ANNALS OF COMMUNICATIONS IN MATHEMATICS
Volume 3, Number 1 (2020), 54-62

DOUBT $\mathcal{N}$-IDEAL THEORY IN BCK-ALGEBRAS BASED ON $\mathcal{N}$-STRUCTURES

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ABSTRACT. The notions of doubt $\mathcal{N}$-subalgebras and doubt $\mathcal{N}$-ideals in BCK-algebras are introduced, and related properties are investigated. Characterizations of a doubt $\mathcal{N}$-subalgebra and a doubt $\mathcal{N}$-ideal are given, and relations between them are discussed.

1. INTRODUCTION

The study of BCK-algebras was introduced by Imai and Iséki [10] in 1966. BCK-algebras have been applied to many branches of mathematics, such as functional analysis, group theory, topology, probability theory.

A crisp set C in a universe $\mathcal{X}$ is a function $\lambda_C : \mathcal{X} \to \{0, 1\}$ yielding the value 0 for elements excluded from the set C and the value 1 for elements belonging to the set C. As a generalization of crisp sets, Zadeh [17] introduced the degree of positive membership in 1965 and defined the fuzzy sets. Jun et al. [12] presented a new function which is called negative-valued function, and developed $\mathcal{N}$-structures as a one of the hybrid models of fuzzy sets. They applied $\mathcal{N}$-structures in BCK-algebras and proposed $\mathcal{N}$-subalgebras and $\mathcal{N}$-ideals [12]. In [11], Jun established the definition of doubt fuzzy subalgebras and ideals in BCK-algebras. After that, many Hybrid models of fuzzy sets were applied in BCK-algebras and other algebraic structures [5, 6, 8, 7, 9, 4, 15, 16, 2, 1, 3, 14, 13].

In this paper, we discuss an $\mathcal{N}$-structure with an application to BCK-algebras. We introduce the notions of doubt $\mathcal{N}$-subalgebras and doubt $\mathcal{N}$-ideals in BCK-algebras, and investigate related properties. Then, we present some characterizations of them by means of doubt level subset. Moreover, relations between a doubt $\mathcal{N}$-subalgebra and a doubt $\mathcal{N}$-ideal in BCK-algebras are discussed.

2. PRELIMINARIES

In this section, we include some basic definitions and preliminary facts about BCK-algebras which are essential for our results.

By a BCK-algebra, we mean an algebra $(\mathcal{X}; *, 0)$ of type (2, 0) satisfying the following axioms for all $x, y, z \in \mathcal{X}$:

(I) $((x * y) * (x * z)) * (z * y) = 0,$

2010 *Mathematics Subject Classification.* 06F35, 03G25, 03B52, 03B05.
Key words and phrases. BCK/BCI-algebra; $\mathcal{N}$-structures; Doubt $\mathcal{N}$-subalgebra; Doubt $\mathcal{N}$-ideal.
Received: February 29, 2020. Accepted: April 25, 2020.
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(II) $(x * (x * y)) * y = 0$,
(III) $x * x = 0$,
(IV) $0 * x = 0$,
(V) $x * y = 0$ and $y * x = 0$ imply $x = y$.

Any BCK-algebra $\mathcal{X}$ satisfies the following axioms for all $x, y, z \in \mathcal{X}$:
(I1) $x * 0 = x$,
(I2) $(x * y) * z = (x * z) * y$,
(I3) $x * y \leq x$,
(I4) $(x * y) * z \leq (x * z) * (y * z)$,
(I5) $x \leq y \Rightarrow x * z \leq y * z, z * y \leq z * x$.
A partial ordering $\leq$ on a BCK-algebra $\mathcal{X}$ can be defined by $x \leq y$ if and only if $x * y = 0$. A non-empty subset K of a BCK-algebra $\mathcal{X}$ is called a subalgebra of $\mathcal{X}$ if $x * y \in K, \forall x, y \in \mathcal{X}$, and an ideal of $\mathcal{X}$ if $\forall x, y \in \mathcal{X}$,

(1) $0 \in K$,
(2) $x * y \in K$ and $y \in K$ imply $x \in K$.

Definition 2.1. [11] A fuzzy set $A = \{(x, \mu_A(x)) \mid x \in \mathcal{X}\}$ in $\mathcal{X}$ is called a doubt fuzzy subalgebra of $\mathcal{X}$ if $\mu_A(x * y) \leq \max\{\mu_A(x), \mu_A(y)\}$ for all $x, y \in \mathcal{X}$.

Definition 2.2. [11] A fuzzy set $A = \{(x, \mu_A(x)) \mid x \in \mathcal{X}\}$ in $\mathcal{X}$ is called a doubt fuzzy ideal of $\mathcal{X}$ if $\mu_A(0) \leq \mu_A(x)$ and $\mu_A(x) \leq \max\{\mu_A(x * y), \mu_A(y)\}$ for all $x, y \in \mathcal{X}$.

Denote by $\mathcal{F}(\mathcal{X}, [-1, 0])$ the collection of functions from a set $\mathcal{X}$ to the interval $[-1, 0]$. We say that, an element of $\mathcal{F}(\mathcal{X}, [-1, 0])$ is a negative-valued function from $\mathcal{X}$ to [-1, 0] (briefly, $\mathcal{N}$-function on $\mathcal{X}$). By an $\mathcal{N}$-structure we mean an ordered pair $(\mathcal{X}, \phi)$, where φ is an $\mathcal{N}$-function on $\mathcal{X}$. In what follows, let $\mathcal{X}$ be a BCK-algebra and φ an $\mathcal{N}$-function on $\mathcal{X}$ unless otherwise specified.

In [12], Jun et al. introduced the concepts of $\mathcal{N}$-subalgebras and $\mathcal{N}$-ideals in BCK-algebras as follows:

Definition 2.3. An $\mathcal{N}$-structure $(\mathcal{X}, \varphi)$ is called an $\mathcal{N}$-subalgebra of $\mathcal{X}$ if for all $x, y \in \mathcal{X}$:
$$\varphi(x * y) \leq \max\{\varphi(x), \varphi(y)\}.$$

Definition 2.4. An $\mathcal{N}$-structure $(\mathcal{X}, \varphi)$ is called an $\mathcal{N}$-ideal of $\mathcal{X}$ if for all $x, y \in \mathcal{X}$:

(1) $\varphi(0) \leq \varphi(x)$,
(2) $\varphi(x) \leq \max\{\varphi(x * y), \varphi(y)\}$.

3. DOUBT $\mathcal{N}$-SUBALGEBRAS AND $\mathcal{N}$-IDEALS

In this section, we introduce doubt $\mathcal{N}$-subalgebras and ideals in BCK-algebras and investigate some of their properties.

Definition 3.1. An $\mathcal{N}$-structure $(\mathcal{X}, \varphi)$ is called a doubt $\mathcal{N}$-subalgebra of $\mathcal{X}$ if for all $x, y \in \mathcal{X}$:
$$\varphi(x * y) \geq \min\{\varphi(x), \varphi(y)\}.$$

Example 3.2. Consider a $BCK-$algebra $X = \{0, a, b, c\}$ with the following Cayley table:

$*$	0	a	b	c
0	0	0	0	0
a	a	0	0	a
b	b	a	0	b
c	c	c	c	0

Let $(\mathcal{X}, \varphi)$ be an $\mathcal{N}$-structure in which φ is given by

$$\varphi(x) = \begin{cases} -0.2, \text{ if } & \text{x= } 0 \\ -0.3, \text{ if } & \text{x= } a, b \\ -0.8, \text{ if } & \text{x= } c. \end{cases}$$

By routine calculation, we know that $(\mathcal{X}, \varphi)$ is a doubt $\mathcal{N}$-subalgebra of $\mathcal{X}$.

For any $\mathcal{N}$-function φ and $\alpha \in [-1, 0]$, we define the set

$$\varphi_\alpha = \{x \in \mathcal{X} : \varphi(x) \geq \alpha\}.$$

Theorem 3.1. *Let $(\mathcal{X}, \varphi)$ be an $\mathcal{N}$-structure over $\mathcal{X}$ and let $\alpha \in [-1, 0]$. If $(\mathcal{X}, \varphi)$ is a doubt $\mathcal{N}$-subalgebra of $\mathcal{X}$, then the nonempty set φ_α is a subalgebra of $\mathcal{X}$.*

Proof. Let $\alpha \in [-1, 0]$ and let $\varphi_\alpha \neq \emptyset$. If $x, y \in \varphi_\alpha$, then $\varphi(x) \geq \alpha$ and $\varphi(y) \geq \alpha$. It follows from Definition 3.1 that

$$\varphi(x * y) \geq \min\{\varphi(x), \varphi(y)\} \geq \alpha.$$

Hence, $x * y \in \varphi_\alpha$, and therefore φ_α is a subalgebra of $\mathcal{X}$. □

Theorem 3.2. *Let $(\mathcal{X}, \varphi)$ be an $\mathcal{N}$-structure over $\mathcal{X}$ and assume that $\emptyset \neq \varphi_\alpha$ is a subalgebra of $\mathcal{X}$ for all $\alpha \in [-1, 0]$. Then, $(\mathcal{X}, \varphi)$ is a doubt $\mathcal{N}$-subalgebra of $\mathcal{X}$.*

Proof. Assume that $\emptyset \neq \varphi_\alpha$ is a subalgebra of $\mathcal{X}$ for all $\alpha \in [-1, 0]$. If there exist $a', b' \in \mathcal{X}$ such that

$$\varphi(a' * b') < \min\{\varphi(a'), \varphi(b')\}.$$

Then by taking

$$\alpha_o \quad = \quad \frac{1}{2}[\varphi(a' * b') + \min\{\varphi(a'), \varphi(b')\}],$$

we have

$$\varphi(a' * b') \quad < \quad \alpha_o < \min\{\varphi(a'), \varphi(b')\}.$$

Hence, $a' * b' \notin \varphi_{\alpha_o}, a' \in \varphi_{\alpha_o}$ and $b' \in \varphi_{\alpha_o}$. This is a contradiction. Therefore, $(\mathcal{X}, \varphi)$ is a doubt $\mathcal{N}$-subalgebra of $\mathcal{X}$. □

Theorem 3.3. *If $(\mathcal{X}, \varphi)$ is a doubt $\mathcal{N}$-subalgebra of $\mathcal{X}$, then $\varphi(0) \geqslant \varphi(x)$ for all $x \in \mathcal{X}$.*

Proof. For any $x \in \mathcal{X}$, we have $\varphi(0) = \varphi(x * x) \geq \min\{\varphi(x), \varphi(x)\} = \varphi(x)$. This completes the proof. □

Theorem 3.4. *If every doubt $\mathcal{N}$-subalgebra $(\mathcal{X}, \varphi)$ of $\mathcal{X}$, satisfies*

$$\varphi(x * y) \geq \varphi((y)$$

for all $x, y \in \mathcal{X}$, then $(\mathcal{X}, \varphi)$ is constant.

Proof. Note that in a BCK-algebra $\mathcal{X}, x * 0 = x$ for all $x \in \mathcal{X}$. Since $\varphi(x * y) \geq \varphi((y)$, we have

$$\varphi(x) = \varphi(x * 0) \geq \varphi(0).$$

It follows from Theorem 3.3 that $\varphi(x) = \varphi(0)$ for all $x, y \in \mathcal{X}$. Therefore, $(\mathcal{X}, \varphi)$ is constant. □

Now, we introduce the notion of doubt $\mathcal{N}$-ideals in BCK-algebras.

Definition 3.3. An $\mathcal{N}$-structure $(\mathcal{X}, \varphi)$ is called a doubt $\mathcal{N}$-ideal of $\mathcal{X}$ if for all $x, y \in \mathcal{X}$:

(1) $\varphi(0) \geq \varphi(x)$,
(2) $\varphi(x) \geq \min\{\varphi(x * y), \varphi(y)\}$.

Example 3.4. Consider a $BCK-$algebra $X = \{0, a, b, c\}$ which is given in Example 3.6. Let $(\mathcal{X}, \varphi)$ be an $\mathcal{N}$-structure in which φ is defined by

$$\varphi(x) = \begin{cases} -0.2, & \text{if} \quad \text{x}= 0 \\ -0.3, & \text{if} \quad \text{x}= a, b \\ -0.8, & \text{if} \quad \text{x}= c. \end{cases}$$

By routine calculation, we know that $(\mathcal{X}, \varphi)$ is a doubt $\mathcal{N}$-ideal of $\mathcal{X}$.

Theorem 3.5. *Let $(\mathcal{X}, \varphi)$ be a doubt $\mathcal{N}$-ideal of $\mathcal{X}$. If $\leq$ is a partial ordering on $\mathcal{X}$, then $\varphi(x) \geq \varphi(y)$ for all $x, y \in \mathcal{X}$ such that $x \leq y$.*

Proof. Let $(\mathcal{X}, \varphi)$ be a doubt $\mathcal{N}$-ideal of $\mathcal{X}$. It is known that $\leq$ is a partial ordering on $\mathcal{X}$ defined by $x \leq y$ if and only if $x * y = 0$ for all $x, y \in \mathcal{X}$. Then,

$$\begin{aligned} \varphi(x) &\geq \min\{\varphi(x * y), \varphi(y)\} \\ &= \min\{\varphi(0), \varphi(y)\} \\ &= \varphi(y). \end{aligned}$$

This completes the proof. □

Theorem 3.6. *Let $(\mathcal{X}, \varphi)$ be a doubt $\mathcal{N}$-ideal of $\mathcal{X}$. Then,*

$$\varphi(x * y) \geq \varphi((x * y) * y) \Leftrightarrow \varphi((x * z) * (y * z)) \geq \varphi((x * y) * z)$$

for all $x, y, z \in \mathcal{X}$.

Proof. Note that

$$\begin{aligned} ((x * (y * z)) * z) * z &= ((x * z) * (y * z)) * z \\ &\leq (x * y) * z \end{aligned}$$

for all $x, y, z \in \mathcal{X}$. Assume that $\varphi(x * y) \geq \varphi((x * y) * y)$ for all $x, y, z \in \mathcal{X}$. It follows from (I2)and Theorem 3.5 that

$$\begin{aligned} \varphi((x * z) * (y * z)) &= \varphi((x * (y * z)) * z) \\ &\geq \varphi(((x * (y * z)) * z) * z) \\ &\geq \varphi((x * y) * z), \end{aligned}$$

for all $x, y, z \in \mathcal{X}$.

Conversely, suppose that

$$\varphi((x * z) * (y * z)) \geq \varphi((x * y) * z) \tag{3.1}$$

for all $x, y, z \in \mathcal{X}$. If we substitute z for y in Equation (3.1), then

$$\begin{aligned} \varphi(x * z) &= \varphi((x * z) * 0) \\ &= \varphi((x * z) * (z * z)) \\ &\geq \varphi((x * z) * z), \end{aligned}$$

for all $x, z \in \mathcal{X}$ by using (III) and (I1.) $\square$

Theorem 3.7. *Let $(\mathcal{X}, \varphi)$ be a doubt $\mathcal{N}$-ideal of $\mathcal{X}$. Then,*

$$\varphi(x * y) \geq \min\{\varphi(x * z), \varphi(z * y)\}$$

for all $x, y, z \in \mathcal{X}$.

Proof. Note that $((x * y) * (x * z)) \leq (z * y)$ for all $x, y, x \in \mathcal{X}$. It follows from Theorem 3.5, that

$$\varphi((x * y) * (x * z)) \geq \varphi(z * y).$$

Now, by Definition 3.3, we have

$$\begin{aligned} \varphi(x * y) &\geq \min\{\varphi((x * y) * (x * z)), \varphi(x * z)\} \\ &\geq \min\{\varphi(x * z), \varphi(z * y)\} \end{aligned}$$

for all $x, y, z \in \mathcal{X}$. This completes the proof. $\square$

Theorem 3.8. *Let $(\mathcal{X}, \varphi)$ be a doubt $\mathcal{N}$-ideal of $\mathcal{X}$. Then,*

$$\varphi(x * (x * y)) \geq \varphi(y)$$

for all $x, y \in \mathcal{X}$.

Proof. Let $(\mathcal{X}, \varphi)$ be a doubt $\mathcal{N}$-ideal of $\mathcal{X}$. Then, for all $x, y \in \mathcal{X}$, we have

$$\begin{aligned} \varphi(x * (x * y)) &\geq \min\{\varphi((x * (x * y)) * y), \varphi(y)\} \\ &= \min\{\varphi((x * y) * (x * y)), \varphi(y)\} \\ &= \min\{\varphi(0), \varphi(y)\} \\ &= \varphi(y). \end{aligned}$$

This completes the proof. $\square$

Theorem 3.9. *Let $(\mathcal{X}, \varphi)$ be an $\mathcal{N}$-structure over $\mathcal{X}$ and let $\alpha \in [-1, 0]$. If $(\mathcal{X}, \varphi)$ is a doubt $\mathcal{N}$-ideal of $\mathcal{X}$, then the nonempty set φ_α is an ideal of $\mathcal{X}$.*

Proof. Assume that $\varphi_\alpha \neq \emptyset$ for $\alpha \in [-1, 0]$. Clearly, $0 \in \varphi_\alpha$. Let $x * y \in \varphi_\alpha$ and $y \in \varphi_\alpha$. Then, $\varphi(x * y) \geq \alpha$ and $\varphi(y) \geq \alpha$. It follows from Definition 3.3 that

$$\varphi(x) \geq \min\{\varphi(x * y), \varphi(y)\} \geq \alpha,$$

so, $x \in \varphi_\alpha$. Therefore, φ_α is an ideal of $\mathcal{X}$. $\square$

Theorem 3.10. *Let $(\mathcal{X}, \varphi)$ be an $\mathcal{N}$-structure over $\mathcal{X}$ and assume that $\emptyset \neq \varphi_\alpha$ is an ideal of $\mathcal{X}$ for all $\alpha \in [-1, 0]$. Then, $(\mathcal{X}, \varphi)$ is a doubt $\mathcal{N}$-ideal of $\mathcal{X}$.*

Proof. Assume that $\emptyset \neq \varphi_\alpha$ is an ideal of $\mathcal{X}$ for all $\alpha \in [-1, 0]$. For any $x \in \mathcal{X}$, let $\varphi(x) = \alpha$. Then, $x \in \varphi_\alpha$, and so φ_α is nonempty. Since φ_α is an ideal of $\mathcal{X}$, so $0 \in \varphi_\alpha$. Hence, $\varphi(0) \geq \alpha = \varphi(x)$ for all $x \in \mathcal{X}$. If there exists $a', b' \in \mathcal{X}$ such that

$$\varphi(a') < \min\{\varphi(a' * b'), \varphi(b')\}.$$

Then, by taking

$$\alpha_1 = \frac{1}{2}[\varphi(a') + \min\{\varphi(a' * b'), \varphi(b')\}],$$

we have

$$\varphi(a') < \alpha_1 < \min\{\varphi(a' * b'), \varphi(b')\}.$$

Hence, $a' \notin \varphi_{\alpha_1}, a' * b' \in \varphi_{\alpha_1}$ and $b' \in \varphi_{\alpha_1}$. This is a contradiction, and so $\varphi(x) \geq \min\{\varphi(x*y), \varphi(y)\}$ for all $x, y \in \mathcal{X}$. Therefore, $(\mathcal{X}, \varphi)$ is a doubt $\mathcal{N}$-ideal of $\mathcal{X}$. □

Theorem 3.11. *Let $(\mathcal{X}, \varphi)$ be an $\mathcal{N}$-structure over $\mathcal{X}$. If the inequality $x * y \leq z$ holds in $\mathcal{X}$, then $\varphi(x) \geq \min\{\varphi(y), \varphi(z)\}$ for all $x, y, z \in \mathcal{X}$.*

Proof. $(\mathcal{X}, \varphi)$ be an $\mathcal{N}$-structure over $\mathcal{X}$ and $x, y, z \in \mathcal{X}$ be such that $x * y \leq z$. Then, $(x * y) * z = 0$, and so

$$\begin{aligned} \varphi(x) &\geq \min\{\varphi(x*y), \varphi(y)\} \\ &\geq \min\{\min\{\varphi((x*y)*z), \varphi(z)\}, \varphi(y)\} \\ &= \min\{\min\{\varphi(0), \varphi(z)\}, \varphi(y)\} \\ &= \min\{\varphi(y), \varphi(z)\}. \end{aligned}$$

This completes the proof. □

Theorem 3.12. *Every doubt $\mathcal{N}$-ideal of $\mathcal{X}$ is a doubt $\mathcal{N}$-subalgebra of $\mathcal{X}$.*

Proof. Let $(\mathcal{X}, \varphi)$ be a doubt $\mathcal{N}$-ideal of $\mathcal{X}$. For any $x, y \in \mathcal{X}$, we have

$$\begin{aligned} \varphi(x*y) &\geq \min\{\varphi((x*y)*x), \varphi(x)\} \\ &= \min\{\varphi((x*x)*y), \varphi(x)\} \\ &= \min\{\varphi(0*y), \varphi(x)\} \\ &= \min\{\varphi(0), \varphi(x)\} \\ &\geq \min\{\varphi(x), \varphi(y)\} \end{aligned}$$

Hence, $(\mathcal{X}, \varphi)$ is a doubt $\mathcal{N}$-subalgebra of $\mathcal{X}$. □

Example 3.5. In Example 3.4, $(\mathcal{X}, \varphi)$ is a doubt $\mathcal{N}$-ideal of $\mathcal{X}$, so that $(\mathcal{X}, \varphi)$ is a doubt $\mathcal{N}$-subalgebra of $\mathcal{X}$.

The converse of Theorem 3.12 is not true in general.

Example 3.6. Consider a $BCK-$algebra $X = \{0, a, b, c, d\}$ with the following Cayley table:

$*$	0	a	b	c	d
0	0	0	0	0	0
a	a	0	0	0	0
b	b	a	0	a	0
c	c	c	c	0	0
d	d	d	d	c	0

Let $(\mathcal{X}, \varphi)$ be an $\mathcal{N}$-structure in which φ is given by

$$\varphi(x) = \begin{cases} 0.0, & \text{if } \quad \text{x= } 0 \\ -0.2, & \text{if } \quad \text{x= } a \\ -0.6, & \text{if } \quad \text{x= } b \\ -0.4, & \text{if } \quad \text{x= } c \\ -0.8, & \text{if } \quad \text{x= } d. \end{cases}$$

Then, $(\mathcal{X}, \varphi)$ is a doubt $\mathcal{N}$-subalgebra of $\mathcal{X}$, but it is not a doubt $\mathcal{N}$-ideal of $\mathcal{X}$ since $\varphi(d) = -0.8 < -0.4 = \min\{\varphi(d*c), \varphi(c)\}$.

We give a condition for a doubt $\mathcal{N}$-subalgebra to be a doubt $\mathcal{N}$-ideal in a BCK-algebra.

Theorem 3.13. *Let $(\mathcal{X}, \varphi)$ be a doubt $\mathcal{N}$-subalgebra of $\mathcal{X}$. If the inequality $x * y \leq z$ holds in $\mathcal{X}$, then $(\mathcal{X}, \varphi)$ is a doubt $\mathcal{N}$-ideal of $\mathcal{X}$.*

Proof. Let $(\mathcal{X}, \varphi)$ be a doubt $\mathcal{N}$-subalgebra of $\mathcal{X}$. Then, from Theorem 3.3, $\varphi(0) \geq \varphi(x)$, for all $x \in \mathcal{X}$. As $x * y \leq z$ holds in $\mathcal{X}$, then from Theorem 3.11, we get $\varphi(x) \geq \min\{\varphi(y), \varphi(z)\}$ for all $x, y, z \in \mathcal{X}$.

Since $x * (x * y) \leq y$ for all $x, y \in \mathcal{X}$, then $\varphi(x) \geq \min\{\varphi(x * y), \varphi(y)\}$. Hence, $(\mathcal{X}, \varphi)$ is a doubt $\mathcal{N}$-ideal of $\mathcal{X}$. □

Theorem 3.14. *Let $(\mathcal{X}, \varphi)$ be a doubt $\mathcal{N}$-ideal of $\mathcal{X}$. Then, the set*

$$\mathcal{H} = \{x \in \mathcal{X} : \varphi(x) = \varphi(0)\}$$

is an ideal of $\mathcal{X}$.

Proof. Obviously, $0 \in \mathcal{H}$. Hence, $\mathcal{H} \neq \emptyset$. Now, let $x, y \in \mathcal{H}$ such that $x * y, y \in \mathcal{H}$. Then, $\varphi(x * y) = \varphi(0) = \varphi(y)$. Now, $\varphi(x) \geq \min\{\varphi(x * y), \varphi(y)\} = \varphi(0)$. Since $(\mathcal{X}, \varphi)$ is a doubt $\mathcal{N}$-ideal of $\mathcal{X}$, $\varphi(0) \geq \varphi(x)$. Therefore, $\varphi(0) = \varphi(x)$. It follows that $x \in \mathcal{H}$, for all $x, y \in \mathcal{X}$. Therefore, $\mathcal{H}$ is an ideal of $\mathcal{X}$. □

For any element $\omega_\alpha \in X$, we consider the set:

$$\varphi_{\omega_\alpha} = \{x \in \mathcal{X} \mid \varphi(x) \geq \varphi(\omega_\alpha)\}.$$

Clearly, $\omega_\alpha \in \varphi_{\omega_\alpha}$. So that $\omega_\alpha \in \varphi_{\omega_\alpha}$ is a nonempty set of $\mathcal{X}$.

Theorem 3.15. *Let ω_α be any element of $\mathcal{X}$. If $(\mathcal{X}, \varphi)$ is a doubt $\mathcal{N}$-ideal of X, then φ_{ω_α} is an ideal of $\mathcal{X}$.*

Proof. Clearly, $0 \in \varphi_{\omega_\alpha}$. Let $x, y \in \mathcal{X}$ be such that $x * y \in \varphi_{\omega_\alpha}$ and $y \in \varphi_{\omega_\alpha}$. Then, $\varphi(x * y) \geq \varphi(\omega_\alpha)$ and $\varphi(y) \geq \varphi(\omega_\alpha)$. It follows that from Definition 3.3, that

$$\varphi(x) \geq \min\{\varphi(x * y), \varphi(y)\} \geq \varphi(\omega_\alpha).$$

Hence, $x \in \varphi_{\omega_t}$, and therefore φ_{ω_α} is an ideal of $\mathcal{X}$. □

Theorem 3.16. *Let $\omega_\alpha \in \mathcal{X}$ and let $(\mathcal{X}, \varphi)$ be an $\mathcal{N}$-structure over $\mathcal{X}$. Then,*

(1) If φ_{ω_α} is an ideal of $\mathcal{X}$, then the following assertion is valid for all $x, y, z \in X$:
 (A1) $\varphi(x) \leq \min\{\varphi(y * z), \varphi(z)\} \Rightarrow \varphi(x) \leq \varphi(y)$.
(2) If $(\mathcal{X}, \varphi)$ satisfies (A1) and
 (A2) $\varphi(0) \geq \varphi(x)$

for all $x \in \mathcal{X}$. Then,φ_{ω_α} is an ideal for all $\omega_\alpha \in Im(\varphi)$.

Proof. (1) Assume that φ_{ω_α} is an ideal of $\mathcal{X}$ for $\omega_\alpha \in \mathcal{X}$. Let $x, y, z \in \mathcal{X}$ be such that $\varphi(x) \leq \min\{\varphi(y * z), \varphi(z)\}$. Then, $y * z \in \varphi_{\omega_\alpha}$ and $z \in \varphi_{\omega_\alpha}$, where $\omega_\alpha = x$. Since φ_{ω_α} is an ideal of $\mathcal{X}$, it follows that $y \in \varphi_{\omega_\alpha}$ for $\omega_\alpha = x$. Hence, $\varphi(y) \geq \varphi(\omega_\alpha) = \varphi(x)$.

(2) Let $\omega_\alpha \in Im(\varphi)$ and suppose that $(\mathcal{X}, \varphi)$ satisfies (A1) and (A2). Clearly, $0 \in \varphi_{\omega_\alpha}$ by (A2). Let $x, y \in \mathcal{X}$ be such that $x * y \in \varphi_{\omega_\alpha}$ and $y \in \varphi_{\omega_\alpha}$. Then, $\varphi(x * y) \geq \varphi(\omega_\alpha)$ and $\varphi(y) \geq \varphi(\omega_\alpha)$, so

$$\min\{\varphi(x * y), \varphi(y)\} \geq \varphi(\omega_\alpha).$$

It follows from (A1) that $\varphi(\omega_\alpha) \leq \varphi(x)$. Thus, $x \in \varphi_{\omega_\alpha}$, and therefore φ_{ω_α} is an ideal of $\mathcal{X}$. □

4. CONCLUSIONS

Doubt $\mathcal{N}$-subalgebras and doubt $\mathcal{N}$-ideals with special properties play an important role in investigating the structure of an algebraic system. In this work, we discussed an $\mathcal{N}$-structure with an application to BCK-algebras. We introduced the notions of doubt $\mathcal{N}$- subalgebras and doubt $\mathcal{N}$-ideals in BCK-algebras, and investigated related properties. We considered some characterizations of a doubt $\mathcal{N}$-subalgebra and a doubt $\mathcal{N}$-ideal in BCK-algebras by means of doubt level subset. Relations between a doubt $\mathcal{N}$-subalgebra and a doubt $\mathcal{N}$-ideal were provided. We believe that our results presented in this paper will give a foundation for further study the algebraic structure of BCK-algebras.

5. ACKNOWLEDGEMENTS

The authors are thankful to the editors and the anonymous reviewers for their valuable suggestions and comments on the manuscript.

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