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THE FAREY MAP EXPLOITED IN THE CONSTRUCTION OF A FAREY MOTHER WAVELET

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ABSTRACT. The wavelet analysis of a function passes through its so-called wavelet transform. Such a transform is mathematically defined as a convolution product of the analyzed function with another analyzing function known as the mother wavelet by involving the scale and the translation parameters. This means that the mother wavelet construction is the starting and major point in the wavelet analysis. Besides, the choice of the mother wavelet remains a major problem in wavelet applications such as statistical series, time series, signal, and image processing. This needs more candidates of mother wavelets to be constructed. The main aim of the present paper is to construct a new mother wavelet by exploiting the well-known Farey map. We showed indeed that such a map may be a mother wavelet owing properties such as admissibility, moments, 2-scale relation, and reconstruction rule already necessary in the wavelet analysis of functions. By a suitable choice of translation-dilation parameters on the original Farey map, we succeeded to prove the main properties of a Farey wavelet analysis. The constructed mother looks to be suitable for many complicated applications such as hyperbolic PDEs.

1. INTRODUCTION

Wavelet analysis was introduced in the 80th decade of the last century in petroleum exploration as a refinement of Fourier analysis which failed in extracting the characteristics of signals subjects of the exploration ([3, 4, 17]). Since then, wavelets have attracted the interest of researchers in both pure and applied fields ([2, 5, 8, 10].

Indeed, wavelets have been shown powerful tools since their discovery in the context of petroleum extraction at the last century. They have been next involved in quasi-all scientific fields ([3, 4, 13, 14]), and also in social and actuarial sciences especially the last decade ([1, 6, 7, 16]).

Wavelets act on signals, images, functions, and time series in an analog way as Fourier analysis by the so-called wavelet transform. A type of transform based on a 'product' of the analyzed object with copies of a source function called the mother/father wavelet, which plays the role of the Fourier sine/cosine.

Wavelets have been also developed independently in the fields of mathematics, quantum physics, electrical engineering, and seismic geology. Next, interchanges between these

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fields have yielded more understanding of their theory and more and more bases as well as applications such as image compression, turbulence, human vision, radar, earthquake, and so on.

Wavelets have been also extended in many new frameworks such as the quantum theory ([9]), analysis on the sphere and manifolds ([3]), statistics ([12, 13, 18]), economics and finance ([1, 12, 18]), and also biosciences ([4, 15, 20]).

For a large community, of non-mathematicians, a wavelet may be defined as a wave function that decays rapidly and which has a zero mean. Wavelet analysis consists of breaking up a signal into parts relatively to approximating functions obtained as shifted and dilated versions of the wavelet [11, 19].

In Fourier analysis, the description of signals is restricted to the global behavior and can not provide information about hidden details due to its lack of time-frequency and/or time-space localization. In 1940, Denis Gabor introduced the so-called windowed Fourier transform (WFT) to overcome this lack. The Fourier modes used in Fourier transform are multiplied by suitable functions localized in time such as Gaussian window. This step permitted to understand new situations. But the situation has changed again with the discovery and/or the emergence of new problems related to irregular variations such as gravitational waves where glitches (which are bursts of noise) remain after filtering. This leads researchers to think about more sophisticated tools for signal processing and leads next to the discovery and development of wavelet analysis. Wavelets thus permit the localization of analyzed signals in both time and frequency. Contrarily to Fourier analysis, wavelets permit also us to analyze nonstationarity, non-seasonality, and irregularity with more precision.

The main common and essential point between all these frameworks is that any wavelet analysis starts with a source function called the mother wavelet, which gives rise next to the wavelet basis, and the wavelet transform. In the present work, we propose to provide a rigorous development of a new wavelet mother by exploiting the characteristics of the well-known Farey map. The remaining parts of the present document will be organized as follows. Section 2 is devoted to the review of wavelet analysis. Section 3 is concerned with the development of our main results dealing with a new type of wavelet function based on the well-known Farey map which we call Farey wavelet. Special characteristics such as admissibility, moments, and reconstruction rules have been established. Section 4 is a conclusion.

2. Wavelets revisited

A wavelet is a wave function that decays rapidly and has a zero average value. Wavelet analysis is a breaking up of a signal into approximating functions (shifted and dilated versions of the wavelet) contained in finite domains [11]. Wavelet analysis was introduced in the early 1980s in the context of signal analysis and petroleum exploration. It aims to give a representation of signals and detect their characteristic. Several methods previously have been used, the most known is the Fourier transform. In Fourier transform further description of signals is limited to the overall behavior and can not provide any information on the details. In digital signal processing, Fourier analysis often requires linear calculation algorithms. In 1940, Denis Gabor introduced the windowed Fourier transform (TFF) to address the problems of time-frequency localization. It consists in calculating the Fourier transform of the signal by multiplying it by a function localized in time (Gaussian window) and then calculating the transform. But the situation has changed with the emergence of new problems especially irregular variation. The major drawback of the TFF is the shape stability and the window's size. Gaussian-type windows can not for example model non-stationary properties. Henceforth the need for analysis using non-linear algorithms, non-stationary signals, and/or non-periodical bases became necessary. Specifically analysis with well-specified characteristics, i.e. localization time and frequency, adaptation to the data, easily implemented advanced algorithms and optimum computation time needs to be developed. This was how wavelet theory was born. It subsequently renewed interest and has been steadily developed in theory and application. Wavelets differ from Fourier methods in that they allow the localization of a signal in both time and frequency. It is a tool that breaks up data into different frequency components or subbands and then studies each component with a resolution that is matched to a specific or proper scale. Unlike the Fourier series, it can be used on non-stationary transient signals with more precise results [13]. This section is devoted to presenting the main ideas on wavelet analysis namely wavelet transforms, multi-resolution analysis, wavelet bases, and algorithms of construction and reconstruction.

In a purely mathematical point of view, a wavelet is a function $\psi \in L^2(\mathbb{R})$ which satisfies the following conditions.

- Admissibility,

$$\mathcal{A}_\psi = \int_{\mathbb{R}^+} |\hat{\psi}(\omega)|^2 \frac{d\omega}{|\omega|} < \infty. \tag{2.1}$$

- Zero mean,

$$\widehat{\psi}(0) = \int_{-\infty}^{+\infty} \psi(u)du = 0. \tag{2.2}$$

- Localization,

$$\|\psi\|_2^2 = \int_{-\infty}^{+\infty} |\psi(u)|^2 du = 1. \tag{2.3}$$

- Enough vanishing moments,

$$p = 0, ..., m-1, \quad \int_{\mathbb{R}} \psi(t) t^p dt = 0. \tag{2.4}$$

To analyze functions/signals by wavelets, one pass by the so-called wavelet transforms. A wavelet transform (WT) is a representation of a time-frequency signal. It replaces Fourier sine/cosine by a wavelet. Generally, there are two types of processing; continuous wavelet transform and discrete wavelet transform.

The CWT is based firstly on the introduction of a translation parameter $u \in \mathbb{R}$ and another parameter $s > 0$ known as the scale to the analyzing wavelet ψ called the mother wavelet. The translation parameter determines the position or the time around which we want to assess the behavior of the signal, while the scale factor is used to assess the signal behavior around the position. That is, it allows us to estimate the frequency of the signal at that point. Let

$$\psi_{s,u}(x) = \frac{1}{\sqrt{s}} \psi(\frac{x-u}{s}). \tag{2.5}$$

The continuous wavelet transform at the position u and the scale s is defined by

$$d_{u,s}(f) = \int_{-\infty}^{\infty} \psi_{u,s}(t) f(t) dt, \quad \forall\, u, s. \tag{2.6}$$

By varying the parameters s and u, we may cover completely all the time-frequency plans. This gives a full and redundant representation of the whole signal to be analyzed (See [17]). This transform is called continuous because of the nature of the parameters s and u that may operate at all levels and positions. The original function f can be reproduced knowing its CWT by the following relationship.

$$f(x)=\frac{1}{\mathcal{A}_\psi}\int\int_{\mathbb{R}} d_{u,s}(f)\psi\left(\frac{x-u}{s}\right)\frac{dsdu}{s^2}. \tag{2.7}$$

The DWT is obtained by restricting the scale and position parameters to a discrete grid. The most known method is the dyadic grid $s=2^{-j}$ and $u=k2^{-j}$, $j,k\in\mathbb{Z}$. In this case, the wavelet copy $\psi_{u,s}$ is usually denoted by $\psi_{j,k}(x)=2^{-j/2}\psi(2^jx-k)$. The DWT of a function f is

$$d_{j,k}=\int_{-\infty}^{\infty}\psi_{j,k}(t)f(t)dt. \tag{2.8}$$

These are often called wavelet coefficients or detail coefficients of the signal S.

It holds that the set $(\psi_{j,k})_{j,k\in\mathbb{Z}}$ constitutes an orthonormal basis of $L^2(\mathbb{R})$ and called wavelet basis. An element $f\in L^2(\mathbb{R})$ is decomposed according to this basis into a series

$$f(t)=\sum_{j=0}^{\infty}\sum_k d_{j,k}\psi_{j,k}(t) \tag{2.9}$$

called the wavelet series of f which replaces the reconstruction formula for the CWT.

It holds in wavelet theory that the previous concepts may induce a functional framework for representing functions by a series of approximations called resolutions. Such framework is known as the multiresolution analysis (MRA) on $\mathbb{R}$. MRA is a family of closed vector subspaces $(V_j)_{j\in\mathbb{Z}}$ of $L^2(\mathbb{R})$. For each $j\in\mathbb{Z}$, V_j is called the approximation at the scale or the level j. More precisely ([17]), a multi-resolution analysis is a countable set of closed subsets $(V_j)_{j\in\mathbb{Z}}$ of $L^2(\mathbb{R})$ that satisfies the following points.

a) $\forall j\in\mathbb{Z};\ .V_j\subset V_{j+1}$.
b) $\bigcap_{j\in\mathbb{Z}} V_j=\{0\}$.
c) $\overline{\bigcup_{j\in\mathbb{Z}} V_j}=L^2(\mathbb{R})$.
d) $\forall j\in\mathbb{Z};\ f\in V_j\Leftrightarrow f(2.)\in V_{j+1}$
e) $\forall j\in\mathbb{Z};\ f\in V_j\Leftrightarrow f(x-k)\in V_j$
f) There exists $\varphi\in V_0$ such that $\{\varphi_{0,k}=\varphi(.-k);\ k\in\mathbb{Z}\}$ is an orthogonal Riesz basis of V_0.

The source function φ is called the scaling function of the MRA or also the wavelet father.

It holds that this function generates all the subspaces V_j's of the MRA by acting dilation/translation parameters. Indeed, the set $\left(\varphi_{j,k}(x)=2^{j/2}\varphi(2^jx-k)\right)_k$ is an orthogonal basis of V_j for all $j,\in\mathbb{Z}$. Moreover, there is an orthogonal supplementary W_j of V_j in V_{j+1}, that is

$$V_{j+1}=V_j\oplus^\perp W_j \tag{2.10}$$

The space W_j, $j\in\mathbb{Z}$ is called detail space at the scale or the level j for which the set $\left(\psi_{j,k}(t)=2^{j/2}\psi(2^jt-k)\right)_k$ is an orthogonal basis.

The strongest point in MRA and wavelet theory is that the scaling function and the analyzing wavelet leads each one to the other. Indeed, recall that φ belongs to $V_0\subset V_1$ and the latter is generated by the basis $(\varphi_{1,k})_k$. Hence, φ is expressed by means of $(\varphi_{1,k})_k$. More precisely, we have the so-called 2-scale relation

$$\varphi(x)=\sqrt{2}\sum_k h_k\varphi(2x-k) \tag{2.11}$$

where the coefficients h_k are

$$h_k = \int_{\mathbb{R}} \varphi(x)\overline{\varphi(2x-k)}dx.$$

It holds that the mother wavelet ψ satisfies

$$\psi(x) = \sqrt{2}\sum_k g_k\varphi(2x-k)$$

where the g_k's are evaluated by

$$g_k = (-1)^k h_{1-k}.$$

For more details, we refer to [11], [13], [17]. These last relations are the starting point to develop next our main results.

3. Main results

In the present paper, our aim is to introduce 'new' wavelet functions (father and mother wavelets) and associated multiresolution analysis using the well-known Farey map. We will show that such a map permits (as in the case of Haar, Morlet, and Gaussian functions) to develop a multiresolution analysis on $\mathbb{R}$ and to conduct important applications and algorithms in applied contexts.

The Faray map (slightly modified) is defined by

$$F(x) = \begin{cases} \dfrac{1}{4\log 2 - 2}\dfrac{x}{1-x} & , \quad 0 \leq x \leq \frac{1}{2}, \\ \dfrac{1}{4\log 2 - 2}\dfrac{1-x}{x} & , \quad \frac{1}{2} \leq x \leq 1. \end{cases}$$

We may define in a general way a generalized Farey map relatively to a gauge function h instead of $\dfrac{1}{4\log 2 - 2}\dfrac{x}{1-x}$ by considering

$$F_h(x) = \begin{cases} h(x) & , \quad 0 \leq x \leq \frac{1}{2}, \\ h(1-x) & , \quad \frac{1}{2} \leq x \leq 1. \end{cases}$$

or more generally

$$F_{h,\overline{h},a}(x) = \begin{cases} h(x) & , \quad 0 \leq x \leq a, \\ \overline{h}(x) & , \quad a \leq x \leq 1, \end{cases} \tag{3.1}$$

relatively to some suitable functions h and $\overline{h}$ and real number a. Figure 1 illustrates the Farey map F.

In this part, we serve the Farey map above to construct a new wavelet function on $\mathbb{R}$ (which may be obviously extended to $\mathbb{R}^n$, $N \geq 2$). To do this we consider a bit of modification of the Farey map above to obtain the modified Farey map which will be denoted by φ and defined as follows

$$\varphi(x) = F(\frac{1+x}{2}) = \begin{cases} \dfrac{1}{(4\log 2 - 2)}\dfrac{1+x}{1-x} & , \quad -1 \leq x \leq 0, \\ \\ \dfrac{1}{(4\log 2 - 2)}\dfrac{1-x}{1+x} & , \quad 0 \leq x \leq 1. \end{cases} \tag{3.2}$$

Related to wavelet theory, the function φ is a wavelet-copy-like (or scaling-function-copy-like) of the Farey map F as in wavelet theory the copies of the mother wavelet and the

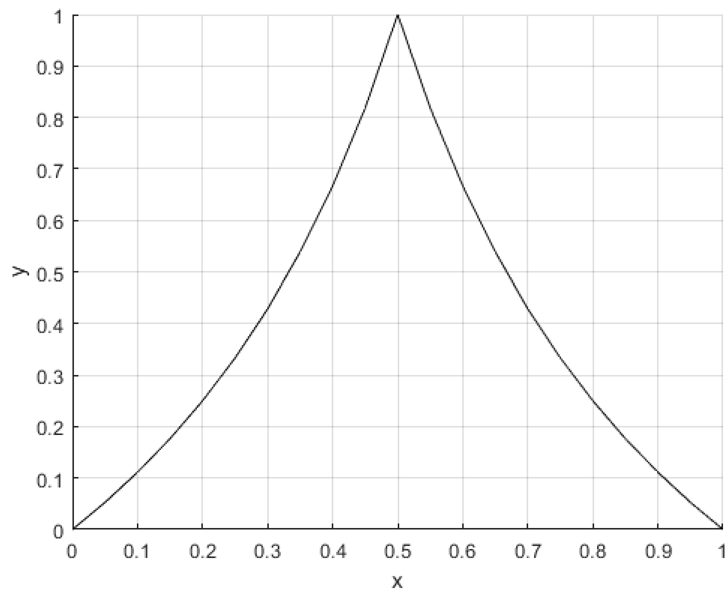

FIGURE 1. The Farey map F.

scaling function are always obtained by means of translation/dilation copies. In our case we remark easily that $\varphi = \sqrt{2}F_{1,-1}$ where for $j, k \in \mathbb{Z}$, $F_{j,k}(x) = \frac{1}{2^{j/2}} F(\frac{x-k}{2^j})$.

In this section, we propose to establish some properties of the modified Farey function φ introduced in (3.2) and thus prove consequently that it serves to construct father and mother wavelets.

Lemma 3.1. *The function φ satisfies $\widehat{\varphi}(0) = 1$.*

Indeed, simple computations yield that

$$\widehat{\varphi}(0) = \frac{1}{2\log 2 - 1} \int_1^2 \frac{2-t}{t} dt = \frac{1}{2\log 2 - 1} \int_1^2 (\frac{2}{t} - 1) dt = 1.$$

The first point in the construction of wavelet mothers and fathers and consequently associated multi-resolution analysis is to check the existence of Riesz basis φ_k (for the eventual space V_0), the admissibility, vanishing moments, and localization for the mother wavelet ψ and next, the famous 2-scale relation. In this direction, we have a series of results.

Lemma 3.2. *The function φ satisfies the 2-scale relation*

$$\varphi(x) = \sqrt{2} \sum_{k \in \mathbb{Z}} h_k \varphi(2x - k),$$

where $h_1 = h_{-1} = \frac{1}{3\sqrt{2}}$, $h_0 = \frac{1}{\sqrt{2}}$ and 0 else.

Proof. Because of the disjointness of the supports of the functions $\varphi(. - k)$, $k \in \mathbb{Z}$ appearing in the 2-scale relation we immediately observe that

$$h_k = 0, \forall k, |k| \geq 2.$$

So the relation in Lemma 3.2 above reads

$$\varphi(x) = \sqrt{2}h_{-1}\varphi(2x+1) + \sqrt{2}h_0\varphi(2x) + \sqrt{2}h_1\varphi(2x-1).$$

Next, evaluating the last equation for suitable values of x we obtain

$$h_1 = h_{-1} = \frac{1}{3\sqrt{2}} \quad \text{and} \quad h_0 = \frac{1}{\sqrt{2}}.$$

Corollary 3.3. *For all $\xi \in \mathbb{R}$, we have*

$$\widehat{\varphi}(\xi) = \mathcal{M}_0(\frac{\xi}{2})\widehat{\varphi}(\frac{\xi}{2}),$$

where $\mathcal{M}_0(\xi) = \frac{1}{2}\left(1 + \frac{2}{3}\cos\xi\right)$.

Proof. By applying the Fourier transform to the 2-scale relation in Lemma 3.2 above we obtain

$$\widehat{\varphi}(\xi) = \frac{1}{2}\left(\frac{1}{3}e^{i\xi/2} + 1 + \frac{1}{3}e^{-i\xi/2}\right)\widehat{\varphi}(\frac{\xi}{2})$$

which reads as

$$\widehat{\varphi}(\xi) = \frac{1}{2}\left(1 + \frac{2}{3}\cos(\frac{\xi}{2})\right)\widehat{\varphi}(\frac{\xi}{2}).$$

Hence, the desired result follows.

Theorem 3.4. *The function ψ defined by*

$$\psi(x) = K_0(\sqrt{2}\sum_{k\in\mathbb{Z}} g_k\varphi(2x-k)),$$

with $g_k = (-1)^{k-1}h_{1-k}$ and $K_0 = \dfrac{4\log 2 - 2}{\sqrt{3 - 4\log 2}}$ satisfies

- **i.** $\widehat{\psi}(\xi) = \mathcal{M}_1(\frac{\xi}{2})\widehat{\varphi}(\frac{\xi}{2})$, *where* $\mathcal{M}_1(\xi) = \frac{K_0}{6}(3 - 2\cos\xi)\,e^{-i\xi}$.
- **ii.** $\|\psi\|_2^2 = 1$.

Proof. Using Lemma 3.1, we get $g_k = 0, \forall, k \leq -1$ and $k \geq 3$. So the relation above reads

$$\psi(x) = K_0\sqrt{2}g_0\varphi(2x) + \sqrt{2}g_1\varphi(2x-1) + \sqrt{2}g_2\varphi(2x-2)$$

with

$$g_0 = -\frac{1}{3\sqrt{2}},\ g_1 = \frac{1}{\sqrt{2}} \quad \text{and} \quad g_2 = -\frac{1}{3\sqrt{2}}.$$

Otherwise,

$$\psi(x) = K_0(-\frac{1}{3}\varphi(2x) + \varphi(2x-1) - \frac{1}{3}\varphi(2x-2)). \tag{3.3}$$

Denote for simplicity $K = \dfrac{1}{(4\log 2 - 2)\sqrt{2}}$. Consequently,

$$\|\psi\|_2^2 = K_0^2(\frac{22}{9K^2}\int_1^2\left(\frac{2-x}{x}\right)^2 dx - \frac{4}{3K^2}\int_0^1\left(\frac{x(1-x)}{(1+x)(2-x)}\right)^2 dx).$$

Now, standard computations yield that

$$\|\psi\|_2^2 = 1.$$

Theorem 3.5. *The function ψ is explicitly expressed by*

$$\psi(x) = K_1 \begin{cases} \frac{1}{3}\frac{1+2x}{1-2x} & ; \quad -\frac{1}{2} \le x \le 0, \\ \frac{1}{3}\frac{1-2x}{1+2x} - \frac{x}{1-x} & ; \quad 0 \le x \le \frac{1}{2}, \\ \frac{x-1}{x} - \frac{1}{3}\frac{1-2x}{3-2x} & ; \quad \frac{1}{2} \le x \le 1, \\ -\frac{1}{3}\frac{3-2x}{1-2x} & ; \quad 1 \le x \le \frac{3}{2}. \end{cases}$$

where $K_1 = \dfrac{1}{\sqrt{3-4\log 2}}$ is the normalization constant.

Proof. We have from Theorem 3.4

$$\psi(x) = K_0(\frac{1}{3}\varphi(2x) - \varphi(2x-1) + \frac{1}{3}\varphi(2x-2)).$$

We now proceed by evaluating the right-hand side quantity piecewise.

- On $[-\frac{1}{2}, 0]$ we get

$\varphi(2x-1) = \varphi(2x-2) = 0$ and $\varphi(2x) = \dfrac{1+2x}{1-2x}$. Consequently,

$$\psi(x) = \frac{K_0}{3(4\log 2 - 3)}\frac{1+2x}{1-2x} = \frac{K_1}{3}\frac{1+2x}{1-2x}.$$

- On $[0, \frac{1}{2}]$ we get similarly to the previous case

$$\psi(x) = \frac{K_0}{4\log 2 - 3}\Big(\frac{1}{3}\frac{1-2x}{1+2x} - \frac{x}{1-x}\Big) = K_1\Big(\frac{1}{3}\frac{1-2x}{1+2x} - \frac{x}{1-x}\Big).$$

- On $[\frac{1}{2}, 1]$ we get

$$\psi(x) = \frac{K_0}{4\log 2 - 3}\Big(-\frac{1-x}{x} - \frac{1}{3}\frac{1-2x}{3-2x}\Big) = K_1\Big(\frac{x-1}{x} - \frac{1}{3}\frac{1-2x}{3-2x}\Big).$$

- On $[1, \frac{3}{2}]$ we get

$$\psi(x) = \frac{K_0}{3(4\log 2 - 2)}\frac{3-2x}{2x-1}) = -\frac{K_1}{3}\frac{3-2x}{1-2x}.$$

Theorem 3.6. *The function $\widetilde{\psi}$ defined by*

$$\widetilde{\psi}(x) = \psi(x) - c\chi_{]-1/2,3/2[}(x)$$

with $c = \dfrac{2\log 2 - 1}{6}$, is admissible with one vanishing moment.

Proof. It remains to show the admissibility and vanishing moments. From equation (3.3) it suffices to show the admissibility of the Farey map F. Denote

$$\mathcal{A}_F = \int_0^\infty \frac{|\widehat{F}(\omega)|^2}{\omega} d\omega,$$

$$H(\omega)=\int_0^{1/2}h(x)e^{i\omega x}dx$$

and

$$G(\omega)=e^{-i\omega}H(\omega).$$

Straightforward computations yield that

$$\widehat{F}(\omega)=H(-\omega)-G(\omega). \tag{3.4}$$

We will now evaluate $\widehat{F}(\omega)$ near the origin $\omega=0$. By applying the equality above, we get

$$\widehat{F}'(0)=-iG(0)-2G(0)=i\frac{1-4\log 2}{4}.$$

Consequently, near 0 we get

$$\widehat{F}(\omega)=i\frac{1-4\log 2}{4}\omega+\omega o(\omega),$$

where $o(\omega)\to 0$ as $\omega\to 0$. Consequently,

$$\int_0^{\eta}\frac{|\widehat{F}(\omega)|^2}{\omega}d\omega<\infty,\ \forall\,\eta>0.$$

On the other hand, using equality (3.4), we get

$$|\widehat{F}(\omega)|\leq 2|G(\omega)|.$$

Next, integration by parts yields that

$$G(\omega)=\frac{e^{-i\omega/2}}{i\omega}+\frac{1}{i\omega}\int_{1/2}^{1}\frac{e^{-i\omega t}}{t^2}dt.$$

Consequently,

$$|G(\omega)|\leq\frac{2}{|\omega|},$$

which means that

$$|\widehat{F}(\omega)|\leq\frac{4}{|\omega|}.$$

As a result, for $\eta>0$ large enough we get

$$\int_{\eta}^{\infty}\frac{|\widehat{F}(\omega)|^2}{\omega}d\omega\leq\int_{\eta}^{\infty}\frac{16}{\omega^3}d\omega<\infty.$$

It holds finally that $\mathcal{A}_F<\infty$.

Figures 2 and 3 below illustrate the functions φ and ψ respectively.

Remark. *The supports of the functions φ and ψ are compact and satisfy $Support(\varphi)=[N_1,N_2]$ and $Support(\varphi)=[\frac{N_1-N_2}{2},\frac{N_2-N_1}{2}]$ with $N_1=-1$ and $N_2=1$, which joins the result of Daubechies on compactly supported wavelets.*

Definition 3.1. Let $g\in L^2(\mathbb{R})$. The system of functions $(g_k=g(.-k);\ k\in\mathbb{Z})$ is called a Riesz basis if there exist positive constants A and B such that for any finite set of integers $\Lambda\subset\mathbb{Z}$ and real numbers λ_k; $k\in\Lambda$, we have

$$A\sum_{k\in\Lambda}\lambda_k^2\leq\|\sum_{k\in\Lambda}\lambda_k g_k\|_2^2\leq B\sum_{k\in\Lambda}\lambda_k^2.$$

Lemma 3.7. *The Farey function F and the modified Farey function φ satisfy the following assertions.*

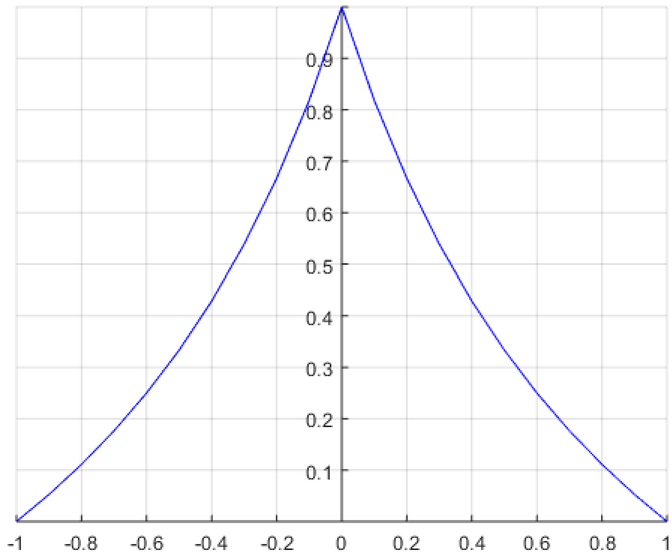

FIGURE 2. Farey scaling function φ.

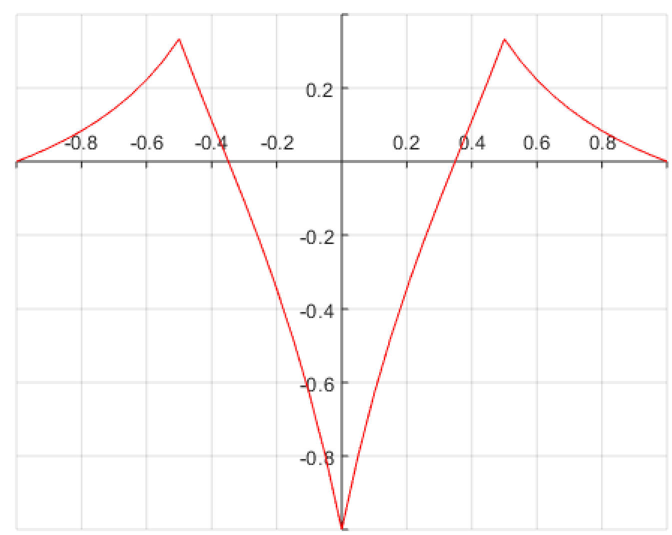

FIGURE 3. Farey wavelet mother ψ.

(1) The system $(F_k = F(. - k),\ k \in \mathbb{Z})$ is orthogonal in $L^2(\mathbb{R})$.
(2) The system $(\varphi_k = \varphi(. - k),\ k \in \mathbb{Z})$ is only a Riesz system in $L^2(\mathbb{R})$.

Proof. The first point is a simple consequence of the disjoint supports of the functions F_k. So, we prove the second. Let $\Lambda \subset \mathbb{Z}$ be a finite set of integers and let $(\lambda_k)_{k\in\Lambda}$ be real numbers. We have

$$\|\sum_{k\in\Lambda}\lambda_k\varphi_k\|_2^2 = \sum_{l,k\in\Lambda}\lambda_l\lambda_k < \varphi_l, \varphi_k > .$$

Due to the supports of the functions φ_k the last equality reads as

$$\|\sum_{k\in\Lambda}\lambda_k\varphi_k\|_2^2 = \sum_{k\in\Lambda}\left(\lambda_k\lambda_{k-1} < \varphi_k, \varphi_{k-1} > + \lambda_k^2\|\varphi_k\|_2^2 + \lambda_k\lambda_{k+1} < \varphi_k, \varphi_{k+1} >\right).$$

As φ_k are positive functions for all k, we obtain

$$\|\sum_{k\in\Lambda}\lambda_k\varphi_k\|_2^2 \geq \sum_{k\in\Lambda}\lambda_k^2\|\varphi_k\|_2^2 = \|\varphi\|_2^2\sum_{k\in\Lambda}\lambda_k^2.$$

On the other hand, using Cauchy-Schwartz inequality we obtain

$$\sum_{k\in\Lambda}(\lambda_k\lambda_{k-1} < \varphi_k, \varphi_{k-1} > + \lambda_k\lambda_{k+1} < \varphi_k, \varphi_{k+1} >) \leq (< \varphi, \varphi_{-1} + \varphi_1 >)\sum_{k\in\Lambda}\lambda_k^2.$$

As a result, by taking $A = \|\varphi\|_2^2$ and $B = \|\varphi\|_2^2 + < \varphi, \varphi_{-1} + \varphi_1 >$ we obtain $0 < A < B < \infty$ and

$$A\sum_{k\in\Lambda}\lambda_k^2 \leq \|\sum_{k\in\Lambda}\lambda_k\varphi_k\|_2^2 \leq B\sum_{k\in\Lambda}\lambda_k^2.$$

So as the Lemma.

Corollary 3.8. *The function $\Gamma_\varphi(\omega) = \sum_{k\in\mathbb{Z}}|\widehat{\varphi}(\omega + 2k\pi)|^2$ is bounded on $\mathbb{R}$.*

Proof. With the same notations in Lemma 3.7 denote

$$H(x) = \sum_{k\in\Lambda}\lambda_k\varphi_k(x),$$

and

$$\widetilde{H}(\xi) = \sum_{k\in\Lambda}\lambda_k e^{-ik\xi}.$$

We have

$$\int_{\mathbb{R}}|\sum_{k\in\Lambda}\lambda_k e^{-ik\xi}|^2dx = \int_{\mathbb{R}}|H(x)|^2dx = \frac{1}{2\pi}\int_{\mathbb{R}}|\widehat{H}(\xi)|^2d\xi.$$

Observe now that

$$\widehat{H}(\xi) = \widetilde{H}(\xi)\widehat{\varphi}(\xi).$$

Hence,

$$\begin{aligned}\int_{\mathbb{R}}|\sum_{k\in\Lambda}\lambda_k e^{-ik\xi}|^2dx &= \frac{1}{2\pi}\int_{\mathbb{R}}|\widetilde{H}(\xi)|^2|\widehat{\varphi}(\xi)|^2d\xi \\ &= \frac{1}{2\pi}\sum_{l\in\mathbb{Z}}\int_{2\pi l}^{2\pi(l+1)}|\widetilde{H}(\xi)|^2|\widehat{\varphi}(\xi)|^2d\xi.\end{aligned}$$

Observing next that $\widetilde{H}$ is 2π-periodic we get

$$\begin{aligned}\int_{\mathbb{R}} |\sum_{k\in\Lambda} \lambda_k e^{-ik\xi}|^2 dx &= \frac{1}{2\pi}\int_0^{2\pi} |\widetilde{H}(\xi)|^2 \sum_{l\in\mathbb{Z}} |\widehat{\varphi}(\xi+2\pi l)|^2 d\xi \\ &= \frac{1}{2\pi}\int_0^{2\pi} |\widetilde{H}(\xi)|^2 \Gamma_\varphi(\xi) d\xi.\end{aligned}$$

Now, as the set $(\varphi_k)_k$ is a Riesz system on $L^2(\mathbb{R})$, we deduce that Γ_φ is bounded.

Definition 3.2. The function Γ_φ is called the overlap function associated to the system (φ_k) or to the function φ.

Proposition 3.9. *Let $\Phi \in L^2(\mathbb{R})$ be defined by its Fourier transform $\widehat{\Phi} = \dfrac{\widehat{\varphi}}{\sqrt{\Gamma_\varphi}}$. Then, the system $(\Phi_k = \Phi(.-k))_{k\in\mathbb{Z}}$ is orthonormal in $L^2(\mathbb{R})$.*

Proof. We have for all $l, k \in \mathbb{Z}$,

$$< \Phi_k, \Phi_l >= \frac{1}{2\pi} < \widehat{\Phi}_k, \widehat{\Phi}_l >= \frac{1}{2\pi}\int_{\mathbb{R}} |\widehat{\Phi}(\xi)|^2 e^{-i(k-l)\xi} d\xi.$$

Consequently,

$$\begin{aligned}< \Phi_k, \Phi_l > &= \frac{1}{2\pi}\sum_{n\in\mathbb{Z}}\int_{2n\pi}^{2(n+1)\pi} |\widehat{\Phi}(\xi)|^2 e^{-i(k-l)\xi} d\xi \\ &= \frac{1}{2\pi}\int_0^{2\pi}\sum_{n\in\mathbb{Z}} |\widehat{\Phi}(\xi+2n\pi)|^2 e^{-i(k-l)\xi} d\xi.\end{aligned}$$

Observe now that

$$\sum_{n\in\mathbb{Z}} |\widehat{\Phi}(\xi+2n\pi)|^2 = \frac{\sum_{n\in\mathbb{Z}} |\widehat{\varphi}(\xi+2n\pi)|^2}{\Gamma_\varphi(\xi)} = 1.$$

We obtain

$$< \Phi_k, \Phi_l >= \frac{1}{2\pi}\int_0^{2\pi} e^{-i(k-l)\xi} d\xi = \delta_{lk}.$$

Lemma 3.10. *The Fourier transform of the modified Farey map φ is expressed by*

$$\widehat{\varphi}(\xi) = 4\cos(\xi)Ci(\xi) - 4\sin(\xi)Si(\xi) - 2\frac{\sin\xi}{\xi};\ \forall\, \xi \neq 0,$$

where $Ci(\xi) = \displaystyle\int_\xi^{2\xi} \frac{\cos t}{t} dt$ and $Si(\xi) = \displaystyle\int_\xi^{2\xi} \frac{\sin t}{t} dt$.

Proof. Assume that $\xi > 0$. Elementary calculus yield that

$$\widehat{\varphi}(\xi) = 2\int_1^2 \frac{2-t}{t}\cos(\xi(t-1))dt,$$

which may be written as

$$\widehat{\varphi}(\xi) = 4\int_1^2 \frac{\cos(\xi(t-1))}{t} dt - 2\int_1^2 \cos(\xi(t-1))dt.$$

The last integral is an easy form. Applying standard trigonometric rules for the first one we obtain

$$\widehat{\varphi}(\xi) = 4\cos(\xi)\int_1^2 \frac{\cos(\xi t)}{t}dt - 4\sin(\xi)\int_1^2 \frac{\sin(\xi t)}{t}dt - 2\frac{\sin\xi}{\xi}.$$

Next, taking $u = \xi t$ for the last integrals we obtain

$$\widehat{\varphi}(\xi) = 4\cos(\xi)Ci(\xi) - 4\sin(\xi)Si(\xi) - 2\frac{\sin\xi}{\xi}.$$

For $\xi < 0$, it suffices to see that $\widehat{\varphi}$ is in fact an even function.

Lemma 3.11. *The modified Farey wavelet mother ψ satisfies*

$$\widehat{\psi}(0) = 0 \ \textit{ and } \ \widehat{x\psi}(0) = \log 2 - \frac{3}{4}.$$

Proof. Simple calculus yields that

$$\widehat{\psi}(0) = \int_0^{1/2} h(x)dx - \int_{1/2}^1 h(1-x)dx = 0.$$

Similarly,

$$\begin{aligned}\widehat{x\psi}(0) &= \int_0^{1/2} xh(x)dx - \int_{1/2}^1 xh(1-x)dx \\ &= \int_{1/2}^1 \frac{(1-2x)(1-x)}{x}dx \\ &= \log 2 - \frac{3}{4}.\end{aligned}$$

4. Conclusion

Wavelet theory has known a great success since its discovery. Mathematically, it provides for function spaces good bases allowing their decomposition into spices associated with different horizons known as the levels of decomposition. A wavelet basis is a family of functions obtained from one function known as the mother wavelet, by translations and dilations. This makes the finding of wavelet mothers of great interest. The analysis of a given function using wavelets passes through the so-called wavelet transform or wavelet coefficient. It is a quantity obtained by a convolution product between the function to be analyzed and the copies of the analyzing wavelet mother.

In the present work, one motivation was to construct indeed a wavelet mother starting from the exploitation of the characteristics of the well-known Farey map. Well-known characteristics in wavelet theory such as admissibility and vanishing moments rules, compact support have been established for the new wavelet. Many extensions may be addressed as future directions for the present work.

- Exploit more the characteristics of the new Farey wavelet such as its continuous wavelet transform, Fourier-Plancherel type rule as well as Parceval formula.
- Associate a discrete wavelet transform for the new Farey wavelet.
- Construct suitable multi-resolution analysis.
- Develop concrete applications of the new wavelet framework to show the utility of the newly constructed wavelet mother.

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