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APPROXIMATION ON THE STANCU VARIANT OF SZÁSZ-MIRAKJAN-KANTOROVICH OPERATORS VIA DUNKL GENERALIZATION OF POST QUANTUM CALCULUS

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ABSTRACT. Our main purpose of this article is to study the approximation properties of Szász-Mirakjan-Kantorovich operators by introducing the non negative parameter $0 \leqq [\alpha]_{p,q} \leqq [\beta]_{p,q}$. For this purpose we define the Stancu variant of Szász-Mirakjan-Kantorovich operators via (p,q)-variant of Dunkl generalization. First we study the Korovkin's type approximation results in weighted spaces. Finally, we obtain the convergence of our new operators in by use of modulus of continuity in Lipschitz class and Petter's K-functionals. The extra parameter p provides more flexibility and a generalized version in approximation rather than q.

1. INTRODUCTION AND PRELIMINARIES

In approximation theory the well-known polynomials given by Bernstein [5] in 1912, while these generalizations in quantum calculus(the q-Bernstein polynomials) have obtained by Lupaş and Phillips ([12], [24]). These types of generalization of approximation have become a very important tools in the study of Mathematical science, Physics, Computer Science and several branches of applied sciences and engineering sciences. By using the preliminary basic facts about the (p,q)-integers, suppose $[s]_{p,q} = \frac{p^s - q^s}{p-q}$, $s = 0, 1, 2, \cdots$, $0 < q < p \leqq 1$, the (p,q)- Bernstein operators was introduced by Mursaleen *et al* [14]:

$$B_s^{p,q}(g;y) = \frac{1}{p^{\frac{s(s-1)}{2}}} \sum_{m=0}^{s} \left[\begin{array}{c} s \\ m \end{array} \right]_{p,q} p^{\frac{m(m-1)}{2}} y^k \prod_{r=0}^{s-m-1} (p^r - q^r y) g\left(\frac{[m]_{p,q}}{p^{m-s}[s]_{p,q}}\right), \; y \in [0,1]. \tag{1.1}$$

The basics forms of exponential functions in (p,q)-analogues are defined as

$$e_s^{p,q}(y) = \sum_{s=0}^{\infty} p^{\frac{s(s-1)}{2}} \frac{y^s}{[s]_{p,q}!}, \; E_r^{p,q}(y) = \sum_{s=0}^{\infty} q^{\frac{s(s-1)}{2}} \frac{y^s}{[s]_{p,q}!},$$

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with the property $e_s^{p,q}(y)E_s^{p,q}(-y)=1$. Note that if $p=1$, $e_s^{p,q}(y)$ and $E_s^{p,q}(y)$ reduce to q-exponential functions. In 1925 for any $y \geqq 0,\ f \in C[0,\infty)$ Szász-Mirakjan introduced the positive linear operators [28] which are the generalized version Bernstein polynomials [5]. By introducing a non-negative parametric variant $\vartheta \geqq 0$, the Dunkl type generalization of Szász-Mirakjan operators [28] was introduced by Sucu [27] and q-analogue by Ben Cheikh *et al* [6]. Içöz [8] introduced the q-Dunkl analogue of Szász operators by generating the exponential functions which are defined by

$$D_\vartheta^q(g;y)=\frac{1}{e_\vartheta^q([s]_q y)}\sum_{m=0}^{\infty}\frac{([s]_q y)^m}{\gamma_\vartheta^q(m)}g\left(\frac{1-q^{2\vartheta\theta_m+m}}{1-q^r}\right). \tag{1.2}$$

where $\vartheta > -\frac{1}{2},\ y \geqq 0,\ 0<q<1$ and $g \in C[0,\infty)$ and $C[0,\infty)$ is the set of all continuous functions defined on $[0,\infty)$.

Most recently the dunkl type generalization for the different operators obtained by the researcher for more details we refer to see the published articles [4, 15, 19, 21, 22, 25, 26]. For most recent work on other related approximation results we refer to see [1, 3, 11, 16, 17, 18]. Very recently Alotaibi *et al* [2] generalised the q-Dunkl analogue of Szász-Mirakjan operators in $(p;q)$-calculus. They introduced the first (p,q)-Dunkl analogue of Szász-Mirakjan operators.

Most recent Nasiruzzaman, *et al* [20], studied the (p,q)-variant of Szász-Mirakjan-Kantorovich operators via (p,q)-Dunkl analogue as follows:

$$\mathcal{D}_\vartheta^{p,q}(g;y)=\frac{[s]_{p,q}}{e_\vartheta^{p,q}([s]_{p,q}y)}\sum_{m=0}^{\infty}\frac{([s]_{p,q}y)^m}{\gamma_\vartheta^{p,q}(m)}p^{-(m+2\vartheta\theta_m)}p^{\frac{m(m-1)}{2}}\int_{q\mathcal{A}}^{q\mathcal{A}+\mathcal{B}}g\left(\frac{t}{qp^{m-1}}\right)d_{p,q}t \tag{1.3}$$

where for $q\in(0,1)$, $p\in(q,1]$ and $\vartheta>-\frac{1}{2}$, the (p,q)-Dunkl analogue of exponential functions is defined as

$$e_\vartheta^{p,q}=\sum_{r=0}^{\infty}p^{\frac{r(r-1)}{2}}\frac{y^r}{\gamma_\vartheta^{p,q}(r)},\qquad y\in[0,\infty) \tag{1.4}$$

$$\gamma_\vartheta^{p,q}(r)=\frac{\prod_{i=0}^{[\frac{r+1}{2}]-1}p^{2\vartheta(-1)^{i+1}+1}\left((p^2)^ip^{2\vartheta+1}-(q^2)^iq^{2\vartheta+1}\right)\prod_{j=0}^{[\frac{r}{2}]-1}p^{2\vartheta(-1)^j+1}\left((p^2)^jp^2-(q^2)^jq^2\right)}{(p-q)^r}, \tag{1.5}$$

$$\gamma_\vartheta^{p,q}(r+1)=\frac{p^{2\vartheta(-1)^{r+1}+1}(p^{2\vartheta\theta_{r+1}+r+1}-q^{2\vartheta\theta_{r+1}+r+1})}{(p-q)}\gamma_\vartheta^{p,q}(r), \tag{1.6}$$

$$\theta_r=\begin{cases}0 & \text{for } r=2\ell,\ \ell=1,2,\cdots,n\\ 1 & \text{for } r=2\ell+1,\ \ell=1,2,\cdots,n.\end{cases} \tag{1.7}$$

and $[\frac{r}{2}]$ denotes the greatest integer function, also we have

$$(u-v)_{p,q}^r=\begin{cases}\prod_{j=0}^{r-1}(p^ju-q^jv) & \text{if } r=1,2,\cdots,n\\ 1 & \text{if } r=0.\end{cases}$$

Lemma 1.1. *For* $g(t)=1,\ t,\ t^2$
$1^*.\ \mathcal{D}_\vartheta^{p,q}(1;y)=1;$

$2^{*}.\ \mathcal{D}_{\vartheta}^{p,q}(t;y) \leqq \frac{2}{[2]_{p,q}}y + \frac{1}{[2]_{p,q}q[s]_{p,q}};$

$3^{*}.\ \mathcal{D}_{\vartheta}^{p,q}(t^2;y) \leqq \frac{3}{[3]_{p,q}}y^2 + \frac{3}{[3]_{p,q}[s]_{p,q}}\left([1+2\vartheta]_{p,q} + \frac{1}{q[s]_{p,q}}\right)y + \frac{1}{[3]_{p,q}q^2[s]_{p,q}^2}.$

2. Auxiliary Operators and estimations of moments

In this section, we construct an Stancu variant of recent investigation by [20], the (p,q)-variant of Szász-Mirakjan-Kantorovich operators via Dunkl formulations as follows:

Definition 2.1. Let $0 \leqq [\alpha]_{p,q} \leqq [\beta]_{p,q}$, then for any $y \in [0,\infty),\ g \in C[0,\infty)\ s \in \mathbb{N}$ and $0 < q < p \leqq 1$, we define

$$\mathcal{R}_m^{p,q}(g;y) = \frac{[s]_{p,q}}{e_{\vartheta}^{p,q}([s]_{p,q}y)} \sum_{m=0}^{\infty} \frac{([s]_{p,q}y)^m}{\gamma_{\vartheta}^{p,q}(m)} p^{-(m+2\vartheta\theta_m)} p^{\frac{m(m-1)}{2}} \int_{q\mathcal{A}}^{q\mathcal{A}+\mathcal{B}} g\left(\frac{[s]_{p,q}t + [\alpha]_{p,q}}{qp^{m-1}([s]_{p,q}+[\beta]_{p,q})}\right) d_{p,q}t. \tag{2.1}$$

Note that If we take $[\alpha]_{p,q} = [\beta]_{p,q} = 0$ in (2.1), then the operators then reduced to [20]. More details for $p=1$ the operators reduce to the operators [9]. If $\vartheta = 0$, these are reduced to the (p,q)-variant of Kantorovich type operators defined by [13].

In order to get the results on uniform convergence of our operators $\mathcal{R}_m^{p,q}(\,\cdot\,;\,\cdot\,)$, we take $q = q_s,\ p = p_s$ satisfying $0 < q_s < 1$ and $q_s < p_s \leqq 1$ such that

$$\lim_{s\to\infty} p_s \to 1, \quad \lim_{s\to\infty} q_s \to 1, \quad \lim_{s\to\infty} p_s^s \to u, \quad \lim_{s\to\infty} q_s^s \to v, \quad (0 < u, v \leqq 1). \tag{2.2}$$

Lemma 2.1. *Take $g(t) = g_i$ such that $g_i = t^{i-1}$ for $i = 1,2,3$. Then we have calculated the following identities :*

(1) $\mathcal{R}_m^{p,q}(g_1;y) = 1$

(2) $\mathcal{R}_m^{p,q}(g_2;y) \leqq \frac{[s]_{p,q}}{[s]_{p,q}+[\beta]_{p,q}}\left(\frac{2}{[2]_{p,q}}y + \frac{1}{[2]_{p,q}q[s]_{p,q}}\right) + \frac{[\alpha]_{p,q}}{[s]_{p,q}+[\beta]_{p,q}};$

(3) $\mathcal{R}_m^{p,q}(g_3;y) \leqq \left(\frac{[s]_{p,q}}{[s]_{p,q}+[\beta]_{p,q}}\right)^2 \left\{\frac{3}{[3]_{p,q}}y^2 + \frac{3}{[3]_{p,q}[s]_{p,q}}\left([1+2\vartheta]_{p,q} + \frac{1}{q[s]_{p,q}}\right)y + \frac{1}{[3]_{p,q}q^2[s]_{p,q}^2}\right\} + \frac{2[\alpha]_{p,q}[s]_{p,q}}{([s]_{p,q}+[\beta]_{p,q})^2}\left\{\frac{2}{[2]_{p,q}}y + \frac{1}{[2]_{p,q}q[s]_{p,q}}\right\} + \left(\frac{[\alpha]_{p,q}}{[s]_{p,q}+[\beta]_{p,q}}\right)^2.$

Proof. In the view of lemma asserted by [20] we easily proved the results for $g(t) = g_1, g_2, g_3$. Thus for $g(t) = g_1$, we get

$$\mathcal{R}_m^{p,q}(g_1;y) = \mathcal{D}_{\vartheta}^{p,q}(g_1;y) = 1;$$

For $g(t) = g_2$, we have

$$\begin{aligned} \mathcal{R}_m^{p,q}(g_2;y) &= \frac{[s]_{p,q}}{[s]_{p,q}+[\beta]_{p,q}}\mathcal{D}_{\vartheta}^{p,q}(g_2;y) + \frac{[\alpha]_{p,q}}{[s]_{p,q}+[\beta]_{p,q}}\mathcal{D}_{\vartheta}^{p,q}(g_1;y) \\ &\leqq \frac{[s]_{p,q}}{[s]_{p,q}+[\beta]_{p,q}}\left(\frac{2}{[2]_{p,q}}y + \frac{1}{[2]_{p,q}q[s]_{p,q}}\right) + \frac{[\alpha]_{p,q}}{[s]_{p,q}+[\beta]_{p,q}}; \end{aligned}$$

Similarly for $g(t) = g_3$, we get

$$\begin{aligned}
\mathcal{R}_m^{p,q}(g_2;y) &= \left(\frac{[s]_{p,q}}{[s]_{p,q}+[\beta]_{p,q}}\right)^2 \mathcal{D}_\vartheta^{p,q}(g_3;y) + \frac{2[\alpha]_{p,q}[s]_{p,q}}{([s]_{p,q}+[\beta]_{p,q})^2}\mathcal{D}_\vartheta^{p,q}(g_2;y) \\
&+ \left(\frac{[\alpha]_{p,q}}{[s]_{p,q}+[\beta]_{p,q}}\right)^2 \mathcal{D}_\vartheta^{p,q}(g_1;y) \\
&\leqq \left(\frac{[s]_{p,q}}{[s]_{p,q}+[\beta]_{p,q}}\right)^2 \left\{\frac{3}{[3]_{p,q}}y^2 + \frac{3}{[3]_{p,q}[s]_{p,q}}\left([1+2\vartheta]_{p,q}+\frac{1}{q[s]_{p,q}}\right)y + \frac{1}{[3]_{p,q}q^2[s]_{p,q}^2}\right\} \\
&+ \frac{2[\alpha]_{p,q}[s]_{p,q}}{([s]_{p,q}+[\beta]_{p,q})^2}\left\{\frac{2}{[2]_{p,q}}y + \frac{1}{[2]_{p,q}q[s]_{p,q}}\right\} + \left(\frac{[\alpha]_{p,q}}{[s]_{p,q}+[\beta]_{p,q}}\right)^2.
\end{aligned}$$

Thus we complete the proof of result asserted by Lemma 2.1. □

Lemma 2.2. *Take $\zeta_i = (t-y)^i$ for $i = 1,\ 2$, then we have the following central moments:*

$$\mathcal{R}_m^{p,q}(\zeta_i;y) \leqq \begin{cases}
\left(\frac{2[s]_{p,q}}{[2]_{p,q}([s]_{p,q}+[\beta]_{p,q})}-1\right)y + \frac{[\alpha]_{p,q}}{[s]_{p,q}+[\beta]_{p,q}} + \frac{1}{q[2]_{p,q}([s]_{p,q}+[\beta]_{p,q})}; & \text{for} \quad i=1 \\
\left\{\frac{3}{[3]_{p,q}}\left(\frac{[s]_{p,q}}{([s]_{p,q}+[\beta]_{p,q})}\right)^2 - \left(\frac{4[s]_{p,q}}{[2]_{p,q}([s]_{p,q}+[\beta]_{p,q})}\right)+1\right\}y^2 & \\
+\left\{\frac{3}{[3]_{p,q}[s]_{p,q}}\left(\frac{[s]_{p,q}}{[s]_{p,q}+[\beta]_{p,q}}\right)^2\left([1+2\vartheta]_{p,q}+\frac{1}{q[s]_{p,q}}\right)\right. & \\
\left.+\frac{4[\alpha]_{p,q}[s]_{p,q}}{[2]_{p,q}([s]_{p,q}+[\beta]_{p,q})^2} - \frac{2}{q[2]_{p,q}([s]_{p,q}+[\beta]_{p,q})} - \frac{2[\alpha]_{p,q}}{[s]_{p,q}+[\beta]_{p,q}}\right\}y & \\
+\frac{1}{[3]_{p,q}q^2[s]_{p,q}([s]_{p,q}+[\beta]_{p,q})} + \frac{2[\alpha]_{p,q}}{[2]_{p,q}q[s]_{p,q}([s]_{p,q}+[\beta]_{p,q})^2} & \\
+\left(\frac{[\alpha]_{p,q}}{[s]_{p,q}+[\beta]_{p,q}}\right)^2 & \text{for} \quad i=2.
\end{cases} \tag{2.3}$$

3. APPROXIMATION IN WEIGHTED SPACES

In this section, we study the Korovkin's type approximation properties of the positive linear operators $\mathcal{R}_m^{p,q}(\,\cdot\,;\,\cdot\,)$ defined by (2.1). We denote the set of all bounded and continuous functions by $C_B[0,\infty)$ equipped with norm $\parallel g \parallel_{C_B} = \sup_{y\in[0,\infty)} \mid g(y) \mid$. We take

$$\mathfrak{E} := \{g(y) : y \in [0,\infty), \frac{g(y)}{1+y^2} \text{ is convergent as } y \to \infty\}.$$

Let

$$B_\sigma[0,\infty) = \{g : |g(y)| \leqq \mathcal{M}_g\sigma(y)\},$$
$$C_\sigma[0,\infty) = \{g : g \in B_\sigma[0,\infty) \cap C[0,\infty)\},$$
$$C_\sigma^k[0,\infty) = \left\{g : g \in C_\sigma[0,\infty) \quad \text{and} \quad \lim_{y\to\infty}\frac{g(y)}{\sigma(y)} = k\right\},$$

where $\sigma(y)$ is the weight function and given as $\sigma(y) = 1+y^2$, k is a constant and $\mathcal{M}_g$ depends on g. $C_\sigma[0,\infty)$ is equipped with the norm $||g||_\sigma = \sup_{y\in[0,\infty)} \frac{|g(y)|}{\sigma(y)}$.

Theorem 3.1. *Let $q_s,\ p_s$ be the real numbers with $q_s \in (0,1)$ and $p_s \in (q_s, 1]$ such that $q_s \to 1$ and $p_s \to 1$ as $s \to \infty$. Then, for every $g \in C[0,\infty) \cap \mathfrak{E}$,*

$$\lim_{s\to\infty} \mathcal{R}_{\vartheta}^{p_s,q_s}(g;y) = g(y)$$

uniformly on each compact subset of $[0,\infty)$.

Proof. For the proof of uniform convergence of the operators $\mathcal{R}_{\vartheta}^{p_s,q_s}$ on each compact subset of $[0,\infty)$, we apply the well known Korovkin's theorem [10]. It is sufficient to show that $\lim_{s\to\infty} \mathcal{R}_{\vartheta}^{p_s,q_s}\big(g_i;y\big) = y^{i-1}$, where $g_i = t^{i-1}$ for $i = 1,2,3$,

Clearly, when $q_s \to 1,\ p_s \to 1$ as $s \to \infty$, then $\frac{1}{[s]_{p_s,q_s}} \to 0$, $\frac{[s]_{p_s,q_s}}{[s]_{p_s,q_s}} \to 1$. Which yields that

$$\lim_{s\to\infty} \mathcal{R}_{\vartheta}^{p_s,q_s}(g_1;y) = 1, \quad \lim_{s\to\infty} \mathcal{R}_{\vartheta}^{p_s,q_s}(g_2;y) = y, \quad \lim_{s\to\infty} \mathcal{R}_{\vartheta}^{p_s,q_s}(g_3;y) = y^2.$$

□

Theorem 3.2. *Let $q_s,\ p_s$ be the real numbers with $0 < q_s < 1$ and $q_s < p_s \leqq 1$ such that $q_s \to 1$ and $p_s \to 1$ as $s \to \infty$. Then, for every $g \in C_{\sigma}^{k}[0,\infty)$, and $g = g_\tau = t^{\tau-1}$ we have*

$$\lim_{s\to\infty} ||\mathcal{R}_{\vartheta}^{p_s,q_s}(g_\tau) - g||_{\sigma} = 0,\ \tau = 1,2,3. \tag{3.1}$$

Proof. Suppose $g(t) \in C_{\sigma}^{k}[0,\infty)$ and $g(t) = g_\tau$, where $g_\tau = t^{\tau-1}$ for $\tau = 1,2,3$. Then from the well known Korovkin's theorem we have $\mathcal{R}_{\vartheta}^{p_s,q_s}(g_\tau;y) \to y^{\tau-1}$ ($s \to \infty$) uniformly for each $\tau = 1,2,3$. Hence, from Lemma 2.1, we have

$$\lim_{s\to\infty} ||\mathcal{R}_{\vartheta}^{p_s,q_s}\ (g_1) - 1||_{\sigma} = 0. \tag{3.2}$$

For $\tau = 2$,

$$\begin{aligned} &||\mathcal{R}_{\vartheta}^{p_s,q_s}\ (g_2) - y||_{\sigma} \\ &\quad= \sup_{y\geqq 0} \frac{|\mathcal{R}_{\vartheta}^{p_s,q_s}(g_2;y) - y|}{1+y^2} \\ &\quad\leqq \left(\frac{2[s]_{p,q}}{[2]_{p,q}([s]_{p,q} + [\beta]_{p,q})} - 1 \right) \sup_{y\geqq 0} \frac{y}{1+y^2} \\ &\quad+ \left\{ \frac{[\alpha]_{p,q}}{[s]_{p,q} + [\beta]_{p,q}} + \frac{1}{q[2]_{p,q}([s]_{p,q} + [\beta]_{p,q})} \right\} \sup_{y\geqq 0} \frac{1}{1+y^2}. \end{aligned}$$

Then

$$\lim_{s\to\infty} ||\mathcal{R}_{\vartheta}^{p_s,q_s}\ (g_2) - y||_{\sigma} = 0. \tag{3.3}$$

Similarly. if we take $\tau = 3$

$$\begin{aligned}
&\left|\left|\mathcal{R}_{\vartheta}^{p_s,q_s}(g_3) - y^2\right|\right|_{\sigma} \\
&\quad = \sup_{y\geqq 0} \frac{\left|\mathcal{R}_{\vartheta}^{p_s,q_s}(g_3;y) - y^2\right|}{1+y^2} \\
&\quad \leqq \left\{ \frac{3}{[3]_{p,q}} \left(\frac{[s]_{p,q}}{([s]_{p,q}+[\beta]_{p,q})} \right)^2 - 1 \right\} \sup_{y\geqq 0} \frac{y^2}{1+y^2} \\
&\quad + \left\{ \left(\frac{[s]_{p,q}}{[s]_{p,q}+[\beta]_{p,q}} \right)^2 \left(\frac{3}{[3]_{p,q}[s]_{p,q}} \right) \left([1+2\vartheta]_{p,q} + \frac{1}{q[s]_{p,q}} \right) \right. \\
&\quad \left. + \frac{4[\alpha]_{p,q}[s]_{p,q}}{[2]_{p,q}([s]_{p,q}+[\beta]_{p,q})^2} \right\} \sup_{y\geqq 0} \frac{y}{1+y^2} \\
&\quad + \left\{ \left(\frac{[s]_{p,q}}{[s]_{p,q}+[\beta]_{p,q}} \right)^2 \frac{1}{[3]_{p,q} q^2 [s]_{p,q}^2} + \frac{2[\alpha]_{p,q}}{q[2]_{p,q}([s]_{p,q}+[\beta]_{p,q})^2} \right. \\
&\quad \left. + \left(\frac{[\alpha]_{p,q}}{[s]_{p,q}+[\beta]_{p,q}} \right)^2 \right\} \sup_{y\geqq 0} \frac{1}{1+y^2},
\end{aligned}$$

$$\lim_{s\to\infty} \left|\left|\mathcal{R}_{\vartheta}^{p_s,q_s}(g_3) - y^2\right|\right|_{\sigma} = 0. \tag{3.4}$$

This completes the proof. □

Let $\omega_b(g;\delta)$ be the modulus of continuity of the function and $g \in \tilde{C}[0,\infty)$, where $\tilde{C}[0,\infty)$, be the space of uniformly continuous functions on $[0,\infty)$, we have

$$\omega_b(g;\delta) = \sup_{|t-y|\leq\delta;} \sup_{y,t\in[0,b]} |\, g(t) - g(y) \,|, \tag{3.5}$$

where it is obvious that $\lim_{\delta\to 0+} \omega_b(g;\delta) = 0$ and for $g \in C[0,\infty)$

$$|\, g(t) - g(y) \,| \leq \left(\frac{|\, t-y \,|}{\delta} + 1 \right) \omega_b(g;\delta). \tag{3.6}$$

Theorem 3.3. *Let $q_s,\ p_s$ be the real numbers with $q_s \in (0,1)$, and $p_s \in (q_s,1]$ such that $q_s \to 1$ and $p_s \to 1$ as $s \to \infty$. Then for every $g \in C_\sigma[0,\infty)$*

$$|\mathcal{R}_{\vartheta}^{p_s,q_s}(g;y) - g(y)| \leqq 2\bigg(\omega_{b+1}(g;\delta_\vartheta(y)) + \mathcal{M}_g(1+b^2)\,(\delta_\vartheta(y))^2 \bigg),$$

where $\delta_\vartheta(y) = \sqrt{\mathcal{R}_{\vartheta}^{p_s,q_s}(\zeta_2;y)}$, $\mathcal{M}_g$ is a constant depending only on g and $\mathcal{R}_{\vartheta}^{p_s,q_s}(\zeta_2;y)$ is defined by Lemma 2.2; and $[0,b+1] \subset [0,\infty), b > 0$.

Proof. Let $y \in [0,b]$ and $t > b+1$, with $t > 0$. Then for $\delta > 0$ we have

$$|\, g(t) - g(y) \,| \leqq \omega_{b+1}(g;|\, t-y \,|) \leqq \left(1 + \frac{|\, t-y \,|}{\delta} \right) \omega_{b+1}(g;\delta). \tag{3.7}$$

By applying the linearity of $\mathcal{R}_{\vartheta}^{p_s,q_s}$ and Cauchy-Schwarz inequality

$$\mathcal{R}_{\vartheta}^{p_s,q_s}\,|g(t) - g(y);y| \leqq \left(\left(1 + \frac{1}{\delta} \mathcal{R}_{\vartheta}^{p_s,q_s}\big((t-y)^2;y\big) \right)^{\frac{1}{2}} \right) \omega_{b+1}(g;\delta). \tag{3.8}$$

For $t-y>1$, we have

$$|g(t)-g(y)| \\ \leqq \mathcal{M}_g\left(2+y^2+t^2\right) \\ \leqq \mathcal{M}_g\left(2+3y^2+2(t-y)^2\right) \leqq 2\mathcal{M}_g(1+b^2)(t-y)^2$$

$$\mathcal{R}_{\vartheta}^{p_s,q_s}\left(|g(t)-g(y)|\,;\right) \leqq 2\mathcal{M}_g(1+b^2)\mathcal{R}_{\vartheta}^{p_s,q_s}\big((t-y)^2;y\big). \tag{3.9}$$

From (3.8) and (3.9), we easily see that

$$\begin{aligned}|\mathcal{R}_{\vartheta}^{p_s,q_s}(g;y)-g(y)| \\ &\leqq \mathcal{R}_{\vartheta}^{p_s,q_s}\,|g(t)-g(y);y| \\ &\leqq \left(\left(1+\frac{1}{\delta}\mathcal{R}_{\vartheta}^{p_s,q_s}\big((t-y)^2;y\big)\right)^{\frac{1}{2}}\right)\omega_{b+1}(g;\delta) \\ &+2\mathcal{M}_g(1+b^2)\mathcal{R}_{\vartheta}^{p_s,q_s}\big((t-y)^2;y\big) \\ &=\left(1+\frac{1}{\delta}\mathcal{R}_{\vartheta}^{p_s,q_s}\big(\zeta_2;y\big)\right)^{\frac{1}{2}}\omega_{b+1}(g;\delta) \\ &+2\mathcal{M}_g(1+b^2)\mathcal{R}_{\vartheta}^{p_s,q_s}\big(\zeta_2;y\big)\end{aligned}$$

if we choose $\delta=\delta_{\vartheta}(y)=\sqrt{\mathcal{R}_{\vartheta}^{p_s,q_s}\big(\zeta_2;y\big)}$, then we get our result. □

For any $g\in C[0,\infty],\ \mathcal{C}>0,\ 0<\nu\leq 1$ and $\gamma_1,\gamma_2\in[0,\infty)$, we recall that

$$Lip_{\mathcal{C}}(\nu)=\{g:|\ g(\gamma_1)-g(\gamma_2)\ |\leq \mathcal{C}\ |\ \gamma_1-\gamma_2\ |^{\nu}\}\,. \tag{3.10}$$

Theorem 3.4. *Let $q_s,\ p_s$ be the real numbers with $q_s\in(0,1)$ and $p_s\in(q_s,1]$ for every positive integer s such that $q_s\to 1$ and $p_s\to 1$ as $s\to\infty$. Then for each $g\in Lip_{\mathcal{C}}(\nu)$, we have*

$$|\mathcal{R}_{\vartheta}^{p_s,q_s}(g;y)-g(y)|\leqq \mathcal{C}\left(\delta_{\vartheta}(y)\right)^{\nu},$$

where $\delta_{\vartheta}(y)$ is defied by theorem 3.3.

Proof. To prove the result, We use Theorem 3.4, the result (3.10) and well-known Hölder inequality. We get

$$\begin{aligned}|\mathcal{R}_{\vartheta}^{p_s,q_s}(g;y)-g(y)| &\leqq \left|\mathcal{R}_{r,\vartheta}^{p_s,q_s}(g(t)-g(y);y)\right| \\ &\leqq \mathcal{R}_{\vartheta}^{p_s,q_s}\left(|g(t)-g(y)|\,;y\right) \\ &\leqq |\ \mathcal{C}\mathcal{R}_{\vartheta}^{p_s,q_s}\left(|t-y|^{\nu};y\right) \\ &\leqq \mathcal{C}\left(\mathcal{R}_{\vartheta}^{p_s,q_s}(g_1;y)\right)^{\frac{2-\nu}{2}}\left(\mathcal{R}_{\vartheta}^{p_s,q_s}(|t-y|^2;y)\right)^{\frac{\nu}{2}} \\ &= \mathcal{C}\left(\mathcal{R}_{\vartheta}^{p_s,q_s}(\zeta_2;y)\right)^{\frac{\nu}{2}}.\end{aligned}$$

This completes the proof of the Theorem 3.4. □

We next denote

$$C_B^2[0,\infty)=\{\psi:\psi\in C_B[0,\infty)\quad\text{and}\quad \psi',\psi''\in C_B[0,\infty)\}\,, \tag{3.11}$$

$$||\psi||_{C_B^2(\mathbb{R}^+)}=||\psi||_{C_B[0,\infty)}+||\psi'||_{C_B[0,\infty)}+||\psi''||_{C_B[0,\infty)}\,, \tag{3.12}$$

$$||\psi||_{C_B[0,\infty)} = \sup_{y\in[0,\infty)} |\psi(y)|. \tag{3.13}$$

Theorem 3.5. *Let $\psi \in C_B^2[0,\infty)$ and $q_s,\ p_s$ be the real numbers with $q_s \in (0,1)$ and $p_s \in (q_s, 1]$ for every positive integer s such that $q_s \to 1$ and $p_s \to 1$ as $s \to \infty$. Then*

$$|\mathcal{R}_\vartheta^{p_s,q_s}(\psi;y) - \psi(y)| \leqq \Theta_\vartheta(y)||\psi||_{C_B^2[0,\infty)}, \tag{3.14}$$

where $\Theta_\vartheta(y) = \delta_\vartheta(y)\left(1 + \frac{\delta_\vartheta(y)}{2}\right)$ and $\delta_\vartheta(y)$ is defined by Theorem 3.3

Proof. From the Taylor series expansion for any $\psi \in C_B^2[0,\infty)$, we have

$$\psi(t) = \psi(y) + \psi'(y)(t-y) + \psi''(\varphi)\,\frac{(t-y)^2}{2} \quad \text{for} \quad \varphi \in (y,t),$$

$$|\psi(t) - \psi(y)| \leqq \mathcal{P} \mid t-y \mid + \frac{1}{2}\mathcal{Q}(t-y)^2,$$

where

$$\mathcal{P} = \sup_{y[0,\infty)} |\psi'(y)| = ||\psi'||_{C_B[0,\infty)} \leqq ||\psi||_{C_B^2[0,\infty)},$$

$$\mathcal{Q} = \sup_{y[0,\infty)} |\psi''(y)| = ||\psi''||_{C_B[0,\infty)} \leqq ||\psi||_{C_B^2[0,\infty)}.$$

Therefore,

$$|\psi(t) - \psi(y)| \leqq \left(\mid t-y \mid + \frac{1}{2}(t-y)^2\right)||\psi||_{C_B^2[0,\infty)}.$$

By applying the linearity of $\mathcal{R}_\vartheta^{p_s,q_s}$, we get

$$\begin{aligned}
&|\mathcal{R}_\vartheta^{p_s,q_s}(\psi;y) - \psi(y)| \\
&\leqq \left(\mathcal{R}_\vartheta^{p_s,q_s}(\mid t-y \mid; y) + \frac{1}{2}\mathcal{R}_\vartheta^{p_s,q_s}\left((t-y)^2; y\right)\right)||\psi||_{C_B^2[0,\infty)} \\
&\leqq \left(\left(\mathcal{R}_\vartheta^{p_s,q_s}(\zeta_2;y)\right)^{\frac{1}{2}} + \frac{1}{2}\mathcal{R}_\vartheta^{p_s,q_s}(\zeta_2;y)\right)||\psi||_{C_B^2[0,\infty)} \\
&= \left(\delta_\vartheta(y) + \frac{(\delta_\vartheta(y))^2}{2}\right)||\psi||_{C_B^2[0,\infty)}.
\end{aligned}$$

Hence this completes the proof of the theorem. □

Let for all $\psi \in C_B^2[0,\infty)$, the Peetre's K-functional $K_2(g;\delta)$ for $\delta > 0$ (see [23]) such as

$$K_2(g;\delta) = \inf_{y\in[0,\infty)} \left\{\left(\delta\left|\left|\psi'' + ||g-\psi||_{C_B[0,\infty)}\right|\right|_{C_B[0,\infty)}\right)\right\}. \tag{3.15}$$

For a given positive constant $\mathcal{L} > 0$

$$K_2(g;\delta) \leqq \mathcal{L}\omega_2(g;\delta^{\frac{1}{2}}),$$

where the second-order modulus of continuity denoted by $\omega_2(g;\delta)$ is defined as

$$\omega_2(g;\delta) = \sup_{0<h<\delta}\ \sup_{y\in[0,\infty)} |g(y) + g(y+2h) - 2g(y+h)|. \tag{3.16}$$

Theorem 3.6. *Let $q_s,\ p_s$ be the real numbers with $o < q_s < 1$ and $q_s < p_s \leqq 1$ for every positive integer s such that $q_s \to 1$ and $p_s \to 1$ as $s \to \infty$. Then for all $g \in C_B[0,\infty)$, we have*

$$|\mathcal{R}_{\vartheta}^{p_s,q_s}(g;y) - g(y)| \leqq 2\mathcal{M}\left\{\omega_2\left(g;\sqrt{\frac{\Theta_\vartheta(y)}{2}}\right) + \min\left(1;\frac{\Theta_\vartheta(y)}{2}\right)||g||_{C_B[0,\infty)}\right\},$$

where $\mathcal{M}$ is a positive constant and $\Theta_\vartheta(y)$ is given in Theorem 3.5.

Proof. We take $\psi \in C_B^2[0,\infty)$ and apply the theorem (3.5). Thus

$$\begin{aligned}|\mathcal{R}_{\vartheta}^{p_s,q_s}(g;y) - g(y)| &\leqq |\mathcal{R}_{\vartheta}^{p_s,q_s}(g-\psi;y)| + |\mathcal{R}_{\vartheta}^{p_s,q_s}(\psi;y) - \psi(y)| + |g(y)-\psi(y)| \\ &\leqq 2||g-\psi||_{C_B[0,\infty)} + \Theta_\vartheta(y)||\psi||_{C_B^2[0,\infty)} \\ &= 2\left(||g-\psi||_{C_B[0,\infty)} + \frac{\Theta_\vartheta(y)}{2}||\psi||_{C_B^2[0,\infty)}\right).\end{aligned}$$

By taking the infimum over all $\psi \in C_B^2[0,\infty)$ and using (3.15), we get

$$|\mathcal{R}_{\vartheta}^{p_s,q_s}(g;y) - g(y)| \leqq 2K_2\left(g;\frac{\Theta_\vartheta(y)}{2}\right).$$

Now from [7] for all $g \in C_B[0,\infty)$, we have the relation:

$$K_2(g;\delta) \leqq \mathcal{M}\{\min(1;\delta) + \omega_2(g;\sqrt{\delta})||g||_{C_B[0,\infty)}\},$$

where $\mathcal{M} > 0$ is an absolute constant. Hence if we choose $\delta = \frac{\Theta_\vartheta(y)}{2}$ then we have desired results. □

Conclusion

This present article is enable to give a generalized version and strong convergence of the paper [9, 13, 20]. We study the approximation results via Dunkl generalization of the Szász-Mirakjan-Kantorovich operators in (p,q)-calculus by introducing the non-negative parameter $[\alpha]_{p,q}$ and $[\beta]_{p,q}$. We obtain the Korovkin's type approximation results in weighted space. Furthermore, by use of modulus of continuity, we estimate the convergence in Lipschitz class and Petter's K-functionals. If we put $[\alpha]_{p,q} = [\beta]_{p,q} = 0$ in (2.1) , then the operators $\mathcal{R}_m^{p,q}$ reduce to recent operators defined by [20].

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