in H_B$. Hence $I \subseteq H_B$.

Corollary 3.8. *If B is a 1-prime (2-prime) of $\mathbb{R}$, then H_B is a prime ideal of $\mathbb{R}$.*

Proof. Straightforward.

The following Example shows that the converse of the Corollary 3.8 need not be true.

Example 3.7. (i) By the Example 3.2, $B = \{a, j\}$ is a bi-ideal and $H_B = \{a\}$ is a prime ideal, but B is not a 1-prime of $\mathbb{R}$. For the bi-ideals $\mathscr{B}_1 = \{a, d\}$ and $\mathscr{B}_2 = \{a, b\}$ and $\mathscr{B}_3 = \{a, i\}$. Now, $\mathscr{B}_1 \cdot \mathscr{B}_2 \cdot \mathscr{B}_3 = \{a\} \subseteq B$ but $\mathscr{B}_1 \nsubseteq B$ and $\mathscr{B}_2 \nsubseteq B$ and $\mathscr{B}_3 \nsubseteq B$.
(ii) Clearly, $B = \{a, f\}$ is a bi-ideal and $H_B = \{a\}$ is a prime ideal, but B is not a 2-prime of $\mathbb{R}$. For $c, e, i \in \mathbb{R}$ and $c\mathbb{R}e\mathbb{R}i = \{a, f\} \subseteq B$ but $c \notin B, e \notin B$ and $i \notin B$.

Theorem 3.9. *The bi-ideal B is a 3-prime of $\mathbb{R}$ if and only if H_B is a prime ideal of $\mathbb{R}$.*

Proof. Let B be any bi-ideal of $\mathbb{R}$ and B be a 3-prime of $\mathbb{R}$. Let us show that H_B is a prime ideal of $\mathbb{R}$. Suppose that $\mathscr{I}_1, \mathscr{I}_2$ and $\mathscr{I}_3$ be the ideals of $\mathbb{R}$ such that $\mathscr{I}_1 \cdot \mathscr{I}_2 \cdot \mathscr{I}_3 \subseteq H_B$. By Lemma 3.7, H_B is the largest two sided ideal of $\mathbb{R}$ such that $H_B \subseteq B$. Thus $\mathscr{I}_1 \subseteq H_B$ or $\mathscr{I}_2 \subseteq H_B$ or $\mathscr{I}_3 \subseteq H_B$.

Conversely, Suppose that H_B is a prime ideal of $\mathbb{R}$. To show that B is a 3-prime of $\mathbb{R}$. For the ideals $\mathscr{I}_1, \mathscr{I}_2$ and $\mathscr{I}_2$ of $\mathbb{R}$ such that $\mathscr{I}_1 \cdot \mathscr{I}_2 \cdot \mathscr{I}_3 \subseteq B$. To show that $\mathscr{I}_1 \subseteq B$ or $\mathscr{I}_2 \subseteq B$ or $\mathscr{I}_3 \subseteq B$. Now, $\mathscr{I}_1 \cdot \mathscr{I}_2 \cdot \mathscr{I}_3 \subseteq \mathscr{I}_1 \cdot \mathscr{I}_2 \cdot \mathscr{I}_3 \subseteq H_B$, since H_B is a prime ideal of $\mathbb{R}$. This implies $\mathscr{I}_1 \subseteq H_B \subseteq B$ or $\mathscr{I}_2 \subseteq H_B \subseteq B$ or $\mathscr{I}_3 \subseteq H_B \subseteq B$. Hence B is a 3-prime of $\mathbb{R}$.

Theorem 3.10. *Let M be a m_3^bsystem and B be a bi-ideal of $\mathbb{R}$ with $B \cap M = \phi$. Then there exists a 3-prime P of $\mathbb{R}$ containing B with $P \cap M = \phi$.*

Proof. Let $X = \big\{J | J$ is a bi-ideal with $B \subseteq J$ and $J \cap M = \phi\big\}$. Clearly X is non-empty. By Zorn's lemma, there exist a maximal element P in X. Let us show that P is a 3-prime of $\mathbb{R}$. By Theorem 3.9, it is enough to prove that H_P is a prime ideal in $\mathbb{R}$. Since $H_P \subseteq P$ and $P \cap M = \phi$ implies that $H_P \cap M = \phi$. Then H_P is a largest ideal in $\mathbb{R}$ such that $H_P \cap M = \phi$. We claim that $\langle a\rangle\langle b\rangle\langle c\rangle \subseteq H_P$. Then $\langle a\rangle \subseteq H_P$ or $\langle b\rangle \subseteq H_P$ or $\langle c\rangle \subseteq H_P$. By proving at a contriduction, If $\langle a\rangle \nsubseteq H_P$, $\langle b\rangle \nsubseteq H_P$ and $\langle c\rangle \nsubseteq H_P$, then $x \in \langle a\rangle \setminus H_P$, $y \in \langle b\rangle \setminus H_P$ and $z \in \langle c\rangle \setminus H_P$. Then $\langle x\rangle \subseteq \langle a\rangle$, $\langle y\rangle \subseteq \langle b\rangle$ and $\langle z\rangle \subseteq \langle c\rangle$. If $\langle a\rangle\langle b\rangle\langle c\rangle \subseteq H_P$ then $\langle x\rangle\langle y\rangle\langle z\rangle \subseteq \langle a\rangle\langle b\rangle\langle c\rangle \subseteq H_P$. Then $(H_P + \langle x\rangle) \cap M \neq \phi$ and $(H_P + \langle y\rangle) \cap M \neq \phi$ and $(H_P + \langle z\rangle) \cap M \neq \phi$. Thus $(H_P + \langle x\rangle)(H_P + \langle y\rangle)(H_P + \langle z\rangle) \subseteq H_P$. Then there exist $m_1 \in (H_P + \langle x\rangle) \cap M$ and $m_2 \in (H_P + \langle y\rangle) \cap M$ and $m_3 \in (H_P + \langle z\rangle) \cap M$ such that $m_1{}' m_2{}' m_3{}' \in M$ for some $m_1{}' \in \langle m_1\rangle \subseteq (H_P + \langle x\rangle)$ and $m_2{}' \in \langle m_2\rangle \subseteq (H_P + \langle y\rangle)$ and $m_3{}' \in \langle m_3\rangle \subseteq (H_P + \langle z\rangle)$. Hence $m_1{}' m_2{}' m_3{}' \in (H_P + \langle x\rangle)(H_P + \langle y\rangle)(H_P + \langle z\rangle) \subseteq H_P$. Which is a contradiction. Thus $\langle a\rangle\langle b\rangle\langle c\rangle \nsubseteq H_P$. Hence H_P is a prime ideal of $\mathbb{R}$. By Theorem 3.9, P is a 3-prime of $\mathbb{R}$. If H_P is not a largest element in X, then there is a maximal ideal P' in $\mathbb{R}$ such that $H_P \subseteq P'$ and $P' \cap M = \phi$. Thus $H_{P'}$ is a prime ideal and P' is the required bi-ideal of $\mathbb{R}$.

4. Types of Semiprime bi-ideals

Definition 4.1. A bi-ideal P of $\mathbb{R}$ is called a
(i) 1-semiprime if $\mathscr{B}^3 \subseteq P$ implies $\mathscr{B} \subseteq P$ for any bi-ideal $\mathscr{B}$ of $\mathbb{R}$.
(ii) 2-semiprime if $a\mathbb{R}a\mathbb{R}a \subseteq P$ implies $a \in P$.
(iii) 3-semiprime if $\mathscr{I}^3 \subseteq P$ implies $\mathscr{I} \subseteq P$ for any ideal $\mathscr{I}$ of $\mathbb{R}$.

Theorem 4.1. *A bi-ideal P of $\mathbb{R}$ is 2-semiprime if and only if $\mathscr{R}^3 \subseteq P$ ($\mathscr{M}^3 \subseteq P, \mathscr{L}^3 \subseteq P$), with $\mathscr{R}$ is a right ideal ($\mathscr{M}$ is a lateral ideal and $\mathscr{L}$ is a left ideal) of $\mathbb{R}$, implies $\mathscr{R} \subseteq P$ ($\mathscr{M} \subseteq P, \mathscr{L} \subseteq P$).*

Theorem 4.2. *If P is a 1-semiprime of $\mathbb{R}$, then P is a 2-semiprime of $\mathbb{R}$.*

The following Example shows that the converse of the Theorem 4.2 need not be true.

Example 4.2. By the Example 3.2, $\mathbb{R} = \{a, b, c, d, e, f, g, h, i, j, k, l, m, n, o, p\}$. Clearly, $P = \{a, b\}$ is a 2-semiprime, but not a 1-semiprime. For the bi-ideal $\mathscr{B} = \{a, c\}$ and $\mathscr{B}^3 \subseteq P$ but $\mathscr{B} \nsubseteq P$.

Theorem 4.3. *If P is a 2-semiprime of $\mathbb{R}$, then P is a 3-semiprime of $\mathbb{R}$.*

Converse of the Theorem 4.3 need not be true as the following Example shows.

Example 4.3. Consider the ternary semiring $\mathbb{R} = \{a, b, c, d, e, f\}$ with the following compositions:

$+$	a	b	c	d	e	f
a	a	b	c	d	e	f
b	b	b	c	d	e	f
c	c	c	c	f	e	f
d	d	d	f	d	e	f
e	e	e	e	e	e	e
f	f	f	f	f	e	f

$\cdot$	a	b	c	d	e	f
a	u	u	u	u	u	u
b	u	v	w	v	w	w
c	u	v	w	v	w	w
d	u	x	y	x	y	y
e	u	x	y	x	y	y
f	u	x	y	x	y	y

$\cdot$	a	b	c	d	e	f
u	a	a	a	a	a	a
v	a	b	c	b	c	c
w	a	b	c	b	c	c
x	a	d	e	d	e	e
y	a	d	e	d	e	e
z	a	d	e	d	e	e

By routine calculation, $P = \{a, e\}$ is a 3-semiprime of $\mathbb{R}$. Now, $f\mathbb{R}f\mathbb{R}f = \{a, e\} \subseteq P$ but $f \notin P$ implies P is not a 2-semiprime of $\mathbb{R}$.

Definition 4.4. A subset N of $\mathbb{R}$ is called a
(i) n_1^b-system if for any $a \in N$, there exist $a_1, a_2, a_3 \in \langle a \rangle_b$ such that $a_1 \cdot a_2 \cdot a_3 \in N$.
(ii) n_2^b-system if for any $a \in N$, there exist $a_1, a_2, a_3 \in \langle a \rangle_r$ or $a_1, a_2, a_3 \in \langle a \rangle_{lat}$ or $a_1, a_2, a_3 \in \langle a \rangle_l$) such that $a_1 \cdot a_2 \cdot a_3 \in N$.
(iii) n_3^b-system if for any $a \in N$, there exist $a_1, a_2, a_3 \in \langle a \rangle$ such that $a_1 \cdot a_2 \cdot a_3 \in N$.

Theorem 4.4. *If P is a bi-ideal of $\mathbb{R}$, then P is a 1-semiprime (2-semiprime, 3-semiprime) if and only if $\mathbb{R} \setminus P$ is an n_1^b-system (n_2^b-system, n_3^b-system).*

Lemma 4.5. *Every n_1^b-system is a n_2^b-system.*

Converse need not be true as the following Example.

Example 4.5. By the Example 3.2,$N = \{c, d, e, f, g, h, i, j, k, l, m, n, o, p\}$ is a n_2^bsystem, but not a n_1^bsystem. For $j \in N$, there is no $x_1, x_2, x_3 \in \langle j \rangle_b$ such that $x_1 \cdot x_2 \cdot x_3 \in N$. Since $\langle j \rangle_b \cdot \langle j \rangle_b \cdot \langle j \rangle_b = \{a\} \notin N$.

Lemma 4.6. *Every n_2^b-system is a n_3^b-system.*

Converse need not be true as the following Example.

Example 4.6. By the Example 4.3, $N = \{b, c, d, f\}$ is a n_3^b-system but not a n_2^b-system of $\mathbb{R}$. For $f \in N$ and $f\mathbb{R}f\mathbb{R}f = e \notin N$.

Corollary 4.7. *If B is a 1-semiprime (2-semiprime) of $\mathbb{R}$, then H_B is a semiprime ideal of $\mathbb{R}$.*

Converse of the above Corollary need not be true as the following Example.

Example 4.7. (i) By the Example 3.2, By routine computation, $H_B = \{a, j\}$, $B = \{a, g, h, j\}$ and $\mathscr{B}_1 = \{a, d\}$. Clearly, H_B is a semiprime ideal, but B is not a 1-semiprime of $\mathbb{R}$ by ${\mathscr{B}_1}^3 = \{a\} \subseteq B$ but $\mathscr{B}_1 \not\subseteq B$.
(ii) By the Example 4.3, Taking $H_B = \{a\}$ is a semiprime ideal of $\mathbb{R}$. For the bi-ideal $B = \{a, d, e\}$ and $f\mathbb{R}f\mathbb{R}f = \{a, e\} \subseteq B$ but $f \notin B$ implies that B is not a 2-semiprime of $\mathbb{R}$.

Theorem 4.8. *The bi-ideal B is a 3-semiprime of $\mathbb{R}$ if and only if H_B is a semiprime ideal of $\mathbb{R}$.*

5. Acknowledgements

The author is obliged the thankful to the reviewer for the numerous and significant suggestions that raised the consistency of the ideas presented in this paper.

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