ANNALS OF COMMUNICATIONS IN MATHEMATICS
Volume 3, Number 4 (2020), 285-292
ISSN: 2582-0818

STABILITY OF FINITE VARIABLE QUARTIC FUNCTIONAL EQUATION IN CLASSICAL METHODS

K. TAMILVANAN, K. LOGANATHAN* AND G. BALASUBRAMANIAN

ABSTRACT. In this work, we investigate the Hyers-Ulam stability by using direct and fixed point methods for the quartic functional equation

$$\begin{aligned}\sum_{b=1}^{p}\phi\left(-v_b+\sum_{a=1;a\neq b}^{p}v_a\right) = \; & 4\sum_{1\leq a<b<c\leq p}\phi\left(v_a+v_b+v_c\right)+(-4p+14)\sum_{a=1;a\neq b}^{p}\phi\left(v_a+v_b\right)\\ & +2\sum_{a=1;a\neq b}^{p}\phi\left(v_a-v_b\right)+(p-8)\phi\left(\sum_{a=1}^{p}v_a\right)+\sum_{b=1}^{p}\phi\left(2v_b\right)\\ & +(2p^2-14p+14)\sum_{a=1}^{p}\phi(v_a)\end{aligned}$$

for positive integer $p \geq 3$.

1. INTRODUCTION

The stability problem of a functional equation became first posed by way of Ulam [13] regarding the stability of group homomorphism which become responded by means of Hyers [6] for Banach spaces. Hyers theorem became generalized through Aoki [2] for additive mapping and through Rassias [11] for linear mappings by using considering an unbounded Cauchy difference. Rassias [11] has provided plenty of have an impact on in the improvement of what we name generalized Hyers-Ulam stability of functional equations.

A generalization of the Th. M. Rassias theorem became acquired with the aid of P. Gavruta cite4through replacing the unbounded Cauchy difference by using a wellknown control function within the spirit of Rassias technique. Won-Gil Park and Jae-Hyeong Bae cite11, delivered the subsequent functional equation

$$f(x+2y)+f(x-2y)=4f(x+y)+4f(x-y)+24f(y)-6f(x) \tag{1.1}$$

and that they mounted the general solution of the functional equation (1.1). It is straightforward to look that the function $f(x)=x^4$ is a solution of the functional equation (ref1.1). Therefore, it is natural that (1.1) is referred to as a quartic functional equation and each

2010 *Mathematics Subject Classification.* 39B52, 39B72, 39B82.
Key words and phrases. Quartic Functional Equation; Fixed Point; Hyers-Ulam Stability; Banach Space.
Received: October 02, 2020. Accepted: December 12, 2020. Published: December 31, 2020.
*Corresponding author.

solution of the quartic functional equation is stated to be quartic mapping. Numerous authors are inspect the stableness for the functional equations in banach and numerous spaces which offers an concept to develop this paper (see [1, 3, 4, 6, 7, 9, 10, 12]).

The aim of this paper is to obtain the Hyers-Ulam stability by using direct and fixed point methods for the quartic functional equation

$$\begin{aligned}\sum_{b=1}^{p}\phi\left(-v_b+\sum_{a=1;a\neq b}^{p}v_a\right) = \quad & 4\sum_{1\leq a<b<c\leq p}\phi\left(v_a+v_b+v_c\right)+(-4p+14)\sum_{a=1;a\neq b}^{p}\phi\left(v_a+v_b\right)\\ & +2\sum_{a=1;a\neq b}^{p}\phi\left(v_a-v_b\right)+(p-8)\phi\left(\sum_{a=1}^{p}v_a\right)+\sum_{b=1}^{p}\phi\left(2v_b\right)\\ & +(2p^2-14p+14)\sum_{a=1}^{p}\phi(v_a) \qquad (1.2)\end{aligned}$$

for positive integer $p \geq 3$ in Banach space.

Theorem (Alternative of fixed point): Suppose that for a complete generalized metric space (A, d) and a strictly contractive mapping $\Gamma : A \to A$ with Lipschitz constant L. Then, for each given element $u \in A$ either

(B1) $d(\Gamma^i u, \Gamma^{i+1}u) = +\infty \quad \forall i \geq 0$, or

(B2) There exists natural number i_0 such that

i) $d(\Gamma^i u, \Gamma^{i+1}u) < \infty \quad \forall i \geq i_0$;

ii) the sequence $(\Gamma^i u)$ is convergent to a fixed point v^* of Γ;

iii) v^* is the unique fixed point of Γ in the set $B = \{v \in A; d(\Gamma^{i_0}u, v) < \infty\}$;

iv) $d(v^*, v) \leq \frac{1}{1-L}d(v, \Gamma v) \quad \forall v \in B$.

Consider E be a normed space and F be a Banach space. For notational handiness, we define a function $\phi : E \to F$ by

$$\begin{aligned}D\phi\left(v_1,v_2,\cdots,v_p\right) \quad & =\sum_{b=1}^{p}\phi\left(-v_b+\sum_{a=1;a\neq b}^{p}v_a\right)-4\sum_{1\leq a<b<c\leq p}\phi\left(v_a+v_b+v_c\right)\\ & -(-4p+14)\sum_{a=1;a\neq b}^{p}\phi\left(v_a+v_b\right)-2\sum_{a=1;a\neq b}^{p}\phi\left(v_a-v_b\right)\\ & -(p-8)\phi\left(\sum_{a=1}^{p}v_a\right)-\sum_{b=1}^{p}\phi\left(2v_b\right)-(2p^2-14p+14)\sum_{a=1}^{p}\phi(v_a)\end{aligned}$$

for all $v_1, v_2, \cdots, v_p \in E$.

2. Stability Result for (1.2) in Banach space using Direct Method

Theorem 2.1. *Let $i \in \{-1, 1\}$. Let $\zeta : E^p \to [0, \infty)$ be a function such that* $\sum_{r=0}^{\infty} \frac{\zeta(2^{ri}v_1, 2^{ri}v_2, \cdots, 2^{ri}v_p)}{2^{4ri}}$ *converges in $\mathbb{R}$ and*

$$\lim_{r\to\infty}\frac{\zeta(2^{ri}v_1,2^{ri}v_2,\cdots,2^{ri}v_p)}{2^{4ri}}=0 \quad \forall\, v_1,v_2,\cdots,v_p\in E. \qquad (2.1)$$

If $\phi : E \to F$ be a function fulfils

$$||D\phi(v_1,v_2,\cdots,v_p)|| \leq \zeta(v_1,v_2,\cdots,v_p) \quad \forall\, v_1,v_2,\cdots,v_p\in E, \qquad (2.2)$$

then there exist a unique quartic function $Q_4 : E \to F$ which fulfils (1.2) and

$$||\phi(v) - Q_4(v)|| \leq \frac{1}{16} \sum_{r=\frac{1-i}{2}}^{\infty} \frac{\nu(2^{ri})}{2^{4ri}} \tag{2.3}$$

where $\nu(v) = \zeta(v, 0, \cdots, 0) \quad \forall v \in E$. The function Q_4 is given by

$$Q_4(v) = \lim_{r\to\infty} \frac{\phi(2^{ri}v)}{2^{4ri}} \quad \forall v \in E. \tag{2.4}$$

Proof. Assume that $i = 1$. Replacing $(v_1, v_2, \cdots, v_p)$ by $(v, 0, \cdots, 0)$ in (2.2), we get

$$\|16\phi(v) - \phi(2v)\| \leq \zeta(v, 0, \cdots, 0) \quad \forall v \in E. \tag{2.5}$$

It follows from (2.5) that

$$\|\frac{\phi(2v)}{2^4} - \phi(v)\| \leq \frac{1}{2^4}\zeta(v, 0, \cdots, 0) \quad \forall v \in E. \tag{2.6}$$

Switching v through $2v$ and dividing by 2^4 in (2.6), we arrive

$$\|\frac{\phi(2^2v)}{2^8} - \frac{\phi(2v)}{2^4}\| \leq \frac{1}{2^8}\zeta(2v, 0, \cdots, 0) \quad \forall v \in E. \tag{2.7}$$

Adding (2.6) and (2.7), we have

$$\|\frac{\phi(2^2v)}{2^4} - \phi(v)\| \leq \frac{1}{2^4}\left(\zeta(v, 0, \cdots, 0) + \frac{\zeta(2v, 0, \cdots, 0)}{2^4}\right) \quad \forall v \in E.$$

In general for any integer $s > 0$, one can easy to verify that

$$\|\frac{\phi(2^sv)}{2^{4s}} - \phi(v)\| \leq \frac{1}{2^4}\sum_{r=0}^{\infty}\frac{\nu(2^rv)}{2^{4r}} \quad \forall v \in E. \tag{2.8}$$

In order to show the convergence of the sequence $\{\frac{\phi(2^sv)}{2^{4s}}\}$, replacing v by 2^tv and dividing 2^{4t} in (2.8), for $s, t > 0$, we get

$$\|\frac{\phi(2^{s+t}v)}{2^{4(s+t)}} - \frac{\phi(2^tv)}{2^{4v}}\| \leq \frac{1}{2^4}\sum_{r=0}^{s-1}\frac{\nu(2^{r+t}v)}{2^{4(r+t)}} \to 0 \;\; as \;\; t \to \infty \tag{2.9}$$

for all $v \in E$. Therefore, $\{\frac{\phi(2^sv)}{2^{4s}}\}$ is a Cauchy sequence. As F is complete, there exists a mapping $Q_4 : E \to F$ such that

$$Q_4(v) = \lim_{s\to\infty}\frac{\phi(2^sv)}{2^{4s}} \quad \forall v \in E.$$

Passing $s \to \infty$ in (2.8) we see that (2.3) holds for $v \in E$. To show that Q_4 fulfils (1.2), switching $(v_1, v_2, \cdots, v_p)$ by $(2^tv, 2^tv, \cdots, 2^tv)$ and dividing 2^{4t} in (2.2), we arrive

$$\frac{1}{2^{4t}}\|Q_4(2^tv, 2^tv, \cdots, 2^tv)\| \leq \frac{1}{2^{4t}}\zeta(2^tv, 2^tv, \cdots, 2^tv) \quad \forall v_1, v_2, \cdots, v_p \in E.$$

Letting $t \to \infty$ in above inequality and using the definition of $Q_4(v)$, we see that $Q_4(v_1, v_2, \cdots, v_p) = 0$. Hence Q_4 satisfies (1.2) for all $v \in E$. To show that Q_4 is unique. Let R_4 be the another quartic mapping fulfilling (1.2) and (2.3), then

$$\|Q_4(v) - R_4(v)\| \leq \frac{1}{2^{4t}}\{\|Q_4(2^tv) - \phi(2^t)\| + \|\phi(2^tv) - R_4(2^tv))\|\}$$
$$\leq \frac{1}{2^4}\sum_{r=0}^{\infty}\frac{\nu(2^{r+t}v)}{2^{4(r+t)}} \to 0 \;\; as \;\; t \to \infty$$

for all $v \in E$. Hence Q_4 is unique. Similarly, we can derive the stability results for $i = -1$. $\square$

Corollary 2.2. *Let a and b be a non-negative real numbers. Let $\phi : E \to F$ be a function fulfilling*

$$||D\phi(v_1, v_2, v_3, \cdots, v_p)|| \le \begin{cases} a, \\ a(\sum_{j=1}^{p} ||v_j||^b), \\ a(\prod_{j=1}^{p} ||v_j||^b + \sum_{j=1}^{p} ||v_j||^{pb}), \end{cases}$$

for all $v_1, v_2, \cdots, v_p \in E$. Then there exists a unique quartic function $Q_4 : E \to F$ such that

$$||\phi(v) - Q_4(v)|| \le \begin{cases} \frac{a}{|15|}, \\ \frac{a||v||^b}{|2^4 - 2^b|}; & b \ne 4, \\ \frac{a||v||^{pb}}{|2^4 - 2^{pb}|}; & b \ne \frac{4}{p}, \end{cases}$$

for all $v \in E$.

3. Stability Result for (1.2) in Banach space using Fixed Point Method

Theorem 3.1. *Let $\phi : E \to F$ be a mapping for which there exists a function $\zeta : E^p \to [0, \infty)$ with the condition*

$$lim_{r\to\infty} \frac{\zeta(\tau_\delta^r v_1, \tau_\delta^r v_2, \cdots, \tau_\delta^r v_p)}{\tau_\delta^{4r}} = 0 \tag{3.1}$$

where

$$\tau_\delta = \begin{cases} 2, \text{ if} & \delta = 0; \\ \frac{1}{2}, \text{ if} & \delta = 1; \end{cases}$$

such that the functional inequality

$$||D\phi(v_1, v_2, \cdots, v_p)|| \le \zeta(v_1, v_2, \cdots, v_p) \quad \forall v_1, v_2, \cdots, v_p \in E. \tag{3.2}$$

If there exist $L = L(\delta)$ such that the function

$$v \to \Upsilon(v) = \zeta\left(\frac{v}{2}, 0, \cdots, 0\right)$$

has the property,

$$\frac{1}{\tau_\delta^4}\Upsilon(\tau_\delta v) = L\Upsilon(v) \quad \forall v \in E. \tag{3.3}$$

Then there exists a unique quartic function $Q_4 : E \to F$ fulfilling (1.2) and

$$||\phi(v) - Q_4(v)|| \le \frac{L^{1-\delta}}{1-L}\Upsilon(v) \tag{3.4}$$

holds for all $v \in E$.

Proof. Suppose $d = \{s/s : E \to F, s(0) = 0)\}$ and define the generalized metric on Φ. $d(s,t) = \inf\{r \in (0,\infty) : ||s(v) - t(v)|| \le r\Upsilon(v), v \in E\}$. It is easy to see that (Φ, d) is complete. Define $\Psi : \Phi \to \Phi$ by

$$\Psi s(v) = \frac{1}{\tau_\delta^4} s(\tau_\delta v) \quad \forall v \in \Phi.$$

Now $s, t \in \Phi$,

$$d(s,t) \le r \Rightarrow ||s(v) - t(v)|| \le r\Upsilon(v) \quad \forall v \in E.$$

$$\Rightarrow ||\frac{1}{\tau_\delta^4}s(\tau_\delta v) - \frac{1}{\tau_\delta^4}t(\tau_\delta v)|| \leq \frac{1}{\tau_\delta^4}r\Upsilon(\tau_\delta v) \quad \forall v \in E.$$
$$\Rightarrow ||\Psi s(v) - \Psi t(v)|| \leq r\Upsilon(v) \quad \forall v \in E.$$
$$\Rightarrow d(\Psi s, \Psi t) \leq rL.$$

This implies $d(\Psi s, \Psi t) \leq Ld(s,t) \quad \forall s, t \in \Phi$. (i,e.,) Ψ is strictly contractive mapping on with Lipschtiz constant L. Switching $(v_1, v_2, \cdots, v_p)$ by $(v, 0, \cdots, 0)$ in (3.2), we obtain

$$||\phi(2v) - 16\phi(v)|| \leq \zeta(v, 0, \cdots, 0) \quad \forall v \in E. \tag{3.5}$$

It is follows from (3.5) that

$$||\phi(v) - \frac{\phi(2v)}{16}|| \leq \frac{\zeta(v, 0, \cdots, 0)}{16} \quad \forall v \in E. \tag{3.6}$$

Utilizing (3.3) for $\delta = 0$, we have

$$||\phi(v) - \frac{\phi(2v)}{16}|| \leq \Upsilon(v) \quad \forall v \in E.$$
$$i.e., d(\phi, \Psi\phi) \leq 1 \Rightarrow d(\phi, \Psi\phi) \leq 1 = L = L^1 < \infty.$$

Again interchanging $v = \frac{v}{2}$ in (3.5) and (3.6), we get

$$||\phi(v) - 16\phi\left(\frac{v}{2}\right)|| \leq \zeta\left(\frac{v}{2}, 0, \cdots, 0\right)$$

and

$$||\phi(v) - 16\phi\left(\frac{v}{2}\right)|| \leq \zeta\left(\frac{v}{2}, 0, \cdots, 0\right) \quad \forall v \in E. \tag{3.7}$$

Utilizing (3.3) for $\delta = 0$, we have

$$||\phi(v) - \frac{\phi(2v)}{16}|| \leq L\Upsilon(v) \quad \forall v \in E. \tag{3.8}$$

(i.e.,) $d(\phi, \Psi\phi) \leq 1 \Rightarrow d(\phi, \Psi\phi) \leq 1 = L^0 < \infty$. In above case, we arrive

$$d(\phi, \Psi\phi) \leq L^{1-\delta}.$$

Therefore, $(B_2(i))$holds. By $(B_2(ii))$, it follows that there exists a fixed point Q_4 of Ψ in E, such that

$$Q_4(v) = lim_{r\to\infty}\frac{\phi(\tau_\delta^r v)}{\tau_\delta^{4r}} \quad \forall v \in E. \tag{3.9}$$

In order to prove $Q_4 : E \to F$ is quartic. Interchanging $(v_1, v_2, \cdots, v_p)$ through $(\tau_\delta^r v_1, \tau_\delta^r v_2, \cdots, \tau_\delta^r v_p)$ in (3.2) and dividing by τ_δ^{4r}, it follows from (3.1) and (3.9), we see that Q_4 fulfils (1.2) for all $v_1, v_2, \cdots, v_p \in E$. Hence Q_4 fulfils (1.2). By $(B_2(iii))$, Q_4 is the unique fixed point of Ψ in the set, $F = \{\phi \in \Phi; d(\Psi\phi, Q_4) < \infty\}$. Utilizing the fixed point alternative result, Q_4 is the unique function such that,

$$||\phi(v) - Q_4(v)|| \leq r\Upsilon(v) \quad \forall v \in E, r > 0.$$

Finally, by $(B_2(iv))$, we reach

$$d(\phi, Q_4) \leq \frac{1}{1-L}d(\phi, \Psi\phi)$$
$$(i.e.,)\ d(\phi, Q_4) \leq \frac{L^{1-\delta}}{1-L}.$$

Hence, we conclude that

$$||\phi(v) - Q_4(v)|| \leq \frac{L^{1-\delta}}{1-L}\Upsilon(v) \quad \forall v \in E.$$

□

Corollary 3.2. *Let $\phi : E \to F$ be a mapping and there exists a real numbers a and b such that*

$$||D\phi(v_1, v_2, \cdots, v_p)|| \leq \begin{cases} a, \\ a(\sum_{j=1}^{p} ||v_j||^b), \\ a(\prod_{j=1}^{p} ||v_j||^b + \sum_{j=1}^{p} ||v_j||^{pb}), \end{cases}$$

for all $v_1, v_2, \cdots, v_p \in E$. Then there exist a unique quartic function $Q_4 : E \to F$ such that

$$||\phi(v) - Q_4(v)|| \leq \begin{cases} \frac{a}{|15|}, \\ \frac{a||v||^b}{|2^4 - 2^b|}; \quad b \neq 4, \\ \frac{a||v||^{pb}}{|2^4 - 2^{pb}|}; \quad b \neq \frac{4}{p}, \end{cases}$$

for all $v \in E$.

Proof. Setting

$$\zeta(v_1, v_2, \cdots, v_p) \leq \begin{cases} a, \\ a(\sum_{j=1}^{p} ||v_j||^b), \\ a(\prod_{j=1}^{p} ||v_j||^b + \sum_{j=1}^{p} ||v_j||^{pb}), \end{cases}$$

for all $v_1, v_2, \cdots, v_p \in E$. Now

$$\frac{\zeta(\tau_\delta^r v_1, \tau_\delta^r v_2, \cdots, \tau_\delta^r v_p)}{\tau_\delta^{4r}} \leq \begin{cases} \frac{a}{\tau_\delta^{4r}}, \\ \frac{a}{\tau_\delta^{4r}} \left\{ \sum_{j=1}^{p} ||\tau_\delta^r v_j||^b \right\}, \\ \frac{a}{\tau_\delta^{4r}} \left\{ \prod_{j=1}^{p} ||\tau_\delta^r v_j||^{pb} + \sum_{j=1}^{p} ||\tau_\delta^r v_j||^{pb} \right\}, \end{cases}$$

$$= \begin{cases} \longrightarrow \quad 0 \quad as \quad r \longrightarrow \infty, \\ \longrightarrow \quad 0 \quad as \quad r \longrightarrow \infty, \\ \longrightarrow \quad 0 \quad as \quad r \longrightarrow \infty, \end{cases}$$

i.e., (3.5) is holds. But we have $\Upsilon(v) = \zeta\left(\frac{v}{2}, 0, \cdots, 0\right)$. Hence

$$\Upsilon(v) = \zeta\left(\frac{v}{2}, 0, \cdots, 0\right) = \begin{cases} a, \\ \frac{a||v||^b}{2^b}, \\ \frac{a||v||^{pb}}{5^{pb}}, \end{cases}$$

$$\frac{1}{\tau_\delta^4} \Upsilon(\tau_\delta v) = \begin{cases} \frac{a}{\tau_\delta^4}, \\ \frac{1}{\tau_\delta^4} \frac{a||v||^b}{2^b}, \\ \frac{1}{\tau_\delta^4} \frac{a||v||^{pb}}{2^{pb}}, \end{cases}$$

$$= \begin{cases} \tau_\delta^{-4} \Upsilon(v), \\ \tau_\delta^{b-4} \Upsilon(v), \\ \tau_\delta^{pb-4} \Upsilon(v), \end{cases}$$

for all $v \in E$. Hence the inequality (1.2) holds for
$L = 2^{-4}$ if $\delta = 0$ *and* $L = \frac{1}{2^{-4}}$ if $\delta = 1$.
$L = 2^{b-4}$ *for* $b < 4$ if $\delta = 0$ *and* $L = \frac{1}{2^{b-4}}$ *for* $b > 4$ if $\delta = 1$.
$L = 2^{pb-4}$ *for* $b < \frac{4}{p}$ if $\delta = 0$ *and* $L = \frac{1}{2^{pb-4}}$ *for* $b > \frac{4}{p}$ if $\delta = 1$.
Now, from (3.5) we prove the following cases:

Case1:$L = 2^{-4}$ if $\delta = 0$.

$$||\phi(v) - Q_4(v)| \leq \frac{L^{1-\delta}}{1-L}\Upsilon(v) = \frac{(2^{-4})}{1-2^{-4}}a = \frac{a}{15}.$$

Case2:$L = \frac{1}{2^{-4}}$ if $\delta = 1$.

$$||\phi(v) - Q_4(v)|| \leq \frac{L^{1-\delta}}{1-L}\Upsilon(v) = \frac{1}{1-2^4}a = \frac{-a}{16}.$$

Case3:$L = 2^4$ for $b < 4$ if $\delta = 0$.

$$||\phi(v) - Q_4(v)|| \leq \frac{L^{1-\delta}}{1-L}\Upsilon(v) = \frac{2^{b-4}}{1-2^{b-4}}\frac{a||v||^b}{2^b} = \frac{a||v||^b}{2^4-2^b}.$$

Case4:$L = \frac{1}{2^{b-4}}$ for $b > 4$ if $\delta = 1$.

$$||\phi(v) - Q_4(v)|| \leq \frac{L^{1-\delta}}{1-L}\Upsilon(v) = \frac{1}{1-\frac{1}{2^{b-4}}}\frac{a||v||^b}{2^b} = \frac{a||v||^b}{2^b-2^4}.$$

Case5:$L = 2^{pb-4}$ for $b < \frac{4}{p}$ if $\delta = 0$.

$$||\phi(v) - Q_4(v)|| \leq \frac{L^{1-\delta}}{1-L}\Upsilon(v) = \frac{2^{pb-4}}{1-2^{pb-4}}\frac{a||v||^{pb}}{2^{pb}} = \frac{a||v||^{pb}}{2^4-2^{pb}}.$$

Case6:$L = \frac{1}{2^{pb-4}}$ for $b > \frac{4}{p}$ if $\delta = 1$.

$$||\phi(v) - Q_4(v)|| \leq \frac{L^{1-\delta}}{1-L}\Upsilon(v) = \frac{1}{1-\frac{1}{2^{pb-4}}}\frac{a||v||^{pb}}{2^{pb}} = \frac{a||v||^{pb}}{2^{pb}-2^4}.$$

Hence the proof. □

REFERENCES

[1] J. Aczel, and J. Dhombres. Functional Equations in Several Variables, Cambridge univ. press, (1989).
[2] T. Aoki. On the stability of the linear transformation in Banach Spaces, J. Math. Soc. Japan 2 (1950), p. 64–66.
[3] J. H. Bae, and W. G. Park. On the solution of a bi-Jensen functional equation and its stability, Bull. Korean Math. Soc. 43(3) (2006), p. 499–507.
[4] A. Bodaghi. Stability of a quartic functional equation, Sci. World J. Art. ID 752146, (2014) 9 pages.
[5] P. Gavruta. A generalization of the Hyers-Ulam-Rassias stability of approximately additive mappings, J. Math. Anal. Appl. 184 (1994), p. 431–436.
[6] D. H. Hyers. On the stability of the linear functional equation, Proc. Natl. Acad. Sci. U.S.A. 27(1941), p. 222–224.
[7] Y. H. Lee, and K. W. Jun. On the stability of approximately additive mappings , Proc. Amer. Math. Soc. 128(2000), p. 1361–1369.
[8] W. G. Park, and J. H. Bae. On the stability a bi-quartic functional equation , Nonlinear Anal. 62(4)(2005), p. 643-654.
[9] C. Park, K. Tamilvanan, G. Balasubramanian, B. Noori, and A. Najati. On a functional equation that has the quadratic-multiplicative property, Open Mathematics, 18(2020), p. 837–845.
[10] C. Park, K. Tamilvanan, B. Noori, M. B. Moghimi, and A. Najati. Fuzzy normed spaces and stability of a generalized quadratic functional equation, AIMS Mathematics, 5(6)(2020), p. 7161-7174.
[11] Th. M. Rassias. On the stability of the linear mapping in Banach spaces, Proc. Amer. Math. Soc. 72(1978), p. 297–300.
[12] K. Tamilvanan, J. R. Lee, and C. Park. Hyers-Ulam stability of a finite variable mixed type quadratic-additive functional equation in quasi-Banach spaces, AIMS Math., 5(6)(2020), p. 5993-6005.
[13] S. M. Ulam. Problems in Modern Mathematics, Science Editions, Wiley, New York (1964).

K. TAMILVANAN
GOVERNMENT ARTS COLLEGE FOR MEN, KRISHNAGIRI-635 001, TAMIL NADU, INDIA.
Email address: tamiltamilk7@gmail.com

K. LOGANATHAN
DEPARTMENT OF MATHEMATICS, ERODE ARTS & SCIENCE COLLEGE, ERODE - 638 009, TAMIL NADU, INDIA.
Email address: loganathankaruppusamy304@gmail.com

G. BALASUBRAMANIAN
GOVERNMENT ARTS COLLEGE FOR MEN, KRISHNAGIRI-635 001, TAMIL NADU, INDIA.
Email address: gbs_geetha@yahoo.com