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FUZZY PARAOPEN SETS AND MAPS IN FUZZY TOPOLOGICAL SPACES

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ABSTRACT. The purpose of this article is to study the concepts of fuzzy paraopen and fuzzy paraclosed sets in fuzzy topological spaces Further, we introduce few class of fuzzy maps namely fuzzy paracontinuous, $*$-fuzzy paracontinuous, fuzzy parairresolute, fuzzy minimal paracontinuous, fuzzy maximal paracontinuous mappings and study their prpoerties.

1. INTRODUCTION

After the introduction of fuzzy sets by L.A. Zadeh[7], the concept of fuzzy topology introduced by C.L.Chang[2] in 1968. The notions of minimal open[6] and maximal open[5] sets explored by F. Nakaoka and N. Oda. Further the idea of paraopen sets was studied by B. M. Ittanagi and S. S. Benchalli[4]. Consequent of this, Ajoy Mukherjee and Kallol Bhandhu Bagchi in [1] introduced the concepts of mean open and mean closed sets and showed that the paraopen and mean open sets are almost same.

In this paper (X, τ) or X stands for fuzzy topological space. The symbols $\lambda, \mu, \gamma, \eta$,... are used to denote fuzzy sets and all other symbols have their usual meaning unless explicitely stated.

2. PRELIMINARIES

Definition 2.1. *A proper nonempty open set U of $X$is said to be a minimal open*[6] *set if ϕ and U are only open sets contained in U.*

Definition 2.2. *A proper nonempty open set U of $X$is said to be a maximal open*[5] *set if X and U are only open sets containing U.*

Definition 2.3. *A open set U of a topological space X is said to be a paraopen* [4]*set if it is neither minimal open nor maximal open set. The family of all paraopen sets in a topological space X is denoted by PaO(X).*

Definition 2.4. *A proper nonempty fuzzy open set α of $X$is said to be a fuzzy minimal open*[3] *set if 0_X and α are only fuzzy open sets contained in α.*

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Definition 2.5. *A proper nonempty fuzzy open set α of Xis said to be a fuzzy maximal open*[3] *set if 1_X and α are only fuzzy open sets containing α.*

Definition 2.6. *Let X and Y be fuzzy topological spaces. A map from fts X to another fts Y is called,*
(i) fuzzy minimal continuous[3]*briefly f. min-continuous) if $f^{-1}(\lambda)$ is a fuzzy open set on X for any fuzzy minimal open set λ on Y.*
(ii)fuzzy maximal continuous[3]*(briefly f. max-continuous) if $f^{-1}(\lambda)$ is a fuzzy open set on X for any fuzzy maximal open set λ on Y.*

3. Fuzzy Paraopen and Some of their Properties

Definition 3.1. *A fuzzy open set μ of a fuzzy topological space X is said to be a fuzzy paraopen set if is neither fuzzy minimal open nor fuzzy maximal open set. The family of all fuzzy paraopen sets in a fuzzy topological space X is denoted by $FPaO(X)$.*

A fuzzy closed set λ of a fuzzy topological space X is said to be a fuzzy paraclosed set if and only if its complement $1_X - \lambda$ is fuzzy paraopen set. The family of all fuzzy paraclosed sets in a fuzzy topological space X is denoted by $FPaC(X)$.

Remark. It is observed from the above definitions that every fuzzy paraopen set is a fuzzy open set and every fuzzy paraclosed set is a fuzzy closed set but not conversely, which is shown by the following example.

Example 3.2. *Let μ_1,μ_2, μ_3 and μ_4 be fuzzy sets on $X = \{a, b, c, d\}$. Then $\mu_1 = \frac{0.0}{a} + \frac{0.3}{b} + \frac{0.5}{c} + \frac{0.0}{d} +$, $\mu_2 = \frac{0.5}{a} + \frac{0.3}{b} + \frac{0.6}{c} + \frac{0.0}{d}$ $\mu_3 = \frac{0.0}{a} + \frac{0.3}{b} + \frac{0.6}{c} + \frac{0.5}{d}$ and $\mu_4 = \frac{0.5}{a} + \frac{0.3}{b} + \frac{0.6}{c} + \frac{0.5}{d}$ be fuzzy sets with $\mathfrak{F}_1 = \{0_X, \mu_1, \mu_2, \mu_3, \mu_4, 1_X\}$, Then $FM_iO(X) = \{\mu_1\}$,$FM_aO(X) = \{\mu_4\}$, $FM_iC(X) = \left\{\mu_4^c\right\}$, $FM_aC(X) = \left\{\mu_1^c\right\}$,$FP_aO(X) = \{\mu_2, \mu_3\}$, $FP_aC(X) = \left\{\mu_2^c, \mu_3^c\right\}$. Here μ_1 is a fuzzy open set but bot a fuzzy paraopen set and μ_4^c is a fuzzy closed set but not a fuzzy paraclosed set.*

Remark. Union and intersection of fuzzy paraopen (resp. fuzzy paraclosed) sets need not be a fuzzy paraopen (resp. fuzzy paraclosed) set which is shown by the following example.

Example 3.3. *In Example 3.2, fuzzy sets μ_2, μ_3 are fuzzy paraopen sets but $\mu_2 \vee \mu_3 = \mu_4$ and $\mu_2 \wedge \mu_3 = \mu_1$ which are not fuzzy paraopen sets.*
Similarly for the fuzzy paraclosed sets μ_2^c,μ_3^c but $\mu_2^c \vee \mu_3^c = \mu_1^c$ and $\mu_2^c \wedge \mu_3^c = \mu_4^c$ which are not fuzzy paraclosed sets.

Theorem 3.1. Let X be a fuzzy topological space and λ be a nonempty proper fuzzy paraopen subset of X. Then there exists a fuzzy minimal open set μ such that $\mu < \lambda$.

Proof. By the definition of fuzzy minimal open set, it is clear that $\mu < \lambda$. □

Theorem 3.2. Let X be a fuzzy topological space and λ be a nonempty proper fuzzy paraopen subset of X. Then there exists a fuzzy maximal open set σ such that $\lambda < \sigma$.

Proof. By the definition of fuzzy maximal open set, it is clear that $\lambda < \sigma$. □

Theorem 3.3. Let X be a fuzzy topological space.
(i)Let λ be a fuzzy paraopen and μ be a fuzzy minimal open set then $\lambda \wedge \mu = 0_X$ or $\mu < \lambda$.
(ii)i)Let λ be a fuzzy paraopen and ϑ be a fuzzy maximal open set then $\lambda \vee \vartheta = 1_X$ or $\lambda < \vartheta$.
(iii)Intersection of fuzzy paraopen sets is either fuzzy paraopen or fuzzy minimal open set.

Proof. (i) Let λ be a fuzzy paraopen and μ be a fuzzy minimal open set in X. Then $\lambda \wedge \mu = 0_X$ or $\lambda \wedge \mu \neq 0_X$. If$\lambda \wedge \mu = 0_X$, then we need not prove anything. Suppose $\lambda \wedge \mu \neq 0_X$. Now we have $\lambda \wedge \mu$ is a fuzzy open set and $\lambda \wedge \mu < \mu$. Hence $\mu < \lambda$.

(ii) Let λ be a fuzzy paraopen and γ be a fuzzy maximal open set in X. Then $\lambda \vee \gamma = 1_X$ or $\lambda \vee \mu \neq 1_X$. If$\lambda \vee \gamma = 1_X$, then we need not prove anything. Suppose $\lambda \vee \gamma \neq 1_X$. Now we have $\lambda \vee \gamma$ is a fuzzy open set and $\gamma < \lambda \vee \gamma$. Since γ is a fuzzy maximal open set, $\lambda \vee \gamma = \gamma$ which implies $\lambda < \gamma$.

(iii)Let λ and ηbe a fuzzy paraopen sets in X. If $\lambda \wedge \eta$ is a fuzzy paraopen set then we need not prove anything. Suppose $\lambda \wedge \eta$ is not a paraopen set. Then by definition, $\lambda \wedge \eta$ is a fuzzy minimal open or fuzzy maximal open set. If $\lambda \wedge \eta$ is a fuzzy minimal open set then we need not prove anything. Suppose $\lambda \wedge \eta$ is a fuzzy maximal open set. Now $\lambda \wedge \eta < \lambda$ and $\lambda \wedge \eta < \eta$ which contradicts the fact that λ and η are fuzzy paraopen sets. Therefore $\lambda \wedge \eta$ is not a fuzzy maximal open set. That is $\lambda \wedge \eta$ must be a fuzzy minimal open set. □

Theorem 3.4. Let X be a fuzzy topological space. A subset ϑ of X is fuzzy paraclosed if and only if it is neither fuzzy maximal closed nor fuzzy minimal closed set.

Proof. By the definition of fuzzy maximal closed set and the fact that the complement of fuzzy minimal open set is fuzzy maximal closed set and the complement of fuzzy maximal open set is fuzzy minimal closed set. □

Theorem 3.5. Let X be a fuzzy topological space and ϑ be a nonempty fuzzy paraclosed subset of X. Then there exists a fuzzy minimal closet set σ such that $\sigma < \vartheta$.

Proof. By the definition of fuzzy minimal closed set it is clear that $\sigma < \vartheta$. □

Theorem 3.6. Let X be a fuzzy topological space and ϑ be a nonempty fuzzy paraclosed subset of X. Then there exists a fuzzy maximal closet set ν such that $\vartheta < \nu$.

Proof. By the definition of fuzzy maximal closed set it is clear that $\vartheta < \nu$. □

Theorem 3.7. Let X be a fuzzy topological space.
(i)Let β be a fuzzy paraclosed and α be a fuzzy minimal closed set then $\beta \wedge \alpha = 0_X$ or $\alpha < \beta$.
(ii)Let β be a fuzzy paraclosed and γ be a fuzzy maximal closed set then $\beta \vee \gamma = 1_X$ or $\beta < \gamma$. (iii)Intersection of fuzzy paraclosed sets is either fuzzy paraclosed or fuzzy minimal closed set.

Proof. (i) Let β be a fuzzy paraclosed and α be a fuzzy minimal closed set in X. Then $(1_X - \beta)$ is fuzzy paraopen and $(1_X - \alpha)$ is fuzzy maximal open set in X. Then by Theorem 3.3(ii) we have $(1_X - \beta) \vee (1_X - \alpha) = X$ or $(1_X - \beta) < (1_X - \alpha)$ which implies $1_X - (\beta \wedge \alpha) = 1_X$ or $\alpha < \beta$. Therefore $\beta \wedge \alpha = 0_X$ or $\alpha < \beta$.

(ii) Let β be a fuzzy paraclosed and γ be a fuzzy maximal closed set in X. Then $(1_X - \beta)$ is fuzzy paraopen and $(1_X - \gamma)$ is fuzzy minimal open sets in X. Then by Theorem 3.3(i) we have $(1_X - \beta) \wedge (1_X - \gamma) = 0_X$ or $1_X - \gamma < 1_X - \beta$ which implies $1_X - (\beta \vee \gamma) = 0_X$ or $\beta < \gamma$. Therefore $\beta \vee \gamma = 1_X$ or $\beta < \gamma$.

(iii)Let β and η be a fuzzy paraclosed sets in X. If $\beta \wedge \eta$ is a fuzzy paraclosed set then we need not prove anything. Suppose $\beta \wedge \eta$ is not a fuzzy paraclosed set. Then by definition, $\beta \wedge \eta$ is a fuzzy minimal closed or fuzzy maximal closed set. If $\beta \wedge \eta$ is a fuzzy minimal closed set then we need not prove anything. Suppose $\beta \wedge \eta$ is a fuzzy maximal closed set. Now $\beta < \beta \wedge \eta$ and $\eta < \beta \wedge \eta$ which contradicts the fact that β and η are fuzzy paraclosed sets. Therefore $\beta \wedge \eta$ is not a fuzzy maximal closed set. That is $\beta \wedge \eta$ must be a fuzzy minimal closed set. □

4. Fuzzy Paracontinuous Maps and Some of Their Properties

Definition 4.1. *Let X and Y be fuzzy topological spaces. A map f from fts X to another fts Y is called*
(i) fuzzy paracontinuous(briefly fp_a-continuous) if $f^{-1}(\lambda)$ is a fuzzy open set on X for every fuzzy paraopen set λ on Y.
(ii)$*$-fuzzy paracontinuous(briefly $*$-fp_a-continuous) if $f^{-1}(\lambda)$ is a fuzzy paraopen set on X for every fuzzy open set λ on Y.
(iii) fuzzy parairresolute(briefly fp_a-irresolute) if $f^{-1}(\lambda)$ is a fuzzy paraopen set on X for every fuzzy paraopen set λ on Y.
(iv) fuzzy minimal paracontinuous(briefly fuzzy min-p_a-continuous) if $f^{-1}(\lambda)$ is a fuzzy paraopen set on X for every fuzzyminimal open set λ on Y.
(v) fuzzy maximal paracontinuous(briefly f.min-p_a-continuous) if $f^{-1}(\lambda)$ is a fuzzy paraopen set on X for every f.maximal open set λ on Y.

Theorem 4.1. Every fuzzy continuous map is fuzzy paracontinuous but not conversely.

Proof. Let $f : X \to Y$ be a fuzzy continuous map. To prove f is fuzzy paracontinuous. Let λ be any fuzzy paraopen set in Y. Since every fuzzy paraopen set is a fuzzy open set, λ is fuzzy open set in Y. Since f is a fuzzy continuous, $f^{-1}(\lambda)$ is fuzzy open set in X. Hence f is a fuzzy paracontinuous. □

Example 4.2. *Let μ_1,μ_2 be fuzzy sets on $X = \{a, b, c\}$ and let λ_1,λ_2,λ_3 be fuzzy sets on $Y = \{x, y, z\}$. Then $\mu_1 = \frac{0.2}{a} + \frac{0.3}{b} + \frac{0.0}{c}$, $\mu_2 = \frac{0.2}{a} + \frac{0.3}{b} + \frac{0.6}{c}$, $\lambda_1 = \frac{0.0}{x} + \frac{0.3}{y} + \frac{0.0}{z}$, $\lambda_2 = \frac{0.2}{x} + \frac{0.3}{y} + \frac{0.0}{z}$ and $\lambda_3 = \frac{0.2}{x} + \frac{0.3}{y} + \frac{0.6}{z}$ are defined as follows: Consider $\mathfrak{F}_1 = \{0_X, \mu_1, \mu_2, 1_X\}$,$\mathfrak{F}_2 = \{0_Y, \lambda_1, \lambda_2, \lambda_3, 1_Y\}$. Let $f : (X, \mathfrak{F}_1) \to (Y, \mathfrak{F}_2)$ be an identity maping. Then f is fuzzy paracontinuous but not fuzzy continuous map because for the fuzzy open set λ_1 on Y, $f^{-1}(\lambda_1) = \lambda_1$ which is not a fuzzy open set on $(X, \mathfrak{F}_1)$.*

Theorem 4.2. Every $*$-fuzzy paracontinuous is fuzzy continuous but not conversely.

Proof. Let $f : X \to Y$ be a $*$-fuzzy paracontinuous map. To prove f is fuzzy continuous. Let λ be a fuzzy open set in Y. Since f is $*$-fuzzy paracontinuous, $f^{-1}(\lambda)$ is fuzzy paraopen set in X. Since every fuzzy paraopen set is a fuzzy open set, $f^{-1}(\lambda)$ is fuzzy open set in X. Hence f is a fuzzy continuous. □

Example 4.3. *Let μ_1,μ_2 and $\mu_3$be fuzzy sets on $X = Y = \{a, b, c\}$. Then $\mu_1 = \frac{1.0}{a} + \frac{0.0}{b} + \frac{0.0}{c}$, $\mu_2 = \frac{1.0}{a} + \frac{0.8}{b} + \frac{0.0}{c}$ and $\mu_3 = \frac{1.0}{a} + \frac{0.8}{b} + \frac{0.5}{c}$ are defined as follows: Consider $\mathfrak{F}_1 = \{0_X, \mu_1, \mu_2, \mu_3, 1_X\}$, Let $f : X \to Y$ be an identity mapping. Then f is fuzzy continuous but not $*$-fuzzy paracontinuous mapping since for the fuzzy open set μ_3 on Y, $f^{-1}(\mu_3) = \mu_3$ which is not a fuzzy paraopen set on X.*

Theorem 4.3. Every $*$-fuzzy paracontinuous is fuzzy paracontinuous but not conversely.

Proof. The proof follows from Theorems 4.1 and 4.2. □

Example 4.4. *In Example 4.3, f is fuzzy paracontinuous map but it is not $*$-fuzzy paracontinuous map.*

Theorem 4.4. Every fuzzy parairresolute map is fuzzy paracontinuous but not conversely.

Proof. Let $f : X \to Y$ be a fuzzy parairresolute map. To prove f is fuzzy paracontinuous. Let λ be any fuzzy paraopen set in Y. Since f is fuzzy parairresolute, $f^{-1}(\lambda)$ is fuzzy paraopen set in X. Since every fuzzy paraopen set is a fuzzy open set, $f^{-1}(\lambda)$ is fuzzy open set in X. Hence f is a fuzzy paracontinuous map. □

Example 4.5. *As described in Example 4.3, consider* $\mathfrak{F}_3 = \{0_X, \mu_2, \mu_3, 1_X\}$ *and* $\mathfrak{F}_1 = \{0_Y, \mu_1, \mu_2, \mu_3, 1_Y\}$. *Let* $f : X \to Y$ *be an identity mapping. Then* f *is fuzzy paracontinuous but not fuzzy parairresolute mapping since for the fuzzy paraopen set* μ_2 *on* Y, $f^{-1}(\mu_2) = \mu_2$ *which is not a fuzzy paraopen set on* X.

Theorem 4.5. Every $*$-fuzzy paracontinuous is fuzzy parairresolute but not conversely.

Proof. Let $f : X \to Y$ be a fuzzy paracontinuous map. To prove f is fuzzy parairresolute. Let λ be a fuzzy paraopen set in Y. Since every fuzzy paraopen set is a fuzzy open set, λ is a fuzzy open set. Since f is $*$-fuzzy paracontinuous, $f^{-1}(\lambda)$ is fuzzy paraopen set in X. Hence f is a fuzzy parairresolute map. □

Example 4.6. *In Example 4.3,* f *is fuzzy parairresolute map but it is not* $*$*-fuzzy paracontinuous map.*

Remark. Fuzzy parairresolute and fuzzy continuous maps are independent of each other.

Example 4.7. *Let* μ_1,μ_2 *be fuzzy sets on* $X = \{a, b, c, d\}$ *and let* λ_1,λ_2,λ_3 *be fuzzy sets on* $Y = \{x, y, z, w\}$. *Then* $\mu_1 = \frac{0.0}{a} + \frac{0.0}{b} + \frac{0.0}{c} + \frac{1.0}{d}$, $\mu_2 = \frac{0.0}{a} + \frac{0.0}{b} + \frac{0.5}{c} + \frac{1.0}{d}$, $\mu_3 = \frac{0.0}{a} + \frac{0.3}{b} + \frac{0.5}{c} + \frac{1.0}{d}$, $\mu_4 = \frac{0.6}{a} + \frac{0.3}{b} + \frac{0.5}{c} + \frac{1.0}{d}$, $\lambda_1 = \frac{0.0}{x} + \frac{0.0}{y} + \frac{0.5}{z} + \frac{0.0}{w}$, $\lambda_2 = \frac{0.0}{x} + \frac{0.0}{y} + \frac{0.5}{z} + \frac{1.0}{w}$, $\lambda_3 = \frac{0.0}{x} + \frac{0.3}{y} + \frac{0.5}{z} + \frac{1.0}{w}$, *are defined as follows: Consider* $\mathfrak{F}_1 = \{0_X, \mu_1, \mu_2, \mu_3, \mu_4, 1_X\}$,$\mathfrak{F}_2 = \{0_Y, \lambda_1, \lambda_2, \lambda_3, 1_Y\}$. *Let* $f : X \to Y$ *be an identity maping. Then* f *is fuzzy parairresolute but not fuzzy continuous because for the fuzzy open set* λ_1 *on* Y, $f^{-1}(\lambda_1) = \lambda_1$ *which is not a fuzzy open set on* X.
Let μ_1,μ_2, μ_3 *be fuzzy sets on* $X = \{a, b, c\}$ *and let* λ_1,λ_2,λ_3 *be fuzzy sets on* $Y = \{x, y, z\}$. *Then* $\mu_1 = \frac{0.1}{a} + \frac{0.1}{b} + \frac{0.1}{c}$, $\mu_2 = \frac{0.2}{a} + \frac{0.2}{b} + \frac{0.2}{c}$, $\mu_3 = \frac{0.6}{a} + \frac{0.6}{b} + \frac{0.6}{c}$, $\lambda_1 = \frac{0.1}{x} + \frac{0.0}{y} + \frac{0.1}{z}$, $\lambda_2 = \frac{0.6}{x} + \frac{0.0}{y} + \frac{0.6}{z}$, $\lambda_3 = \frac{0.6}{x} + \frac{0.6}{y} + \frac{0.6}{z}$ *are defined as follows: Consider* $\mathfrak{F}_1 = \{0_X, \mu_1, \mu_2, \mu_3, 1_X\}$,$\mathfrak{F}_2 = \{0_Y, \lambda_1, \lambda_1, \lambda_3, 1_Y\}$. *Let* $f : X \to Y$ *be a fuzzy mapping defined as* $f(a) = f(b) = f(c) = y$. *Then* f *is fuzzy continuous but not fuzzy parairreolute because for the fuzzy paraopen set* λ_2 *on* Y, $f^{-1}(\lambda_3) = 0_X$ *which is not a fuzzy paraopen set on* X.

Theorem 4.6. Every fuzzy minimal paracontinuous map is fuzzy minimal continuous but not conversely.

Proof. Let $f : X \to Y$ be a fuzzy minimal paracontinuous map. To prove f is fuzzy minimal continuous. Let ϑ be any fuzzy minimal open set in Y. Since f is fuzzy minimal paracontinuous, $f^{-1}(\vartheta)$ is fuzzy paraopen set in X. Since every fuzzy paraopen set is a fuzzy open set,$f^{-1}(\vartheta)$ is a fuzzy open set in X. Hence f is a fuzzy minimal continuous. □

Example 4.8. *From Example 4.3,* f *is fuzzy minimal continous but it is not a fuzzy minimal precontinuous, since for the fuzzy minimal open* μ_1 *on* Y, $f^{-1}(\mu_1) = \mu_1$ *which is not a fuzzy paraopen set on* X.

Remark. Fuzzy minimal paracontinuous and fuzzy paracontinuous(resp. fuzzy continuous) are independent of each other.

Example 4.9. *Let* μ_1,μ_2 *be fuzzy sets on* $X = \{a, b, c\}$ *and let* λ_1,λ_2,λ_3 *be fuzzy sets on* $Y = \{x, y, z\}$. *Then* $\mu_1 = \frac{0.5}{a} + \frac{0.0}{b} + \frac{0.0}{c}$, $\mu_2 = \frac{0.5}{a} + \frac{0.6}{b} + \frac{0.0}{c}$, $\mu_3 = \frac{0.5}{a} + \frac{0.6}{b} + \frac{0.1}{c}$, $\lambda_1 = \frac{0.5}{x} + \frac{0.6}{y} + \frac{0.0}{z}$, $\lambda_2 = \frac{0.5}{x} + \frac{0.6}{y} + \frac{0.9}{z}$, $\lambda_3 = \frac{0.5}{x} + \frac{0.7}{y} + \frac{0.0}{z}$ *and* $\lambda_4 = \frac{0.5}{x} + \frac{0.7}{y} + \frac{0.9}{z}$ *are defined as follows: Consider* $\mathfrak{F}_1 = \{0_X, , \mu_1, \mu_2, \mu_3, 1_X\}$,$\mathfrak{F}_2 = \{0_Y, \lambda_1, \lambda_1, \lambda_3, \lambda_4, 1_Y\}$. *Let* $f : X \to Y$ *be an identity maping. Then* f *is fuzzy minimal paracontinuous but not fuzzy paracontinuous(resp. fuzzy continuous) map because for the fuzzy paraopen set* λ_3 *on* Y, $f^{-1}(\lambda_3) = \lambda_3$ *which is not a fuzzy open set on* X. *In Example 4.3,* f *is fuzzy paracontinuous but not fuzzy minimal paracontinuous.*

Theorem 4.7. Every fuzzy maximal paracontinuous is fuzzy maximal continuous but not conversely.

Proof. Let $f : X \to Y$ be a fuzzy maximal paracontinuous map. To prove f is fuzzy maximal continuous. Let β be any fuzzy maximal open set in Y. Since f is fuzzy maximal paracontinuous, $f^{-1}(\beta)$ is fuzzy paraopen set in X. Since every fuzzy paraopen set is a fuzzy open set,$f^{-1}(\beta)$ is a fuzzy open set in X. Hence f is a fuzzy maximal continuous. □

Example 4.10. *In Example 4.3, f is fuzzy maximal continuous but it is not fuzzy maximal paracontinuous map.*

Remark. Fuzzy maximal paracontinuous and fuzzy paracontinuous(resp. fuzzy continuous) are independent of each other.

Example 4.11. *Let μ_1,μ_2 be fuzzy sets on $X = \{a, b, c, d\}$ and let λ_1,λ_2,λ_3 be fuzzy sets on $Y = \{x, y, z, w\}$. Then $\mu_1 = \frac{0.0}{a} + \frac{0.0}{b} + \frac{0.0}{c} + \frac{0.8}{d}$, $\mu_2 = \frac{0.0}{a} + \frac{0.0}{b} + \frac{0.6}{c} + \frac{0.8}{d}$, $\mu_3 = \frac{0.0}{a} + \frac{0.5}{b} + \frac{0.6}{c} + \frac{0.8}{d}$, $\mu_4 = \frac{0.1}{a} + \frac{0.5}{b} + \frac{0.6}{c} + \frac{0.8}{d}$, $\lambda_1 = \frac{0.0}{x} + \frac{0.0}{y} + \frac{0.4}{z} + \frac{0.0}{w}$, $\lambda_2 = \frac{0.0}{x} + \frac{0.0}{y} + \frac{0.4}{z} + \frac{0.8}{w}$, $\lambda_3 = \frac{0.0}{x} + \frac{0.5}{y} + \frac{0.6}{z} + \frac{0.8}{w}$, are defined as follows: Consider $\mathfrak{F}_1 = \{0_X, \mu_1, \mu_2, \mu_3, \mu_4, 1_X\}, \mathfrak{F}_2 = \{0_Y, \lambda_1, \lambda_2, \lambda_3, 1_Y\}$. Let $f : X \to Y$ be an identity maping. Then f is fuzzy maximal paracontinuous but not fuzzy paracontinuous(resp. fuzzy continuous) map because for the fuzzy paraopen set λ_2 on Y, $f^{-1}(\lambda_2) = \lambda_2$ which is not a fuzzy open set on X. In Example 4.3, f is fuzzy paracontinuous(resp. fuzzy continuous) but not fuzzy maximal paracontinuous.*

Remark. Fuzzy minimal paracontinuous and fuzzy maximal paracontinuous are independent of each other.

Example 4.12. *In Example 4.9, f is fuzzy minimal paracontinuous map but it is not fuzzy maximal paracontinuous map. From Example 4.11, f is fuzzy maximal paracontinuous map but it is not fuzzy minimal paracontinuous map.*

Theorem 4.8. Let X and Y be fuzzy topological spaces. A map $f : X \to Y$ is a fuzzy paracontinuous if and only if the inverse image of each fuzzy paraclosed set in Y is a fuzzy closed set in X.

Proof. The proof follows from the definition and fact that the complement of fuzzy paraopen set is fuzzy paraclosed set. □

Theorem 4.9. Let X and Y be fuzzy topological spaces and A be a nonempty fuzzy subset of X. If $f : X \to Y$ is fuzzy paracontinuous then the restriction map $f_A : A \to Y$ is a fuzzy paracontinuous.

Proof. Let $f : X \to Y$ be a fuzzy paracontinuous map and $A \subset X$. To prove f_A is a fuzzy paracontinuous. Let λ be a fuzzy paraopen set in Y. Since f is fuzzy paracontinuous, $f^{-1}(\lambda)$ is a fuzzy open set in X. By the definition of relative topology $f_A^{-1}(\lambda) = A \wedge f^{-1}(\lambda)$. Therefore $A \wedge f^{-1}(\lambda)$ is a fuzzy open set in A. Hence f_A is a fuzzy paracontinuous. □

Remark. The composition of fuzzy paracontinuous maps need not be fuzzy paracontinuous.

Example 4.13. *Let $X = Y = Z = \{a, b, c, d\}$ and the fuzzy sets $\mu_1 = \frac{0.0}{a} + \frac{0.0}{b} + \frac{0.1}{c} + \frac{0.0}{d}$, $\mu_2 = \frac{0.0}{a} + \frac{0.0}{b} + \frac{0.1}{c} + \frac{0.5}{d}$, $\mu_3 = \frac{0.0}{a} + \frac{0.6}{b} + \frac{0.1}{c} + \frac{0.5}{d}$ and $\mu_4 = \frac{0.4}{a} + \frac{0.6}{b} + \frac{0.1}{c} + \frac{0.5}{d}$ are defined as follows: Consider $\mathfrak{F}_1 = \{0_X, \mu_1, \mu_2, 1_X\}$, $\mathfrak{F}_2 = \{0_Y, \mu_1, \mu_2, \mu_3, 1_Y\}$ and $\mathfrak{F}_3 = \{0_Z, \mu_1, \mu_3, \mu_4, 1_Z\}$. Let $f : X \to Y$ and $g : Y \to Z$ be identity mappings. Then f and g are fuzzy paracontinuous maps $g \circ f : X \to Z$ is not fuzzy paracontinuous, since for the fuzzy paraopen set μ_3 in Z, $f^{-1}(\mu_3) = \mu_3$ which is not fuzzy open set in X.*

Theorem 4.10. If $f : X \to Y$ is fuzzy continuous and $g : Y \to Z$ is fuzzy paracontinuous. Then $g \circ f : X \to Z$ is a fuzzy paracontinuous.

Proof. Let ϑ be any fuzzy paraopen set in Z. Since g is fuzzy paracontinuous, $g^{-1}(\vartheta)$ is a fuzzy open set in Y. Again since f is fuzzy continuos, $f^{-1}(g^{-1}(\vartheta)) = (g \circ f)^{-1}(\vartheta)$ is a fuzzy open set in X. Hence $g \circ f$ is a fuzzy paracontinuous. □

Theorem 4.11. Let X and Y be fuzzy topological spaces. A map $f : X \to Y$ is $*$-fuzzy paracontinuous if and only if the inverse image of each fuzzy closed set in Y is a fuzzy paraclosed set in X.

Proof. The proof follows from the definition and fact that the complement of fuzzy paraopen set is fuzzy paraclosed set. □

Remark. Let X and Y be fuzzy topological spaces. If $f : X \to Y$ is fuzzy $*$-fuzzy paracontinuous, then the restriction map $f_A : A \to Y$ need not be $*$-fuzzy paracontinuous.

Example 4.14. *Let $X = Y = Z = \{a, b, c\}$ and the fuzzy sets $\mu_1 = \frac{0.6}{a} + \frac{0.0}{b} + \frac{0.0}{c}$, $\mu_2 = \frac{0.6}{a} + \frac{0.2}{b} + \frac{0.0}{c}$ and $\mu_3 = \frac{0.6}{a} + \frac{0.2}{b} + \frac{0.5}{c}$ are defined as follows: Consider $\mathfrak{F} = \{0_X, \mu_1, \mu_2, \mu_3, 1_X\}$ and $\mathfrak{F}_1 = \{0_Y, \mu_2, 1_Y\}$. Let $\beta = \frac{0.0}{a} + \frac{0.2}{b} + \frac{0.8}{c}$ be a fuzzy set with $\mathfrak{F}_\beta = \{0_\beta, \mu_4, \mu_5, \mu_6, \beta\}$ where $\mu_4 = \frac{0.0}{a} + \frac{0.2}{b} + \frac{0.0}{c}$ and $\mu_5 = \frac{0.0}{a} + \frac{0.2}{b} + \frac{0.5}{c}$. Let $f : X \to Y$ be an identity map. Then f is $*$-fuzzy paracontinuous but $f_\beta : \mathfrak{F}_\beta \to Y$ is not a $*$-fuzzy paracontinuous, since for the fuzzy open set μ_2 in Y, $f^{-1}(\mu_2) = \mu_2$ which is not a fuzzy paraopen set in $\mathfrak{F}_\beta$.*

Theorem 4.12. If $f : X \to Y$ and $g : Y \to Z$ is $*$-fuzzy paracontinuous. Then $g \circ f : X \to Z$ is a $*$-fuzzy paracontinuous.

Proof. Let ϑ be any fuzzy paraopen set in Z. Since every fuzzy paraopen set is a fuzzy open set, $g^{-1}(\vartheta)$ is a fuzzy paraopen set in Y. Again since f is fuzzy $*$-fuzzy paracontinuous, $f^{-1}(g^{-1}(\vartheta)) = (g \circ f)^{-1}(\vartheta)$ is a fuzzy paraopen set in X. Hence $g \circ f$ is a $*$-fuzzy paracontinuous. □

Theorem 4.13. If $f : X \to Y$ is fuzzy paracontinuous and $g : Y \to Z$ is $*$-fuzzy paracontinuous. Then $g \circ f : X \to Z$ is a fuzzy paracontinuous(resp. fuzzy continuous).

Proof. Let ϑ be any fuzzy paraopen(resp. fuzzy open) set in Z. Since every fuzzy paraopen set is a fuzzy open set, ϑ is a fuzzy open set in Z. Since g is a $*$-fuzzy paracontinuous, $g^{-1}(\vartheta)$ is a fuzzy paraopen set in Y. Again since f is fuzzy paracontinuous, $f^{-1}(g^{-1}(\vartheta)) = (g \circ f)^{-1}(\vartheta)$ is a fuzzy open set in X. Hence $g \circ f$ is fuzzy paracontinuous(resp. fuzzy continuous) map. □

Theorem 4.14. Let X and Y be fuzzy topological spaces. A map $f : X \to Y$ is fuzzy parairresolute if and only if the inverse image of each fuzzy are paraclosed set in Y is a fuzzy paraclosed set in X.

Proof. The proof follows from the definition and fact that the complement of fuzzy paraopen set is fuzzy paraclosed set. □

Remark. Let X and Y be fuzzy topological spaces. If $f : X \to Y$ is fuzzy parairresolute. Then the restriction map $f_A : A \to Y$ need not be fuzzy parairresolute.

Example 4.15. *In Example 4.3, let $\beta = \frac{0.0}{a} + \frac{0.0}{b} + \frac{0.6}{c}$ be a fuzzy set with $\mathfrak{F}_\beta = \{0_\beta, \mu_4, \beta\}$ where $\mu_4 = \frac{0.0}{a} + \frac{0.0}{b} + \frac{0.5}{c}$. Let $f : X \to Y$ be an identity map. Then f is fuzzy parairresolute but $f_\beta : \mathfrak{F}_\beta \to Y$ is not a fuzzy parairresolute, since for the fuzzy paraopen set μ_2 in Y, $f^{-1}(\mu_2) = \mu_2$ which is not a fuzzy paraopen set in $\mathfrak{F}_\beta$.*

Theorem 4.15. If $f : X \to Y$ is fuzzy paracontinuous and $g : Y \to Z$ is fuzzy parairresolute. Then $g \circ f : X \to Z$ is a fuzzy paracontinuous.

Proof. Let ϑ be a fuzzy paraopen set in Z. Since g is a fuzzy parairresolute $g^{-1}(\vartheta)$ is a fuzzy paraopen set in Y. Again since f is fuzzy paracontinuous, $f^{-1}(g^{-1}(\vartheta)) = (g \circ f)^{-1}(\vartheta)$ is a fuzzy open set in X. Hence $g \circ f$ is fuzzy paracontinuous. □

Theorem 4.16. If $f : X \to Y$ and $g : Y \to Z$ are fuzzy parairresolute. Then $g \circ f : X \to Z$ is a fuzzy parairresolute.

Proof. Let ϑ be a fuzzy paraopen set in Z. Since g is a fuzzy parairresolute $g^{-1}(\vartheta)$ is a fuzzy paraopen set in Y. Again since f is fuzzy parairresolute, $f^{-1}(g^{-1}(\vartheta)) = (g \circ f)^{-1}(\vartheta)$ is a fuzzy paraopen set in X. Hence $g \circ f$ is fuzzy paracontinuous. □

Theorem 4.17. If $f : X \to Y$ is $*$-fuzzy paracontinuous and $g : Y \to Z$ is fuzzy parairresolute. Then $g \circ f : X \to Z$ is a fuzzy parairresolute.

Proof. Let ϑ be a fuzzy paraopen set in Z. Since g is a fuzzy parairresolute, $g^{-1}(\vartheta)$ is a fuzzy paraopen set in Y. Since every fuzzy paraopen set is a fuzzy open set, we have $g^{-1}(\vartheta)$ is a fuzzy open set in Y. Again since f is $*$-fuzzy paracontinuous, $f^{-1}(g^{-1}(\vartheta)) = (g \circ f)^{-1}(\vartheta)$ is a fuzzy paraopen set in X. Hence $g \circ f$ is fuzzy parairresolute. □

Theorem 4.18. If $f : X \to Y$ is fuzzy parairresolute and $g : Y \to Z$ is $*$-fuzzy paracontinuous. Then $g \circ f : X \to Z$ is a fuzzy parairresolute.

Proof. Let ϑ be a fuzzy paraopen set in Z. Since every fuzzy paraopen set is a fuzzy open set, ϑ is a fuzzy open set in Z. Since g is a fuzzy paracontinuous, $g^{-1}(\vartheta)$ is a fuzzy paraopen set in Y. Again Since f is fuzzy parairresolute, $f^{-1}(g^{-1}(\vartheta)) = (g \circ f)^{-1}(\vartheta)$ is a fuzzy paraopen set in X. Hence $g \circ f$ is fuzzy parairresolute mapping. □

Theorem 4.19. Let X and Y be fuzzy topological spaces. A map $f : X \to Y$ is fuzzy minimal fuzzy paracontinuous if and only if the inverse image of each fuzzy maximal closed set in Y is a fuzzy paraclosed set in X.

Proof. The proof follows from the definition and fact that the complement of fuzzy minimal open set is fuzzy maximal closed set and the complement of fuzzy paraopen set is fuzzy paraclosed set. □

Remark. The composition of fuzzy minimal paracontinuous maps need not be a fuzzy minimal paracontinuous.

Example 4.16. *Let $X = Y = Z = \{a, b, c, d\}$ and the fuzzy sets $\alpha_1 = \frac{0.0}{a} + \frac{0.0}{b} + \frac{0.1}{c} + \frac{0.3}{d}$, $\alpha_2 = \frac{0.0}{a} + \frac{0.6}{b} + \frac{0.1}{c} + \frac{0.3}{d}$, $\alpha_3 = \frac{0.1}{a} + \frac{0.6}{b} + \frac{0.1}{c} + \frac{0.3}{d}$ and $\alpha_4 = \frac{0.2}{a} + \frac{0.6}{b} + \frac{0.1}{c} + \frac{0.3}{d}$ are defined as follows: Consider $\mathfrak{F}_1 = \{0_X, \alpha_1, \alpha_2, \alpha_3, 1_X\}$, $\mathfrak{F}_2 = \{0_Y, \alpha_2, \alpha_3, \alpha_4, 1_Y\}$ and $\mathfrak{F}_3 = \{0_Z, \alpha_3, \alpha_4, 1_Z\}$. Let $f : X \to Y$ and $g : Y \to Z$ be identity mappings. Then f and g are fuzzy minimal paracontinuous maps $g \circ f : X \to Z$ is not fuzzy minimal paracontinuous, since for the fuzzy minimal open set α_3 in Z, $f^{-1}(\alpha_3) = \alpha_3$ which is not fuzzy paraopen set in X.*

Theorem 4.20. If $f : X \to Y$ is fuzzy parairresolute and $g : Y \to Z$ is fuzzy minimal paracontinuous. Then $g \circ f : X \to Z$ is a fuzzy minimal paracontinuous.

Proof. Let η be a fuzzy minimal open set in Z. Since g is fuzzy minimal paracontinuous, $g^{-1}(\eta)$ is a fuzzy paraopen set in Y. Again since f is fuzzy parairresolute, $f^{-1}(g^{-1}(\eta)) = (g \circ f)^{-1}(\eta)$ is a fuzzy paraopen set in X. Hence $g \circ f$ is fuzzy minimal paracontinuous map. □

Theorem 4.21. If $f : X \to Y$ is fuzzy paracontinuous and $g : Y \to Z$ is fuzzy minimal paracontinuous. Then $g \circ f : X \to Z$ is a fuzzy minimal paracontinuous.

Proof. Let η be a fuzzy minimal open set in Z. Since g is fuzzy minimal paracontinuous, $g^{-1}(\eta)$ is a fuzzy paraopen set in Y. Again since f is fuzzy paracontinuous, $f^{-1}(g^{-1}(\eta)) = (g \circ f)^{-1}(\eta)$ is a fuzzy open set in X. Hence $g \circ f$ is fuzzy minimal paracontinuous mapping. □

Theorem 4.22. If $f : X \to Y$ is fuzzy parairresolute and $g : Y \to Z$ is $*$-fuzzy paracontinuous, then $g \circ f : X \to Z$ is a fuzzy minimal paracontinuous.

Proof. Let η be a fuzzy minimal open set in Z. Since every fuzzy minimal open set is a fuzzy open set, η is an open set in Z. Since f is $*$-fuzzy paracontinuous, $g^{-1}(\eta)$ is a fuzzy paraopen set in Y. Again since f is fuzzy parairresolute $f^{-1}(g^{-1}(\eta)) = (g \circ f)^{-1}(\eta)$ is a fuzzy paraopen set in X. Hence $g \circ f$ is fuzzy minimal paracontinuous. □

Theorem 4.23. Let X and Y be fuzzy topological spaces. A map $f : X \to Y$ is fuzzy maximal paracontinuous if and only if the inverse image of each fuzzy minimal closed set in Y is a fuzzy paraclosed set in X.

Proof. The proof follows from the definition and fact that the complement of fuzzy maximal open set is fuzzy minimal closed set and the complement of fuzzy paraopen set is fuzzy paraclosed set. □

Remark. The composition of fuzzy maximal paracontinuous maps need not be a fuzzy maximal paracontinuous.

Example 4.17. *Let $X = Y = Z = \{a, b, c, d\}$ and the fuzzy sets $\alpha_1 = \frac{0.0}{a} + \frac{0.1}{b} + \frac{0.0}{c} + \frac{0.0}{d}$, $\alpha_2 = \frac{0.0}{a} + \frac{0.1}{b} + \frac{0.8}{c} + \frac{0.0}{d}$, $\alpha_3 = \frac{0.0}{x} + \frac{0.1}{y} + \frac{0.8}{z} + \frac{0.2}{w}$ and $\alpha_4 = \frac{0.3}{x} + \frac{0.1}{y} + \frac{0.8}{z} + \frac{0.2}{w}$ are defined as follows: Consider $\mathfrak{F}_1 = \{0_X, \alpha_2, \alpha_3, \alpha_4, 1_X\}$, $\mathfrak{F}_2 = \{0_Y, \alpha_1, \alpha_2, \alpha_3, 1_Y\}$ and $\mathfrak{F}_3 = \{0_Z, \alpha_1, \alpha_2, 1_Z\}$. Let $f : X \to Y$ and $g : Y \to Z$ be identity mappings. Then f and g are fuzzy maximal paracontinuous maps $g \circ f : X \to Z$ is not fuzzy maximal paracontinuous, since for the fuzzy maximal open set α_2 in Z, $f^{-1}(\alpha_2) = \alpha_2$ which is not fuzzy paraopen set in X.*

Theorem 4.24. If $f : X \to Y$ is fuzzy parairresolute and $g : Y \to Z$ is fuzzy maximal paracontinuous. Then $g \circ f : X \to Z$ is a fuzzy maximal paracontinuous.

Proof. Let γ be a fuzzy maximal open set in Z. Since g is fuzzy maximal paracontinuous, $g^{-1}(\gamma)$ is a fuzzy paraopen set in Y. Again since f is fuzzy parairresolute, $f^{-1}(g^{-1}(\gamma)) = (g \circ f)^{-1}(\gamma)$ is a fuzzy paraopen set in X. Hence $g \circ f$ is fuzzy maximal paracontinuous. □

Theorem 4.25. If $f : X \to Y$ is fuzzy paracontinuous and $g : Y \to Z$ is fuzzy maximal paracontinuous, then $g \circ f : X \to Z$ is a fuzzy maximal continuous.

Proof. Let γ be a fuzzy maximal open set in Z. Since g is fuzzy maximal paracontinuous, $g^{-1}(\gamma)$ is a fuzzy paraopen set in Y. Again since f is fuzzy paracontinuous, $f^{-1}(g^{-1}(\gamma)) = (g \circ f)^{-1}(\gamma)$ is a fuzzy open set in X. Hence $g \circ f$ is fuzzy maximal continuous. □

Theorem 4.26. If $f : X \to Y$ is fuzzy parairresolute and $g : Y \to Z$ is $*$-fuzzy paracontinuous, then $g \circ f : X \to Z$ is a fuzzy maximal paracontinuous.

Proof. Let γ be a fuzzy maximal open set in Z. Since every fuzzy maximal open set is a fuzzy open set, γ is a fuzzy open set in Z. Since g is $*$-fuzzy paracontinuous, $g^{-1}(\gamma)$ is a fuzzy paraopen set in Y. Again since f is fuzzy parairresolute, $f^{-1}(g^{-1}(\gamma)) = (g \circ f)^{-1}(\gamma)$ is a fuzzy paraopen set in X. Hence $g \circ f$ is fuzzy maximal paracontinuous. □

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