ANNALS OF COMMUNICATIONS IN MATHEMATICS
Volume 1, Number 1 (2018), 65-73
ISSN: 2582-0818
© http://www.technoskypub.com

CUBIC SUBALGEBRAS OF BCH-ALGEBRAS

TAPAN SENAPATI* AND K. P. SHUM

ABSTRACT. In this paper, the notion of cubic subalgebras of BCH-algebras are introduced. Some characterization of cubic subalgebras of BCH-algebras are given. The homomorphic image and inverse image of cubic subalgebras are studied and investigated some related properties.

1. INTRODUCTION

Extending the concept of fuzzy sets (FSs), many scholars introduced various notions of higher-order FSs. Among them, interval-valued fuzzy sets (IVFSs) provides with a flexible mathematical framework to cope with imperfect and imprecise information. Moreover, Jun et al. [14] introduced the concept of cubic sets, as a generalization of fuzzy set and interval-valued fuzzy set. Jun et al. [15] applied the notion of cubic sets to a group, and introduced the notion of cubic subgroups. Also, Muhiuddin et al. applied the notion of cubic sets to BCK/BCI-algebras on different aspects (see for e.g., [17], [18], [19]).

The notions of BCK/BCI-algebras [11] were initiated by Imai and Iseki in 1966 as a generalization of the concept of set-theoretic difference and propositional calculus. It is known that the class of BCK-algebras is a proper subclass of the class of BCI-algebras. Senapati et al. [23, 24, 25, 26, 27, 28] have done lot of works on these algebras. In 1983, Hu and Li [9, 10] introduced the notion of a BCH-algebra, which is a generalization of the notions of BCK and BCI-algebras. They have studied a few properties of these algebras. Certain other properties have been studied by Ahmad [1], Chaudhry [5], Chaudhry et al. [6], Dudek and Thomys [8], Roh et. al. [20, 21] and Dar et al. [7]. Based on (bipolar) fuzzy set Jun et al. [12, 13] introduced fuzzy closed ideals and (bipolar) fuzzy filters in BCH-algebras. Using the algebraic structure of soft sets, Kazanc et al. [16] introduced soft BCH-algebras and some of their properties and structural characteristics are discussed and studied. Borumand Saeid et al. [3, 4] introduced the concept of Smarandache BCH-algebras and fuzzy n-fold ideals in BCH-algebras and investigated some of their useful properties.

The objective of this paper is to introduce the concept of cubic set to subalgebras of BCH-algebras. The notion of cubic subalgebras of BCH-algebras are defined and lot of

2010 *Mathematics Subject Classification.* 06F35, 03G25, 08A72.
Key words and phrases. BCH-algebra; Subalgebra; Cubic set; Cubic subalgebra.
*Corresponding author.

properties are investigated. Section 2 recalls some definitions, viz., BG-algebra, subalgebra and refinement of unit interval. In Section 3, subalgebras of cubic sets are defined with some its properties. In Section 4, homomorphism of cubic subalgebras and some of its properties are studied. In Section 5, a conclusion of the proposed work is given.

2. Preliminaries

An algebra $(X, *, 0)$ of type $(2, 0)$ is called a BCH-algebra [9] if it satisfies the following axioms, for all $x, y, z \in X$

1. $x * x = 0$
2. $x * y = 0$ and $y * x = 0$ imply $x = y$,
3. $(x * y) * z = (x * z) * y$.

Any BCH-algebra X satisfies the following axioms:

(i) $x * 0 = x$,
(ii) $(x * (x * y)) * y = 0$,
(iii) $0 * (x * y) = (0 * x) * (0 * y)$,
(iv) $0 * (0 * (0 * x)) = 0 * x$,
(v) $x \leq y$ implies $0 * x = 0 * y$,

for all $x, y, z \in X$ [6].

A non-empty subset S of a BG-algebra X is called a subalgebra of X if $x * y \in S$, for all $x, y \in S$. A mapping $f : X \to Y$ of BCH-algebras is called a homomorphism if $f(x * y) = f(x) * f(y)$ for all $x, y \in X$. Note that if $f : X \to Y$ is a homomorphism, then $f(0) = 0$.

We now review some fuzzy logic concepts as follows:

Let X be the collection of objects denoted generally by x. Then a fuzzy set [29] A in X is defined as $A = \{< x, \mu_A(x) >: x \in X\}$ where $\mu_A(x)$ is called the membership value of x in A and $0 \leq \mu_A(x) \leq 1$.

An interval-valued fuzzy set [30] A over X is an object having the form $A = \{\langle x, \tilde{\mu}_A(x)\rangle : x \in X\}$, where $\tilde{\mu}_A(x) : X \to D[0, 1]$, where $D[0, 1]$ is the set of all subintervals of $[0, 1]$. The intervals $\tilde{\mu}_A(x)$ denote the intervals of the degree of membership of the element x to the set A, where $\tilde{\mu}_A(x) = [\mu^-_A(x), \mu^+_A(x)]$ for all $x \in X$.

The determination of maximum and minimum between two real numbers is very simple but it is not simple for two intervals. Biswas [2] described a method to find max/sup and min/inf between two intervals or a set of intervals.

Definition 2.1. [2] Consider two elements $D_1, D_2 \in D[0, 1]$. If $D_1 = [a^-_1, a^+_1]$ and $D_2 = [a^-_2, a^+_2]$, then $rmin(D_1, D_2) = [\min(a^-_1, a^-_2), \min(a^+_1, a^+_2)]$ which is denoted by $D_1 \wedge^r D_2$. Thus, if $D_i = [a^-_i, a^+_i] \in D[0, 1]$ for i=1,2,3,4,..., then we define $rsup_i(D_i) = [\sup\limits_i(a^-_i), \sup\limits_i(a^+_i)]$, i.e, $\vee^r_i D_i = [\vee_i a^-_i, \vee_i a^+_i]$. Now we call $D_1 \geq D_2$ iff $a^-_1 \geq a^-_2$ and $a^+_1 \geq a^+_2$. Similarly, the relations $D_1 \leq D_2$ and $D_1 = D_2$ are defined.

Based on the (interval valued) fuzzy sets, Jun et al. [14] introduced the notion of (internal, external) cubic sets, and investigated several properties.

Definition 2.2. [14] Let X be a nonempty set. A cubic set A in X is a structure $A = \{\langle x, \tilde{\mu}_A(x), \nu_A\ (x)\rangle : x \in X\}$ which is briefly denoted by $A = (\tilde{\mu}_A, \nu_A)$ where $\tilde{\mu}_A = [\mu^-_A, \mu^+_A]$ is an IVFS in X and ν_A is a fuzzy set in X.

3. CUBIC SUBALGEBRAS OF BCH-ALGEBRAS

In what follows, let X denote a BCH-algebra unless otherwise specified. Combined the definitions of subalgebras over crisp set and the idea of cubic set, cubic subalgebras of BCH-algebras are defined below.

Definition 3.1. Let $A = (\tilde{\mu}_A, \nu_A)$ be cubic set in X, where X is a subalgebra, then the set A is cubic subalgebra over the binary operator $*$ if it satisfies the following conditions:

(F1) $\tilde{\mu}_A(x * y) \geq rmin\{\tilde{\mu}_A(x), \tilde{\mu}_A(y)\}$

(F2) $\nu_A(x * y) \leq \max\{\nu_A(x), \nu_A(y)\}$

for all $x, y \in X$.

Let us illustrate this definition using the following example.

Example 3.2. Let X=$\{0, a, b, c, d\}$ be a BCH-algebra with the following Cayley table:

$*$	0	a	b	c	d
0	0	0	d	c	b
a	a	0	d	c	b
b	b	b	0	d	c
c	c	c	b	0	d
d	d	d	c	b	0

Define a cubic set $A = (\tilde{\mu}_A, \nu_A)$ in X by

$$\tilde{\mu}_A(x) = \begin{cases} [0.5, 0.7], & \text{if } x = 0 \\ [0.3, 0.6], & \text{if } x = a, c \\ [0.2, 0.4], & \text{if } x = b, d \end{cases} \quad \text{and} \quad \nu_A(\mathrm{x}) = \begin{cases} 0.2, & \text{if } x = 0 \\ 0.5, & \text{if } x = a, c \\ 0.7, & \text{if } x = b, d \end{cases}$$

All the conditions of Definition 3.1 have been satisfy by the set A. Thus $A = (\tilde{\mu}_A, \nu_A)$ is a cubic subalgebra of X.

Proposition 3.1. *If $A = (\tilde{\mu}_A, \nu_A)$ is a cubic subalgebra in X, then for all $x \in X$, $\tilde{\mu}_A(0) \geq \tilde{\mu}_A(x)$ and $\nu_A(0) \leq \nu_A(x)$. Thus, $\tilde{\mu}_A(0)$ and $\nu_A(0)$ are the upper bounds and lower bounds of $\tilde{\mu}_A(x)$ and $\nu_A(x)$ respectively.*

Proof. For all $x \in X$, we have, $\tilde{\mu}_A(0) = \tilde{\mu}_A(x * x) \geq rmin\{\tilde{\mu}_A(x), \tilde{\mu}_A(x)\} = \tilde{\mu}_A(x)$ and $\nu_A(0) = \nu_A(x * x) \leq \max\{\nu_A(x), \nu_A(x)\} = \nu_A(x)$. □

Theorem 3.2. *Let $A = (\tilde{\mu}_A, \nu_A)$ be a cubic subalgebra of X. If there exists a sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} \tilde{\mu}_A(x_n) = [1, 1]$ and $\lim_{n\to\infty} \nu_A(x_n) = 0$. Then $\tilde{\mu}_A(0) = [1, 1]$ and $\nu_A(0) = 0$.*

Proof. By Proposition 3.1, $\tilde{\mu}_A(0) \geq \tilde{\mu}_A(x)$ for all $x \in X$, therefore, $\tilde{\mu}_A(0) \geq \tilde{\mu}_A(x_n)$ for every positive integer n. Consider, $[1, 1] \geq \tilde{\mu}_A(0) \geq \lim_{n\to\infty} \tilde{\mu}_A(x_n) = [1, 1]$. Hence, $\tilde{\mu}_A(0) = [1, 1]$.

Again, by Proposition 3.1, $\nu_A(0) \leq \nu_A(x)$ for all $x \in X$, thus $\nu_A(0) \leq \nu_A(x_n)$ for every positive integer n. Now, $0 \leq \nu_A(0) \leq \lim_{n\to\infty} \nu_A(x_n) = 0$. Hence, $\nu_A(0) = 0$. □

Proposition 3.3. *If a cubic set $A = (\tilde{\mu}_A, \nu_A)$ in X is a cubic subalgebra, then for all $x \in X$, $\tilde{\mu}_A(0 * x) \geq \tilde{\mu}_A(x)$ and $\nu_A(0 * x) \leq \nu_A(x)$.*

Proof. For all $x \in X$, $\tilde{\mu}_A(0 * x) \geq rmin\{\tilde{\mu}_A(0), \tilde{\mu}_A(x)\} = rmin\{\tilde{\mu}_A(x * x), \tilde{\mu}_A(x)\} \geq rmin\ \{rmin\{\tilde{\mu}_A(x), \tilde{\mu}_A(x)\}, \tilde{\mu}_A(x)\} = \tilde{\mu}_A(x)$ and $\nu_A(0 * x) \leq \max\{\nu_A(0), \nu_A(x)\} = \nu_A(x)$. □

$\tilde{\mu}_A(x) = [\mu_A^-(x), \mu_A^+(x)]$ represents the membership value and $\nu_A(x)$ represents the non-membership value of x in cubic set A. But $\mu_A^-(x)$, $\mu_A^+(x)$ are the membership value and $\nu_A(x)$ are the non-membership value of x to some fuzzy sets. These values can also form a fuzzy subalgebra, proved in the following theorem.

Theorem 3.4. *A cubic set $A = (\tilde{\mu}_A, \nu_A)$ in X is a cubic subalgebra of X iff μ_A^-, μ_A^+ and ν_A are fuzzy subalgebras of X.*

Proof. Let μ_A^-, μ_A^+ and ν_A be fuzzy subalgebras of X and $x, y \in X$. Then $\mu_A^-(x * y) \geq \min\{\mu_A^-(x), \mu_A^-(y)\}$, $\mu_A^+(x * y) \geq \min\{\mu_A^+(x), \mu_A^+(y)\}$ and $\nu_A(x * y) \leq \max\{\nu_A(x), \nu_A(y)\}$. Now, $\tilde{\mu}_A(x * y) = [\mu_A^-(x * y), \mu_A^+(x * y)] \geq [min\{\mu_A^-(x), \mu_A^-(y)\}, min\{\mu_A^+(x), \mu_A^+(y)\}] = rmin\{[\mu_A^-(x), \mu_A^+(x)], [\mu_A^-(y), \mu_A^+(y)]\} = rmin\{\tilde{\mu}_A(x), \tilde{\mu}_A(y)\}$. Therefore, A is a cubic subalgebra of X.

Conversely, assume that, $A = (\tilde{\mu}_A, \nu_A)$ is a cubic subalgebra of X. For any $x, y \in X$, $[\mu_A^-(x * y), \mu_A^+(x * y)] = \tilde{\mu}_A(x * y) \geq rmin\{\tilde{\mu}_A(x), \tilde{\mu}_A(y)\} = rmin\{[\mu_A^-(x), \mu_A^+(x)], [\mu_A^-(y), \mu_A^+(y)] = [\min\{\mu_A^-(x), \mu_A^-(y)\}, \min\{\mu_A^+(x), \mu_A^+(y)\}]$. Thus $\mu_A^-(x * y) \geq \min\{\mu_A^-(x), \mu_A^-(y)\}$, $\mu_A^+(x*y) \geq \min\{\mu_A^+(x), \mu_A^+(y)\}$ and $\nu_A(x*y) \leq \max\{\nu_A(x), \nu_A(y)\}$. Hence, μ_A^-, μ_A^+ and ν_A are fuzzy subalgebras of X. $\square$

Theorem 3.5. *Let $A = (\tilde{\mu}_A, \nu_A)$ be a cubic subalgebra of X and let $n \in \mathbb{N}$ (the set of natural numbers). Then*

(i) $\tilde{\mu}_A(\prod^n x * x) \geq \tilde{\mu}_A(x)$, for any odd number n,
(ii) $\nu_A(\prod^n x * x) \leq \nu_A(x)$, for any odd number n,
(iii) $\tilde{\mu}_A(\prod^n x * x) = \tilde{\mu}_A(x)$, for any even number n,
(iv) $\nu_A(\prod^n x * x) = \nu_A(x)$, for any even number n.

Proof. Let $x \in X$ and assume that n is odd. Then $n = 2p - 1$ for some positive integer p. We prove the theorem by induction.

Now $\tilde{\mu}_A(x * x) = \tilde{\mu}_A(0) \geq \tilde{\mu}_A(x)$ and $\nu_A(x * x) = \nu_A(0) \leq \nu_A(x)$. Suppose that $\tilde{\mu}_A(\prod^{2p-1} x*x) \geq \tilde{\mu}_A(x)$ and $\nu_A(\prod^{2p-1} x*x) \leq \nu_A(x)$. Then by assumption, $\tilde{\mu}_A(\prod^{2(p+1)-1} x* x) = \tilde{\mu}_A(\prod^{2p+1} x * x) = \tilde{\mu}_A(\prod^{2p-1} x * (x * (x * x))) = \tilde{\mu}_A(\prod^{2p-1} x * x) \geq \tilde{\mu}_A(x)$ and $\nu_A(\prod^{2(p+1)-1} x*x) = \nu_A(\prod^{2p+1} x*x) = \nu_A(\prod^{2p-1} x*(x*(x*x))) = \nu_A(\prod^{2p-1} x*x) \leq \nu_A(x)$, which proves (i) and (ii). Proofs are similar for the cases (iii) and (iv). $\square$

The sets $\{x \in X : \tilde{\mu}_A(x) = \tilde{\mu}_A(0)\}$ and $\{x \in X : \nu_A(x) = \nu_A(0)\}$ are denoted by $I_{\tilde{\mu}_A}$ and I_{ν_A} respectively. These two sets are also subalgebra of X.

Theorem 3.6. *Let $A = (\tilde{\mu}_A, \nu_A)$ be a cubic subalgebra of X, then the sets $I_{\tilde{\mu}_A}$ and I_{ν_A} are subalgebras of X.*

Proof. Let $x, y \in I_{\tilde{\mu}_A}$. Then $\tilde{\mu}_A(x) = \tilde{\mu}_A(0) = \tilde{\mu}_A(y)$ and so, $\tilde{\mu}_A(x*y) \geq rmin\{\tilde{\mu}_A(x), \tilde{\mu}_A(y)\} = \tilde{\mu}_A(0)$. By using Proposition 3.1, we know that $\tilde{\mu}_A(x * y) = \tilde{\mu}_A(0)$ or equivalently $x * y \in I_{\tilde{\mu}_A}$.

Again, let $x, y \in I_{\nu_A}$. Then $\nu_A(x) = \nu_A(0) = \nu_A(y)$ and so, $\nu_A(x*y) \leq \max\{\nu_A(x), \nu_A(y)\} = \nu_A(0)$. Again, by Proposition 3.1, we know that $\nu_A(x * y) = \nu_A(0)$ or equivalently $x * y \in I_{\nu_A}$. Hence, the sets $I_{\tilde{\mu}_A}$ and I_{ν_A} are subalgebras of X. $\square$

Theorem 3.7. *Let B be a nonempty subset of X and $A = (\tilde{\mu}_A, \nu_A)$ be cubic set in X defined by* $\tilde{\mu}_A(x) = \begin{cases} [\alpha_1, \alpha_2], & \text{if } x \in B \\ [\beta_1, \beta_2], & \text{otherwise} \end{cases}$ *and* $\nu_A(x) = \begin{cases} \gamma, & \text{if } x \in B \\ \delta, & \text{otherwise} \end{cases}$ *for all $[\alpha_1, \alpha_2], [\beta_1, \beta_2] \in D[0,1]$ and γ, $\delta \in [0,1]$ with $[\alpha_1, \alpha_2] \geq [\beta_1, \beta_2]$ and $\gamma \leq \delta$. Then A is a cubic subalgebra of X if and only if B is a subalgebra of X. Moreover, $I_{\tilde{\mu}_A} = B = I_{\nu_A}$.*

Proof. Let A be a cubic subalgebra of X. Let $x, y \in X$ be such that $x, y \in B$. Then $\tilde{\mu}_A(x * y) \geq rmin\{\tilde{\mu}_A(x), \tilde{\mu}_A(y)\} = rmin\{[\alpha_1, \alpha_2], [\alpha_1, \alpha_2]\} = [\alpha_1, \alpha_2]$ and $\nu_A(x * y) \leq \max\{\nu_A(x), \nu_A(y)\} = \max\{\gamma, \gamma\} = \gamma$. So $x * y \in B$. Hence, B is a subalgebra of X.

Conversely, suppose that B is a subalgebra of X. Let $x, y \in X$. Consider two cases
Case (i) If $x, y \in B$ then $x * y \in B$, thus $\tilde{\mu}_A(x * y) = [\alpha_1, \alpha_2] = rmin\{\tilde{\mu}_A(x), \tilde{\mu}_A(y)\}$ and $\nu_A(x * y) = \gamma = \max\{\nu_A(x), \nu_A(y)\}$.
Case (ii) If $x \notin B$ or, $y \notin B$, then $\tilde{\mu}_A(x * y) \geq [\beta_1, \beta_2] = rmin\{\tilde{\mu}_A(x), \tilde{\mu}_A(y)\}$ and $\nu_A(x * y) \leq \delta = \max\{\nu_A(x), \nu_A(y)\}$.

Hence, A is a cubic subalgebra of X.

Now, $I_{\tilde{\mu}_A} = \{x \in X, \tilde{\mu}_A(x) = \tilde{\mu}_A(0)\} = \{x \in X, \tilde{\mu}_A(x) = [\alpha_1, \alpha_2]\} = B$ and $I_{\nu_A} = \{x \in X, \nu_A(x) = \nu_A(0)\} = \{x \in X, \nu_A(x) = \gamma\} = B$. □

Definition 3.3. Let $A = (\tilde{\mu}_A, \nu_A)$ be a cubic set in X. For $[s_1, s_2] \in D[0,1]$ and $t \in [0,1]$, the set $U(\tilde{\mu}_A : [s_1, s_2]) = \{x \in X : \tilde{\mu}_A(x) \geq [s_1, s_2]\}$ is called upper $[s_1, s_2]$-level of A and $L(\nu_A : t) = \{x \in X : \nu_A(x) \leq t\}$ is called lower t-level of A.

Theorem 3.8. *If $A = (\tilde{\mu}_A, \nu_A)$ is a cubic subalgebra of X, then the upper $[s_1, s_2]$-level and lower t-level of A are subalgebras of X.*

Proof. Let $x, y \in U(\tilde{\mu}_A : [s_1, s_2])$. Then $\tilde{\mu}_A(x) \geq [s_1, s_2]$ and $\tilde{\mu}_A(y) \geq [s_1, s_2]$. It follows that $\tilde{\mu}_A(x*y) \geq rmin\{\tilde{\mu}_A(x), \tilde{\mu}_A(y)\} \geq [s_1, s_2]$ so that $x*y \in U(\tilde{\mu}_A : [s_1, s_2])$. Hence, $U(\tilde{\mu}_A : [s_1, s_2])$ is a subalgebra of X.

Let $x, y \in L(\nu_A : t)$. Then $\nu_A(x) \leq t$ and $\nu_A(y) \leq t$. It follows that $\nu_A(x * y) \leq \max\{\nu_A(x), \nu_A(y)\} \leq t$ so that $x * y \in L(\nu_A : t)$. Hence, $L(\nu_A : t)$ is a subalgebra of X. □

Theorem 3.9. *Let $A = (\tilde{\mu}_A, \nu_A)$ be a cubic set in X, such that the sets $U(\tilde{\mu}_A : [s_1, s_2])$ and $L(\nu_A : t)$ are subalgebras of X for every $[s_1, s_2] \in D[0,1]$ and $t \in [0,1]$. Then $A = (\tilde{\mu}_A, \nu_A)$ is a cubic subalgebra of X.*

Proof. Let for every $[s_1, s_2] \in D[0,1]$ and $t \in [0,1]$, $U(\tilde{\mu}_A : [s_1, s_2])$ and $L(\nu_A : t)$ are subalgebras of X. In contrary, let $x_0, y_0 \in X$ be such that $\tilde{\mu}_A(x_0 * y_0) < rmin\{\tilde{\mu}_A(x_0), \tilde{\mu}_A(y_0)\}$. Let $\tilde{\mu}_A(x_0) = [\theta_1, \theta_2]$, $\tilde{\mu}_A(y_0) = [\theta_3, \theta_4]$ and $\tilde{\mu}_A(x_0 * y_0) = [s_1, s_2]$. Then $[s_1, s_2] < rmin\{[\theta_1, \theta_2], [\theta_3, \theta_4]\} = [\min\{\theta_1, \theta_3\}, \min\{\theta_2, \theta_4\}]$. So, $s_1 < \min\{\theta_1, \theta_3\}$ and $s_2 < \min\{\theta_2, \theta_4\}$. Let us consider, $[\rho_1, \rho_2] = \frac{1}{2}\Big[\tilde{\mu}_A(x_0 * y_0) + rmin\{\tilde{\mu}_A(x_0), \tilde{\mu}_A(y_0)\}\Big] = \frac{1}{2}\Big[[s_1, s_2] + [\min\{\theta_1, \theta_3\}, \min\{\theta_2, \theta_4\}]\Big] = \Big[\frac{1}{2}(s_1 + \min\{\theta_1, \theta_3\}), \frac{1}{2}(s_2 + \min\{\theta_2, \theta_4\})\Big]$. Therefore, $\min\{\theta_1, \theta_3\} > \rho_1 = \frac{1}{2}(s_1 + \min\{\theta_1, \theta_3\}) > s_1$ and $\min\{\theta_2, \theta_4\} > \rho_2 = \frac{1}{2}(s_2 + \min\{\theta_2, \theta_4\}) > s_2$. Hence, $[\min\{\theta_1, \theta_3\}, \min\{\theta_2, \theta_4\}] > [\rho_1, \rho_2] > [s_1, s_2]$, so that $x_0 * y_0 \notin U(\tilde{\mu}_A : [s_1, s_2])$ which is a contradiction, since $\tilde{\mu}_A(x_0) = [\theta_1, \theta_2] \geq [\min\{\theta_1, \theta_3\}, \min\{\theta_2, \theta_4\}] > [\rho_1, \rho_2]$ and $\tilde{\mu}_A(y_0) = [\theta_3, \theta_4] \geq [\min\{\theta_1, \theta_3\}, \min\{\theta_2, \theta_4\}] > [\rho_1, \rho_2]$. This implies $x_0 * y_0 \in U(\tilde{\mu}_A : [s_1, s_2])$. Thus $\tilde{\mu}_A(x * y) \geq rmin\{\tilde{\mu}_A(x), \tilde{\mu}_A(y)\}$ for all $x, y \in X$.

Again, let $x_0, y_0 \in X$ be such that $\nu_A(x_0*y_0) > max\{\nu_A(x_0), \nu_A(y_0)\}$. Let $\nu_A(x_0) = \eta_1$, $\nu_A(y_0) = \eta_2$ and $\nu_A(x_0 * y_0) = t$. Then $t > max\{\eta_1, \eta_2\}$. Let us consider, $t_1 = \frac{1}{2}[\nu_A(x_0*y_0)+max\{\nu_A(x_0), \nu_A(y_0)\}]$. We get that $t_1 = \frac{1}{2}(t+\max\{\eta_1, \eta_2\})$. Therefore, $\eta_1 < t_1 = \frac{1}{2}(t + \max\{\eta_1, \eta_2\}) < t$ and $\eta_2 < t_1 = \frac{1}{2}(t + \max\{\eta_1, \eta_2\}) < t$. Hence, $\max\{\eta_1, \eta_2\} < t_1 < t = \nu_A(x_0*y_0)$, so that $x_0*y_0 \notin L(\nu_A : t)$ which is a contradiction, since $\nu_A(x_0) = \eta_1 \leq \max\{\eta_1, \eta_2\} < t_1$ and $\nu_A(y_0) = \eta_2 \leq \max\{\eta_1, \eta_2\} < t_1$. This implies $x_0, y_0 \in L(\nu_A : t)$. Thus $\nu_A(x * y) \leq max\{\nu_A(x), \nu_A(y)\}$ for all $x, y \in X$. □

Theorem 3.10. *Any subalgebra of X can be realized as both the upper $[s_1, s_2]$-level and lower t-level of some cubic subalgebra of X.*

Proof. Let P be a cubic subalgebra of X, and A be cubic set on X defined by

$$\tilde{\mu}_A(x) = \begin{cases} [\alpha_1, \alpha_2], & \text{if } x \in P \\ [0, 0], & \text{otherwise} \end{cases} \quad \text{and} \quad \nu_\mathrm{A}(\mathrm{x}) = \begin{cases} \beta, & \text{if } x \in P \\ 1, & \text{otherwise} \end{cases}$$

for all $[\alpha_1, \alpha_2] \in D[0, 1]$ and $\beta \in [0, 1]$. We consider the following cases:
Case (i) If $x, y \in P$, then $\tilde{\mu}_A(x) = [\alpha_1, \alpha_2]$, $\nu_A(x) = \beta$ and $\tilde{\mu}_A(y) = [\alpha_1, \alpha_2]$, $\nu_A(y) = \beta$. Thus, $\tilde{\mu}_A(x * y) = [\alpha_1, \alpha_2] = rmin\{[\alpha_1, \alpha_2], [\alpha_1, \alpha_2]\} = rmin\{\tilde{\mu}_A(x), \tilde{\mu}_A(y)\}$ and $\nu_A(x * y) = \beta = \max\{\beta, \beta\} = \max\{\nu_A(x), \nu_A(y)\}$.
Case (ii) If $x \in P$ and $y \notin P$ then $\tilde{\mu}_A(x) = [\alpha_1, \alpha_2]$, $\nu_A(x) = \beta$ and $\tilde{\mu}_A(y) = [0, 0]$, $\nu_A(y) = 1$. Thus, $\tilde{\mu}_A(x * y) \geq [0, 0] = rmin\{[\alpha_1, \alpha_2], [0, 0]\} = rmin\{\tilde{\mu}_A(x), \tilde{\mu}_A(y)\}$ and $\nu_A(x * y) \leq 1 = \max\{\beta, 1\} = \max\{\nu_A(x), \nu_A(y)\}$.
Case (iii) If $x \notin P$ and $y \in P$ then $\tilde{\mu}_A(x) = [0, 0]$, $\nu_A(x) = 1$ and $\tilde{\mu}_A(y) = [\alpha_1, \alpha_2]$, $\nu_A(y) = \beta$. Thus, $\tilde{\mu}_A(x * y) \geq [0, 0] = rmin\{[0, 0], [\alpha_1, \alpha_2]\} = rmin\{\tilde{\mu}_A(x), \tilde{\mu}_A(y)\}$ and $\nu_A(x * y) \leq 1 = \max\{1, \beta\} = \max\{\nu_A(x), \nu_A(y)\}$.
Case (iv): If $x \notin P$ and $y \notin P$ then $\tilde{\mu}_A(x) = [0, 0]$, $\nu_A(x) = 1$ and $\tilde{\mu}_A(y) = [0, 0]$, $\nu_A(y) = 1$. Now $\tilde{\mu}_A(x * y) \geq [0, 0] = rmin\{[0, 0], [0, 0]\} = rmin\{\tilde{\mu}_A(x), \tilde{\mu}_A(y)\}$ and $\nu_A(x * y) \leq 1 = \max\{1, 1\} = \max\{\nu_A(x), \nu_A(y)\}$.

Therefore, A is a cubic subalgebra of X. □

Theorem 3.11. *Let P be a subset of X and A be cubic set on X which is given in the proof of Theorem 3.10. If A be realized as lower level subalgebra and upper level subalgebra of some cubic subalgebra of X, then P is a cubic subalgebra of X.*

Proof. Let A be a cubic subalgebra of X, and $x, y \in P$. Then $\tilde{\mu}_A(x) = [\alpha_1, \alpha_2] = \tilde{\mu}_A(y)$ and $\nu_A(x) = \beta = \nu_A(y)$. Thus $\tilde{\mu}_A(x*y) \geq rmin\{\tilde{\mu}_A(x), \tilde{\mu}_A(y)\} = rmin\{[\alpha_1, \alpha_2], [\alpha_1, \alpha_2]\} = [\alpha_1, \alpha_2]$ and $\nu_A(x * y) \leq \max\{\nu_A(x), \nu_A(y)\} = \max\{\beta, \beta\} = \beta$, which imply that $x * y \in P$. Hence, the theorem. □

4. HOMOMORPHISM OF CUBIC SUBALGEBRAS

In this section, homomorphism of cubic subalgebra is defined and some results are studied.

Let f be a mapping from a set X into a set Y. Let $B = (\tilde{\mu}_B, \nu_B)$ be cubic set in Y. Then the inverse image of B, is defined as $f^{-1}(B) = \{\langle x, f^{-1}(\tilde{\mu}_B), f^{-1}(\nu_B)\rangle : x \in X\}$ with the membership function and non-membership function respectively are given by $f^{-1}(\tilde{\mu}_B)(x) = \tilde{\mu}_B(f(x))$ and $f^{-1}(\nu_B)(x) = \nu_B(f(x))$. It can be shown that $f^{-1}(B)$ is cubic set.

Theorem 4.1. *Let $f : X \to Y$ be a homomorphism of BCH-algebras. If $B = (\tilde{\mu}_B, \nu_B)$ is a cubic subalgebra of Y, then the preimage $f^{-1}(B) = \{\langle x, f^{-1}(\tilde{\mu}_B), f^{-1}(\nu_B)\rangle : x \in X\}$ of B under f is a cubic subalgebra of X.*

Proof. Assume that $B = (\tilde{\mu}_B, \nu_B)$ is a cubic subalgebra of Y and let $x, y \in X$. Then $f^{-1}(\tilde{\mu}_B)(x*y) = \tilde{\mu}_B(f(x*y)) = \tilde{\mu}_B(f(x)*f(y)) \geq rmin\{\tilde{\mu}_B(f(x), \tilde{\mu}_B(f(y))\} = rmin\{f^{-1}(\tilde{\mu}_B)\ (x), f^{-1}(\tilde{\mu}_B)(y)\}$ and $f^{-1}(\nu_B)(x*y) = \nu_B(f(x*y)) = \nu_B(f(x)*f(y)) \leq \max\{\nu_B(f(x), \nu_B(f(y))\} = \max\{f^{-1}(\nu_B)(x), f^{-1}(\nu_B)(y)\}$. Therefore, $f^{-1}(B) = \{\langle x, f^{-1}(\tilde{\mu}_B), f^{-1}(\nu_B)\rangle : x \in X\}$ is a cubic subalgebra of X. □

Definition 4.1. A cubic set A in the BCH-algebra X is said to have the rsup-property and inf-property if for any subset T of X there exist $t_0 \in T$ such that $\tilde{\mu}_A(t_0) = rsup_{t_0\in T}\tilde{\mu}_A(t)$ and $\nu_A(t_0) = \inf\limits_{t_0\in T} \nu_A(t)$ respectively.

Definition 4.2. Let f be a mapping from the set X to the set Y. If $A = (\tilde{\mu}_A, \nu_A)$ is cubic set in X, then the image of A under f, denoted by $f(A)$, and is defined as $f(A) = \{\langle x, f_{rsup}(\tilde{\mu}_A), f_{\inf}(\nu_A)\rangle : x \in Y\}$, where

$$f_{rsup}(\tilde{\mu}_A)(y) = \begin{cases} rsup_{x\in f^{-1}(y)}\tilde{\mu}_A(x), & \text{if } f^{-1}(y) \neq \phi \\ [0,0], & \text{otherwise} \end{cases}$$

and

$$f_{\inf}(\nu_A)(y) = \begin{cases} \inf\limits_{x\in f^{-1}(y)} \nu_A(x), & \text{if } f^{-1}(y) \neq \phi \\ 1, & \text{otherwise.} \end{cases}$$

Theorem 4.2. *Let $f : X \to Y$ be a homomorphism from a BCH-algebra X onto a BCH-algebra Y. If $A = (\tilde{\mu}_A, \nu_A)$ is a cubic subalgebra of X, then the image $f(A) = \{\langle x, f_{rsup}(\tilde{\mu}_A), f_{\inf}(\nu_A)\rangle : x \in Y\}$ of A under f is a cubic subalgebra of Y.*

Proof. Let $A = (\tilde{\mu}_A, \nu_A)$ be a cubic subalgebra of X and let $y_1, y_2 \in Y$. We know that, $\{x_1 * x_2 : x_1 \in f^{-1}(y_1)$ and $x_2 \in f^{-1}(y_2)\} \subseteq \{x \in X : x \in f^{-1}(y_1 * y_2)\}$. Now,

$$\begin{aligned} f_{rsup}(\tilde{\mu}_A)(y_1*y_2) &= rsup\{\tilde{\mu}_A(x) : x \in f^{-1}(y_1*y_2)\} \\ &\geq rsup\{\tilde{\mu}_A(x_1*x_2) : x_1 \in f^{-1}(y_1)\ \ and\ \ x_2 \in f^{-1}(y_2)\} \\ &\geq rsup\{rmin\{\tilde{\mu}_A(x_1), \tilde{\mu}_A(x_2)\} : x_1 \in f^{-1}(y_1), x_2 \in f^{-1}(y_2)\} \\ &= rmin\{rsup\{\tilde{\mu}_A(x_1) : x_1 \in f^{-1}(y_1)\}, \\ &= \qquad rsup\{\tilde{\mu}_A(x_2) : x_2 \in f^{-1}(y_2)\}\} \\ &= rmin\{f_{rsup}(\tilde{\mu}_A)(y_1), f_{rsup}(\tilde{\mu}_A)(y_2)\} \end{aligned}$$

and

$$\begin{aligned} f_{\inf}(\nu_A)(y_1*y_2) &= \inf\{\nu_A(x) : x \in f^{-1}(y_1*y_2)\} \\ &\leq \inf\{\nu_A(x_1*x_2) : x_1 \in f^{-1}(y_1)\ \ and\ \ x_2 \in f^{-1}(y_2)\} \\ &\leq \inf\{\max\{\nu_A(x_1), \nu_A(x_2)\} : x_1 \in f^{-1}(y_1), x_2 \in f^{-1}(y_2)\} \\ &= \max\{\inf\{\nu_A(x_1) : x_1 \in f^{-1}(y_1)\}, \\ &= \qquad \inf\{\nu_A(x_2) : x_2 \in f^{-1}(y_2)\}\} \\ &= \max\{f_{\inf}(\nu_A)(y_1), f_{\inf}(\nu_A)(y_2)\}. \end{aligned}$$

Hence, $f(A) = \{\langle x, f_{rsup}(\tilde{\mu}_A), f_{\inf}(\nu_A)\rangle : x \in Y\}$ is a cubic subalgebra of Y. □

5. CONCLUSIONS

To investigate the structure of an algebraic system, it is clear that subalgebras with special properties play an important role. In the present paper, we considered the notions of cubic subalgebras of BCH-algebras and investigated some of their useful properties. The homomorphism of subalgebras has been introduced and some important properties are of it are also studied. It is our hope that this work would other foundations for further study of the theory of BCH-algebras.

6. ACKNOWLEDGEMENTS

The authors would like to express their sincere thanks to the anonymous referees for their valuable suggestion and comments and that improved this paper.

REFERENCES

[1] B. Ahmad. On classification of BCH-algebras, Math. Japon., 35 (1990), 801804.
[2] R. Biswas. Rosenfeld's fuzzy subgroups with interval valued membership function, Fuzzy Sets and Systems, 63(1) (1994), 87-90.
[3] A. Borumand Saeid, and A. Namdar. Smarandache BCH-algebras, World Applied Sciences Journal, 7 (2009), 77-83.
[4] A. Borumand Saeid, A. Namdar, and M. K. Rafsanjani. Fuzzy n-fold ideals in BCH-algebras, Neural Computing and Applications, 19 (2010), 775-783.
[5] M. A. Chaudhry. On BCH-algebras, Math. Japon., 36(4) (1991), 665-676.
[6] M. A. Chaudhry, and H. Fakhar-Ud-Din. On some classes of BCH-algebras, Int. J. Math. Math. Sci., 25(3) (2001), 205-211.
[7] K. H. Dar, and M. Akram. On endomorphisms of BCH-algebras, An. Univ. Craiova Ser. Mat. Inform., 33 (2006), 227-234.
[8] W. A. Dudek, and J. Thomys. On decompositions of BCH-algebras, Math. Japon., 35(6) (1990), 1131-1138.
[9] Q. P. Hu, and X. Li. On BCH-algebras, Math. Sem. Notes, 11(2) (1983), 313-320.
[10] Q. P. Hu, and X. Li. On proper BCH-algebras, Math. Japon., 30(4) (1985), 659-661.
[11] Y. Huang. BCI-algebra, Science Press, Beijing, 2006.
[12] Y. B. Jun. Fuzzy closed ideals and fuzzy filters in BCH-algebras, J. Fuzzy Math., 7(2) (1999), 435-444.
[13] Y. B. Jun, and C. H. Park. Filters of BCH-algebras based on bipolar-valued fuzzy sets, International Mathematical Forum, 4(13) (2009), 631-643.
[14] Y. B. Jun, C. S. Kim, and K. O. Yang. Cubic sets, Ann. Fuzzy Math. Inform., 4(1) (2012), 83-98.
[15] Y. B. Jun, S. T. Jung, and M. S. Kim. Cubic subgroups, Ann. Fuzzy Math. Inform., 2(1) (2011), 915.
[16] O. Kazanc, S. Ylmaz, and S. Yamak. Soft sets and soft BCH-algebras, Hacet. J. Math. Stat., 39(2) (2010), 205-217.
[17] G. Muhiuddin, Sun Shin Ahn, Chang Su Kim and Young Bae Jun. Stable Cubic Sets, Journal of Computational Analysis and Applications, 23(5) (2017), 802–819 .
[18] G. Muhiuddin, Feng Feng and Young Bae Jun. Subalgebras of BCK/BCI-Algebras Based on Cubic Soft Sets, The Scientific World Journal, Volume 2014, Article ID 458638, 9 pages (2014).
[19] G. Muhiuddin and Abdullah M. Al-roqi. Cubic soft sets with applications in BCK/BCI-algebras, Annals of Fuzzy Mathematics and Informatics, 8(2) (2014), 291–304.
[20] E. H. Roh, S. Y. Kim, and Y. B. Jun. On a problem in BCH-algebras, Math. Japon., 52 (2000), 279-283.
[21] E. H. Roh, Y. B. Jun, and Q. Zhang. Special subset in BCH-algebras, Far East J. Math. Sci., 3 (2001), 723-729.
[22] A. Rosenfeld. Fuzzy Groups, J. Math. Anal. Appl., 35 (1971), 512-517.
[23] T. Senapati, M. Bhowmik, and M. Pal. Atanassov's intuitionistic fuzzy translations of intuitionistic fuzzy H-ideals in BCK/BCI-algebras, Notes on Intuitionistic Fuzzy Sets, 19(1) (2013), 32-47.
[24] T. Senapati, M. Bhowmik, M. Pal, and B. Davvaz. Atanassov's intuitionistic fuzzy translations of intuitionistic fuzzy subalgebras and ideals in BCK/BCI-algebras, Eurasian Math. J., 6(1) (2015), 96-114.
[25] T. Senapati, and K.P. Shum, Cubic commutative ideals of BCK-algebras, Missouri J. Math. Sci., 30(1) (2018), 5-19.

[26] T. Senapati, and K.P. Shum, Cubic implicative ideals of BCK-algebras, Missouri J. Math. Sci., **29**(2) (2017), 125-138.

[27] T. Senapati, M. Bhowmik, and M. Pal. Fuzzy dot subalgebras and fuzzy dot ideals of B-algebras, Journal of Uncertain Systems, 8(1) (2014), 22-30.

[28] T. Senapati, Y.B. Jun, G. Muhiuddin and K. P. Shum. Cubic intuitionistic structures applied to ideals of BCI-algebras, Analele Univ. Ovidius, Constanta, Ser. Mat., (2019) (In press).

[29] L. A. Zadeh. Fuzzy sets, Information and Control, 8 (1965), 338-353.

[30] L. A. Zadeh. The concept of a linguistic variable and its application to approximate reasoning. I, Inform. Sci., 8 (1975), 199-249.

TAPAN SENAPATI
DEPARTMENT OF APPLIED MATHEMATICS WITH OCEANOLOGY AND COMPUTER PROGRAMMING, VIDYASAGAR UNIVERSITY, MIDNAPORE 721102, INDIA
Email address: math.tapan@gmail.com

K. P. SHUM
INSTITUTE OF MATHEMATICS, YUNNAN UNIVERSITY, KUNMING 650091, PEOPLE'S REPUBLIC OF CHINA
Email address: kpshum@ynu.edu.cn