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SUP-HESITANT FUZZY SUBALGEBRAS AND ITS TRANSLATIONS AND EXTENSIONS

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ABSTRACT. In BCK/BCI-algebras, the notion of Sup-hesitant fuzzy subalgebra is introduced, and related properties are investigated. Characterizations of a Sup-hesitant fuzzy subalgebra are discussed. Sup-hesitant fuzzy translation and Sup-hesitant fuzzy extension of Sup-hesitant fuzzy subalgebras are introduced, and their relations are investigated.

1. INTRODUCTION

As a generalization of fuzzy set, the notions of Atanassov's intuitionistic fuzzy set, type 2 fuzzy set and fuzzy multiset etc. are introduced and studied by several researchers. Hesitant fuzzy sets are another generalization of fuzzy sets, and it is introduced by Torra and Narukawa in [10, 11]. They discussed the relationship between hesitant fuzzy sets and intuitionistic fuzzy sets. Hesitant fuzzy sets are applied to algebraic structures (see [1, 3, 4, 5, 6, 8, 9]).

In this paper, we introduce the notion of sup-hesitant fuzzy subalgebras and investigate several related properties in BCK/BCI-algebras. We consider characterizations of Sup-hesitant fuzzy subalgebras. We also discuss Sup-hesitant fuzzy translation and Sup-hesitant fuzzy extension of Sup-hesitant fuzzy subalgebras. We investigate relations between Sup-hesitant fuzzy translation and Sup-hesitant fuzzy extension.

2. PRELIMINARIES

A BCK/BCI-algebra is an important class of logical algebras introduced by K. Iséki and was extensively investigated by several researchers.

An algebra $(X;*,0)$ of type $(2,0)$ is called a *BCI-algebra* if it satisfies the following conditions:

(I) $(\forall x,y,z \in X)\,(((x*y)*(x*z))*(z*y)=0)$,
(II) $(\forall x,y \in X)\,((x*(x*y))*y=0)$,
(III) $(\forall x \in X)\,(x*x=0)$,
(IV) $(\forall x,y \in X)\,(x*y=0, y*x=0 \Rightarrow x=y)$.

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If a BCI-algebra X satisfies the following identity:

(V) $(\forall x \in X)\,(0 * x = 0)$,

then X is called a *BCK-algebra*.

Any BCK/BCI-algebra X satisfies the following conditions:

$$(\forall x \in X)\,(x * 0 = x)\,, \tag{2.1}$$

$$(\forall x, y, z \in X)\,(x \leq y \;\Rightarrow\; x * z \leq y * z,\; z * y \leq z * x)\,, \tag{2.2}$$

$$(\forall x, y, z \in X)\,((x * y) * z = (x * z) * y)\,, \tag{2.3}$$

$$(\forall x, y, z \in X)\,((x * z) * (y * z) \leq x * y) \tag{2.4}$$

where $x \leq y$ if and only if $x * y = 0$.

Any BCI-algebra X satisfies the following conditions:

$$(\forall x, y, z \in X)\,(0 * (0 * ((x * z) * (y * z))) = (0 * y) * (0 * x))\,, \tag{2.5}$$

$$(\forall x, y \in X)\,(0 * (0 * (x * y)) = (0 * y) * (0 * x))\,, \tag{2.6}$$

$$(\forall x \in X)\,(0 * (0 * (0 * x)) = 0 * x)\,. \tag{2.7}$$

A subset S of a BCK/BCI-algebra X is called a *subalgebra* of X if $x * y \in S$ for all $x, y \in S$.

We refer the reader to the books [2, 7] for further information regarding BCK/BCI-algebras.

Torra [10] introduced a new extension for fuzzy sets to manage those situations in which several values are possible for the definition of a membership function of a fuzzy set.

Let X be a reference set. Then we define hesitant fuzzy set on X in terms of a function $\mathcal{H}$ that when applied to X returns a subset of $[0;1]$ (see [10, 11]).

3. Sup-hesitant fuzzy subalgebras

In what follows, the power set of $[0,1]$ is denoted by $\mathcal{P}([0,1])$ and

$$\mathcal{P}^*([0,1]) = \mathcal{P}([0,1]) \setminus \{\emptyset\}.$$

For any element $Q \in \mathcal{P}^*([0,1])$, the supremum of Q is denoted by $\sup Q$. For any hesitant fuzzy set $\mathcal{H}$ on X and $Q \in \mathcal{P}^*([0,1])$, consider the set

$$\mathrm{Sup}[\mathcal{H}; Q] := \{x \in X \mid \sup \mathcal{H}(x) \geq \sup Q\}\,.$$

Definition 3.1. Let X be a BCK/BCI-algebra. Given an element $Q \in \mathcal{P}^*([0,1])$, a hesitant fuzzy set $\mathcal{H}$ on X is called a *Sup-hesitant fuzzy subalgebra* of X related to Q (briefly, *Q-Sup-hesitant fuzzy subalgebra* of X) if the set $\mathrm{Sup}[\mathcal{H}; Q]$ is a subalgebra of X. If $\mathcal{H}$ is a Q-Sup-hesitant fuzzy subalgebra of X for all $Q \in \mathcal{P}^*([0,1])$, then we say that $\mathcal{H}$ is a *Sup-hesitant fuzzy subalgebra* of X.

Example 3.2. (1) Let $X = \{0, a, b, c\}$ be a BCK-algebra with the following Cayley table:

$*$	0	a	b	c
0	0	0	0	0
a	a	0	a	0
b	b	b	0	0
c	c	b	a	0

Let $\mathcal{H}$ be a hesitant fuzzy set on X defined by Table 1.
It is routine to verify that $\mathcal{H}$ is a Sup-hesitant fuzzy subalgebra of X.

TABLE 1. Tabular representation of $\mathcal{H}$

X	0	a	b	c
$\mathcal{H}(x)$	$(0.8, 1]$	$(0.3, 0.4) \cup \{0.8\}$	$[0.5, 0.6]$	$(0.3, 0.5) \cup \{0.6\}$

(2) Let $X = \{0, a, b, c, d\}$ be a BCK-algebra with the following Cayley table:

$*$	0	a	b	c	d
0	0	0	0	0	0
a	a	0	0	0	0
b	b	a	0	0	0
c	c	c	c	0	0
d	d	c	c	a	0

Let $\mathcal{H}$ be a hesitant fuzzy set on X defined by Table 2.

TABLE 2. Tabular representation of $\mathcal{H}$

X	0	a	b	c	d
$\mathcal{H}(x)$	$\{0.8, 0.9\}$	$[0.2, 0.3)$	$(0.7, 0.8]$	$\{0.4\} \cup (0.5, 0.6)$	$[0.1, 0.2]$

It is easy to verify that $\mathcal{H}$ is a Sup-hesitant fuzzy subalgebra of X.

(3) Consider a BCI-algebra $X = \{0, 1, a, b, c\}$ with the following Cayley table.

$*$	0	1	a	b	c
0	0	0	c	c	a
1	1	0	c	c	a
a	a	a	0	0	c
b	b	a	1	0	c
c	c	c	a	a	0

Let $\mathcal{H}$ be a hesitant fuzzy set on X defined by Table 3.

TABLE 3. Tabular representation of $\mathcal{H}$

X	0	1	a	b	c
$\mathcal{H}(x)$	$\{0.6, 0.8\}$	$[0.2, 0.6)$	$(0.3, 0.5]$	$\{0.3\} \cup (0.4, 0.5)$	$[0.1, 0.3]$

Then $\mathcal{H}$ is a Q_1-Sup-hesitant fuzzy subalgebra of X with $Q_1 := [0.45, 0.55]$. But it is not a Q_2-Sup-hesitant fuzzy subalgebra of X with $Q_2 := [0.3, 0.4)$ since $\text{Sup}[\mathcal{H}; Q_2] = \{0, 1, a, b\}$ is not a subalgebra of X.

(4) Consider a BCK-algebra $X = \{0, a, b, c, d\}$ with the following Cayley table.

$*$	0	a	b	c	d
0	0	0	0	0	0
a	a	0	0	0	a
b	b	a	0	0	b
c	c	b	a	0	c
d	d	d	d	d	0

TABLE 4. Tabular representation of $\mathcal{H}$

X	0	a	b	c	d
$\mathcal{H}(x)$	$[0.7, 0.8]$	$(0.5, 0.7)$	$[0.1, 0.3]$	$[0.3, 0.5]$	$[0.1, 0.2]$

Let $\mathcal{H}$ be a hesitant fuzzy set on X defined by Table 4.
Then $\mathcal{H}$ is a Q_1-Sup-hesitant fuzzy subalgebra of X with $Q_1 := [0.1, 0.2)$. If we take $Q_2 := (0.25, 0.45)$, then $\mathrm{Sup}[\mathcal{H}; Q_2] = \{0, a, c\}$ which is not a subalgebra of X. Hence $\mathcal{H}$ is not a Q_2-Sup-hesitant fuzzy subalgebra of X.

Theorem 3.1. *A hesitant fuzzy set $\mathcal{H}$ on a BCK/BCI-algebra X is a Sup-hesitant fuzzy subalgebra of X if and only if the following assertion is valid:*

$$(\forall x, y \in X)\left(\sup \mathcal{H}(x * y) \geq \min\{\sup \mathcal{H}(x), \sup \mathcal{H}(y)\}\right). \tag{3.1}$$

Proof. Assume that $\mathcal{H}$ is a Sup-hesitant fuzzy subalgebra of X. If (3.1) does not hold, then

$$\sup \mathcal{H}(x * y) < \sup Q \leq \min\{\sup \mathcal{H}(x), \sup \mathcal{H}(y)\}$$

for some $Q \in \mathcal{P}^*([0, 1])$ and $x, y \in X$. Then $x, y \in \mathrm{Sup}[\mathcal{H}; Q]$ and $x * y \notin \mathrm{Sup}[\mathcal{H}; Q]$. This is a contradiction, and so

$$\sup \mathcal{H}(x * y) \geq \min\{\sup \mathcal{H}(x), \sup \mathcal{H}(y)\}$$

for all $x, y \in X$.

Conversely, suppose that (3.1) is valid. Let $Q \in \mathcal{P}^*([0, 1])$ and $x, y \in \mathrm{Sup}[\mathcal{H}; Q]$. Then $\sup \mathcal{H}(x) \geq \sup Q$ and $\sup \mathcal{H}(y) \geq \sup Q$. It follows from (3.1) that

$$\sup \mathcal{H}(x * y) \geq \min\{\sup \mathcal{H}(x), \sup \mathcal{H}(y)\} \geq \sup Q$$

and that $x * y \in \mathrm{Sup}[\mathcal{H}; Q]$. Hence the set $\mathrm{Sup}[\mathcal{H}; Q]$ is a subalgebra of X, and so $\mathcal{H}$ is a Sup-hesitant fuzzy subalgebra of X. □

Lemma 3.2. *Every Sup-hesitant fuzzy subalgebra $\mathcal{H}$ of a BCK/BCI-algebra X satisfies:*

$$(\forall x \in X)\left(\sup \mathcal{H}(0) \geq \sup \mathcal{H}(x)\right). \tag{3.2}$$

Proof. Using (III) and (3.1) induce

$$\sup \mathcal{H}(0) = \sup \mathcal{H}(x * x) \geq \min\{\sup \mathcal{H}(x), \sup \mathcal{H}(x)\} = \sup \mathcal{H}(x)$$

for all $x \in X$. □

Proposition 3.3. *Let $\mathcal{H}$ be a Sup-hesitant fuzzy subalgebra of a BCK-algebra X. For any elements $a_1, a_2, \cdots, a_n \in X$, if there exists $a_k \in \{a_1, a_2, \cdots, a_n\}$ such that $a_1 = a_k$, then*

$$(\forall x \in X)\left(\sup \mathcal{H}((\cdots((a_1 * a_2) * a_3) * \cdots) * a_n) \geq \sup \mathcal{H}(x)\right).$$

Proof. Using (2.3), (III) and (V), we have $(\cdots((a_1 * a_2) * a_3) * \cdots) * a_n = 0$. Thus

$$\sup \mathcal{H}((\cdots((a_1 * a_2) * a_3) * \cdots) * a_n) = \sup \mathcal{H}(0) \geq \sup \mathcal{H}(x)$$

for all $x \in X$ by Lemma 3.2. □

Theorem 3.4. *A hesitant fuzzy set $\mathcal{H}$ on a BCK/BCI-algebra X is a Sup-hesitant fuzzy subalgebra of X if and only if the set*

$$U_{\sup}(\mathcal{H}; t) := \{x \in X \mid \sup \mathcal{H}(x) \geq t\}$$

is a subalgebra of X for all $t \in [0,1]$.

Proof. Assume that $\mathcal{H}$ is a Sup-hesitant fuzzy subalgebra of X. Let $x, y \in U_{\sup}(\mathcal{H}; t)$ for $t \in [0,1]$. Then $\sup \mathcal{H}(x) \geq t$ and $\sup \mathcal{H}(y) \geq t$. It follows from (3.1) that

$$\sup \mathcal{H}(x * y) \geq \min\{\sup \mathcal{H}(x), \sup \mathcal{H}(y)\} \geq t.$$

Hence $x * y \in U_{\sup}(\mathcal{H}; t)$, and therefore $U_{\sup}(\mathcal{H}; t)$ is a subalgebra of X.

Conversely, suppose that $U_{\sup}(\mathcal{H}; t)$ is a subalgebra of X. For any $x, y \in X$, let $\sup \mathcal{H}(x) = t_x$ and $\sup \mathcal{H}(y) = t_y$. Taking $t := \min\{t_x, t_y\}$ implies that $x, y \in U_{\sup}(\mathcal{H}; t)$, and thus $x * y \in U_{\sup}(\mathcal{H}; t)$. Hence

$$\sup \mathcal{H}(x * y) \geq t = \min\{t_x, t_y\} = \min\{\sup \mathcal{H}(x), \sup \mathcal{H}(y)\}.$$

Therefore $\mathcal{H}$ is a Sup-hesitant fuzzy subalgebra of X by Theorem 3.1. □

Given a hesitant fuzzy set $\mathcal{H}$ on a BCK/BCI-algebra X, let

$$\top := 1 - \sup\{\sup \mathcal{H}(x) \mid x \in X\}.$$

Definition 3.3. Let $\mathcal{H}$ be a hesitant fuzzy set on a BCK/BCI-algebra X and $t \in [0, \top]$. A hesitant fuzzy set $\mathcal{H}_T^t : X \to \mathcal{P}([0,1])$ is called a *Sup-hesitant fuzzy t-translation* of $\mathcal{H}$ if $\sup \mathcal{H}_T^t(x) = \sup \mathcal{H}(x) + t$ for all $x \in X$.

Theorem 3.5. *Let $\mathcal{H}$ be a Sup-hesitant fuzzy subalgebra of a BCK/BCI-algebra X and let $t \in [0, \top]$. Then the Sup-hesitant fuzzy t-translation $\mathcal{H}_T^t$ of $\mathcal{H}$ is a Sup-hesitant fuzzy subalgebra of X.*

Proof. For any $x, y \in X$, we have

$$\begin{aligned}
&\sup \mathcal{H}_T^t(x * y) = \sup \mathcal{H}(x * y) + t \geq \min\{\sup \mathcal{H}(x), \sup \mathcal{H}(y)\} + t \\
&= \min\{\sup \mathcal{H}(x) + t, \sup \mathcal{H}(y) + t\} = \min\{\sup \mathcal{H}_T^t(x), \sup \mathcal{H}_T^t(y)\}.
\end{aligned}$$

Therefore $\mathcal{H}_T^t$ is a Sup-hesitant fuzzy subalgebra of X. □

Theorem 3.6. *Let $\mathcal{H}$ be a hesitant fuzzy set on a BCK/BCI-algebra X such that its Sup-hesitant fuzzy t-translation is a Sup-hesitant fuzzy subalgebra of X for some $t \in [0, \top]$. Then $\mathcal{H}$ is a Sup-hesitant fuzzy subalgebra of X.*

Proof. Assume that $\mathcal{H}_T^t$ is a Sup-hesitant fuzzy subalgebra of X for some $t \in [0, \top]$. Then

$$\begin{aligned}
&\sup \mathcal{H}(x * y) + t = \sup \mathcal{H}_T^t(x * y) \geq \min\{\sup \mathcal{H}_T^t(x), \sup \mathcal{H}_T^t(y)\} \\
&= \min\{\sup \mathcal{H}(x) + t, \sup \mathcal{H}(y) + t\} = \min\{\sup \mathcal{H}(x), \sup \mathcal{H}(y)\} + t
\end{aligned}$$

for all $x, y \in X$. It follows that $\sup \mathcal{H}(x * y) \geq \min\{\sup \mathcal{H}(x), \sup \mathcal{H}(y)\}$. Hence $\mathcal{H}$ is a Sup-hesitant fuzzy subalgebra of X. □

Definition 3.4. Let $\mathcal{H}$ and $\mathcal{G}$ be hesitant fuzzy sets on a set X. If $\sup \mathcal{H}(x) \leq \sup \mathcal{G}(x)$ for all $x \in X$, then we say that $\mathcal{G}$ is a *Sup-hesitant fuzzy extension* of $\mathcal{H}$.

Definition 3.5. Let $\mathcal{H}$ and $\mathcal{G}$ be hesitant fuzzy sets on a BCK/BCI-algebra X. Then $\mathcal{G}$ is called a *Sup-hesitant fuzzy extension* of $\mathcal{H}$ based on a subalgebra of X (briefly, *Sup-hesitant fuzzy S-extension* of $\mathcal{H}$) if the following assertions are valid.

(i) $\mathcal{G}$ is a Sup-hesitant fuzzy extension of $\mathcal{H}$.
(ii) If $\mathcal{H}$ is a Sup-hesitant fuzzy subalgebra of X, then so is $\mathcal{G}$.

Theorem 3.7. *If $\mathcal{H}$ is a Sup-hesitant fuzzy subalgebra of X, then its Sup-hesitant fuzzy t-translation is a Sup-hesitant fuzzy S-extension of $\mathcal{H}$ for all $t \in [0, \top]$.*

Proof. Straightforward. □

The converse of Theorem 3.7 is false as seen in the following example.

Example 3.6. Let $X = \{0, 1, 2, 3, 4\}$ be a set and a binary operation $*$ is given by the following Cayley table.

$*$	0	1	2	3	4
0	0	0	0	0	0
1	1	0	1	0	0
2	2	2	0	2	0
3	3	1	3	0	1
4	4	4	4	4	0

Then X is a BCK-algebra. Let $\mathcal{H}$ be a hesitant fuzzy set on X defined by Table 5.

TABLE 5. Tabular representation of $\mathcal{H}$

X	0	1	2	3	4
$\mathcal{H}(x)$	$[0.6, 0.8]$	$(0.3, 0.5)$	$[0.1, 0.3]$	$[0.4, 0.6]$	$[0.1, 0.2]$

Then $\mathcal{H}$ is a Sup-hesitant fuzzy subalgebra of X. Let $\mathcal{G}$ be a hesitant fuzzy set on X given by Table 6.

TABLE 6. Tabular representation of $\mathcal{H}$

X	0	1	2	3	4
$\mathcal{G}(x)$	$[0.6, 0.84]$	$(0.3, 0.56)$	$[0.1, 0.38]$	$[0.4, 0.67]$	$[0.1, 0.22]$

Then $\mathcal{G}$ is a Sup-hesitant fuzzy S-extension of $\mathcal{H}$. But it is not the Sup-hesitant fuzzy t-translation of $\mathcal{H}$ for all $t \in [0, \top]$.

For a hesitant fuzzy set $\mathcal{H}$ on a BCK/BCI-algebra X, $t \in [0, \top]$ and $k \in [0, 1]$ with $k \geq t$, let

$$U_{\sup}(\mathcal{H}; k - t) := \{x \in X \mid \sup \mathcal{H}(x) \geq k - t\}.$$

If $\mathcal{H}$ is a Sup-hesitant fuzzy subalgebra of X, then it is clear that $U_{\sup}(\mathcal{H}; k - t)$ is a subalgebra of X. But, if we do not give a condition that $\mathcal{H}$ is a Sup-hesitant fuzzy subalgebra of X then $U_{\sup}(\mathcal{H}; k - t)$ is not a subalgebra of X as seen in the following example.

Example 3.7. Consider a BCK-algebra $X = \{0, 1, 2, 3, 4\}$ with the following Cayley table.

$*$	0	1	2	3	4
0	0	0	0	0	0
1	1	0	0	0	0
2	2	1	0	0	0
3	3	1	1	0	0
4	4	3	3	1	0

Let $\mathcal{H}$ be a hesitant fuzzy set on X defined by Table 7.
Then $\mathcal{H}$ is not a Sup-hesitant fuzzy subalgebra of X since

$$\sup \mathcal{H}(4 * 2) = 0.3 < 0.5 = \min\{\sup \mathcal{H}(4), \sup \mathcal{H}(2)\}.$$

TABLE 7. Tabular representation of $\mathcal{H}$

X	0	1	2	3	4
$\mathcal{H}(x)$	$[0.6, 0.7)$	$(0.3, 0.4)$	$[0.1, 0.6)$	$[0.2, 0.3]$	$(0.1, 0.5]$

If we take $t = 0.1$ and $k = 0.5$, then $U_{\sup}(\mathcal{H}; k - t) = U_{\sup}(\mathcal{H}; 0.4) = \{0, 1, 2, 4\}$ is not a subalgebra of X since $4 * 2 = 3 \notin U_{\sup}(\mathcal{H}; 0.4)$.

Theorem 3.8. *Let $\mathcal{H}$ be a hesitant fuzzy set on a BCK/BCI-algebra X and $t \in [0, \top]$. Then the Sup-hesitant fuzzy t-translation of $\mathcal{H}$ is a Sup-hesitant fuzzy subalgebra of X if and only if $U_{\sup}(\mathcal{H}; k - t)$ is a subalgebra of X for all $k \geq t$.*

Proof. The necessity is clear. Suppose that $U_{\sup}(\mathcal{H}; k - t)$ is a subalgebra of X for all $k \geq t$. Assume that there exist $r \in [0, 1]$ and $a, b \in X$ such that

$$\sup \mathcal{H}_T^t(a * b) < r \leq \min\{\sup \mathcal{H}_T^t(a), \sup \mathcal{H}_T^t(b)\}.$$

Then $\sup \mathcal{H}_T^t(a) \geq r$ and $\sup \mathcal{H}_T^t(b) \geq r$, that is, $\sup \mathcal{H}(a) \geq r - t$ and $\sup \mathcal{H}(b) \geq r - t$. Hence $a, b \in U_{\sup}(\mathcal{H}; r - t)$, and so $a * b \in U_{\sup}(\mathcal{H}; r - t)$. It follows that $\sup \mathcal{H}(a * b) \geq r - t$, i.e., $\sup \mathcal{H}_T^t(a * b) \geq r$, a contradiction. Therefore

$$\sup \mathcal{H}_T^t(x * y) \geq \min\{\sup \mathcal{H}_T^t(x), \sup \mathcal{H}_T^t(y)\}$$

for all $x, y \in X$. Thus the Sup-hesitant fuzzy t-translation of $\mathcal{H}$ is a Sup-hesitant fuzzy subalgebra of X. □

Theorem 3.9. *Let $\mathcal{H}$ be a Sup-hesitant fuzzy subalgebra of a BCK/BCI-algebra X and $t_1, t_2 \in [0, \top]$. If $t_1 \geq t_2$, then the Sup-hesitant fuzzy t_1-translation of $\mathcal{H}$ is a Sup-hesitant fuzzy S-extension of the Sup-hesitant fuzzy t_2-translation of $\mathcal{H}$.*

Proof. Straightforward. □

Theorem 3.10. *Let $\mathcal{H}$ be a Sup-hesitant fuzzy subalgebra of a BCK/BCI-algebra X and $t \in [0, \top]$. For every Sup-hesitant fuzzy S-extension $\mathcal{G}$ of the Sup-hesitant fuzzy t-translation of $\mathcal{H}$, there exists $k \in [0, \top]$ such that $k \geq t$ and $\mathcal{G}$ is a Sup-hesitant fuzzy S-extension of the Sup-hesitant fuzzy k-translation of $\mathcal{H}$.*

Proof. If $\mathcal{H}$ is a Sup-hesitant fuzzy subalgebra of a BCK/BCI-algebra X and $t \in [0, \top]$, then the Sup-hesitant fuzzy t-translation $\mathcal{H}_T^t$ of $\mathcal{H}$ is a Sup-hesitant fuzzy subalgebra of X. If $\mathcal{G}$ is a Sup-hesitant fuzzy S-extension of $\mathcal{H}_T^t$, then there exists $k \in [0, \top]$ such that $k \geq t$ and $\sup \mathcal{G}(x) \geq \sup \mathcal{H}_T^k(x)$ for all $x \in X$. □

Theorem 3.10 is illustrated in the following example.

Example 3.8. Consider a BCK-algebra $X = \{0, a, b, c, d\}$ with the following Cayley table.

$*$	0	a	b	c	d
0	0	0	0	0	0
a	a	0	0	a	a
b	b	b	0	b	b
c	c	c	c	0	c
d	d	d	d	d	0

Let $\mathcal{H}$ be a hesitant fuzzy set on X defined by Table 8.

TABLE 8. Tabular representation of $\mathcal{H}$

X	0	a	b	c	d
$\mathcal{H}(x)$	$[0.2, 0.7]$	$(0.3, 0.4)$	$[0.1, 0.2]$	$[0.3, 0.5]$	$[0, 0.1]$

TABLE 9. Tabular representation of $\mathcal{H}$

X	0	a	b	c	d
$\mathcal{H}^t_T(x)$	$[0.2, 0.9]$	$(0.3, 0.6)$	$[0.1, 0.4]$	$[0.3, 0.7]$	$[0, 0.3]$

Then $\mathcal{H}$ is a Sup-hesitant fuzzy subalgebra of X and $\top = 0.3$. If we take $t = 0.2$, then the Sup-hesitant fuzzy t-translation $\mathcal{H}^t_T$ of $\mathcal{H}$ is given by Table 9.
Let $\mathcal{G}$ be a hesitant fuzzy set on X defined by Table 10.

TABLE 10. Tabular representation of $\mathcal{H}$

X	0	a	b	c	d
$\mathcal{G}(x)$	$[0.2, 0.94]$	$(0.3, 0.63)$	$[0.1, 0.55]$	$[0.3, 0.88]$	$[0, 0.37]$

Then $\mathcal{G}$ is a Sup-hesitant fuzzy subalgebra of X which is Sup-hesitant fuzzy extension of $\mathcal{H}^t_T$, and so $\mathcal{G}$ is a Sup-hesitant fuzzy S-extension of $\mathcal{H}^t_T$. But $\mathcal{G}$ is not a Sup-hesitant fuzzy k-translation of $\mathcal{H}$ for all $k \in [0, \top]$. If we take $k = 0.23$, then the Sup-hesitant fuzzy k-translation $\mathcal{H}^k_T$ of $\mathcal{H}$ is given by Table 11.

TABLE 11. Tabular representation of $\mathcal{H}$

X	0	a	b	c	d
$\mathcal{H}^k_T(x)$	$[0.2, 0.93]$	$(0.3, 0.63)$	$[0.1, 0.43]$	$[0.3, 0.73]$	$[0, 0.33]$

Note that $\mathcal{G}(x) \geq \mathcal{H}^k_T(x)$ for all $x \in X$, and thus $\mathcal{G}$ is a Sup-hesitant fuzzy S-extension of the Sup-hesitant fuzzy k-translation $\mathcal{H}^k_T$ of $\mathcal{H}$.

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