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GENERALIZATIONS OF FUZZY QUASI OPEN SETS AND CONNECTEDNESS BETWEEN FUZZY SETS IN FUZZY BITOPOLOGICAL SPACES

G. SARAVANAKUMAR, A. VADIVEL*, S. MURUGAMBIGAI AND M. KAMARAJ

Abstract. In this paper we introduce and study fuzzy quasi e (resp. e^*, a, β, δs and δp)-open sets, fuzzy quasi e (resp. e^*, a, β, δs and δp)-closed sets, fuzzy quasi e (resp. e^*, a, β, δs and δp)-connectedness between fuzzy sets fuzzy quasi e (resp. e^*, a, β, δs and δp)–separated sets in fuzzy bitopological spaces.

1. INTRODUCTION

The concept of fuzzy set was introduced by Zadeh [29] provided a natural foundation for building new branches in mathematics. Fuzzy sets have applications in many fields such as information [23] and control [24]. In 1968 chang [6] introduced fuzzy topological space using fuzzy sets. Kandil [14] defined and studied the concept of fuzzy bitopological spaces as a generalization of bitopological spaces [16] in fuzzy setting. Since then many results from classical topology are being extended in both fuzzy topological and fuzzy bitopological spaces ([3], [4], [12], [14], [15], [18]-[21], [28]) and their properties were also investigated. The initiations of e-open sets, e^*-open sets and a-open sets in topological spaces are due to Ekici [[9],[10],[11]]. In fuzzy topology, e-open sets were introduced by Seenivasan in 2015 [22]. In 1971 Datta [7] introduced and studied quasi semiopen sets in bitopological spaces. Using it concepts of fuzzy quasi semiopen sets and connectedness between fuzzy sets in fuzzy bitopological spaces were defined and studied [26]. The purpose of this paper is to generalize some of the concepts of [8, 13, 25] in fuzzy bitopological spaces using fuzzy e (resp. e^*, a, β, δs and δp)-open sets.

2. PRELIMINARIES

We recall the following definition.

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*Corresponding author.

Definition 2.1. A fuzzy subset λ in an fts (X, τ) is called fuzzy regular open (fro, for short) [2] if $\lambda = IntCl(\lambda)$ and a regular closed set if $\lambda = ClInt(\lambda)$.

Definition 2.2. [22] The fuzzy δ interior of subset λ of X is the union of all fuzzy regular open sets contained in λ and fuzzy δ closure of subset λ of X is the intersection of all fuzzy regular closed sets containing λ.

Definition 2.3. [27] A subset λ is called fuzzy δ open if $\lambda = \delta Int(\lambda)$. The complement of fuzzy δ open set is called fuzzy δ closed (i.e., $\lambda = \delta Cl(\lambda)$.)

Definition 2.4. A subset λ is called fuzzy δ-pre open [1] (resp. fuzzy δ-semi open [17], fuzzy e-open [22]) if $\lambda \leq IntCl_\delta(\lambda)$ (resp. $\lambda \leq ClInt_\delta(\lambda)$, $\lambda \leq IntCl_\delta(\lambda) \vee ClInt_\delta(\lambda)$)

Definition 2.5. [1, 17, 22] The complement of a fuzzy δ-preopen set (resp. fuzzy δ-semiopen set, fuzzy e-open) is called fuzzy δ-preclosed (resp. fuzzy δ-semiclosed, fuzzy e- closed).

Definition 2.6. [1, 17, 22] The intersection of all fuzzy δ-preclosed (resp. fuzzy δ-semiclosed, fuzzy e-closed) sets containing λ is called fuzzy δp (resp. δs, e)-closure of λ and is denoted by $f\delta pCl(\lambda)$ (resp. $f\delta sCl(\lambda)$, $feCl(\lambda)$ and the union of all fuzzy δ-preopen (resp. fuzzy δ-semiopen, fuzzy e-open) sets contained in λ is called fuzzy δp (resp. δs, e)-interior of λ and is denoted by $f\delta pInt(\lambda)$ (resp. $f\delta sInt(\lambda)$, $feInt(\lambda)$).

Definition 2.7. A fuzzy bitopological space [14] (in short fbts) in an ordered triple (X, τ_1, τ_2) where τ_1 and τ_2 are fuzzy topologies on X and the members of τ_1 (or τ_2) are called τ_1-fuzzy (or τ_2-fuzzy) open sets.

A fuzzy set λ in a fbts (X, τ_1, τ_2) is called τ_i-fuzzy closed if its complement $1-\lambda$ or λ' is τ_i-fuzzy open for $i = 1, 2$.

Definition 2.8. [5] In a fbts (X, τ_1, τ_2) a fuzzy set λ is said to be fuzzy quasi-open (in short fqo) if $\lambda = \mu \vee \eta$ for some $\mu \in \tau_1$ and $\eta \in \tau_2$.

In this paper we shall denote the family of τ_i-fuzzy e (resp. e^*, a, β, δs and δp)-open (τ_i-fuzzy e (resp. e^*, a, β, δs and δp)-closed) sets in fbts (X, τ_1, τ_2) by τ_i^{feo} (resp. $\tau_i^{fe^*o}$, τ_i^{fao}, $\tau_i^{f\beta o}$, $\tau_i^{f\delta so}$ and $\tau_i^{f\delta po}$) (τ_i^{fec} (resp. $\tau_i^{fe^*c}$, τ_i^{fac}, $\tau_i^{f\beta c}$, $\tau_i^{f\delta sc}$ and $\tau_i^{f\delta pc}$)) for $i = 1,\ 2$.

3. FUZZY QUASI e (RESP. e^*, a, β, δs AND δp)-OPEN SETS IN FUZZY BITOPOLOGICAL SPACES

Definition 3.1. In a fbts (X, τ_1, τ_2) a fuzzy set λ is said to be fuzzy quasi e (resp. e^*, a, β, δs and δp)-open (in short $fqeo$) (resp., in short fqe^*o, $fqao$, $fq\beta o$, $fq\delta so$ and $fq\delta po$)) if $\lambda = \mu \vee \eta$ for some $\mu \in \tau_1^{feo}$ and $\eta \in \tau_2^{feo}$ (resp. $\mu \in \tau_1^{fe^*o}$ and $\eta \in \tau_2^{fe^*o}$, $\mu \in \tau_1^{fao}$ and $\eta \in \tau_2^{fao}$, $\mu \in \tau_1^{f\beta o}$ and $\eta \in \tau_2^{f\beta o}$, $\mu \in \tau_1^{f\delta so}$ and $\eta \in \tau_2^{f\delta so}$, $\mu \in \tau_1^{f\delta po}$ and $\eta \in \tau_2^{f\delta po}$).

Remark.

(i) Every fuzzy quasi-open set is fuzzy quasi e-open sets.
(ii) Every fuzzy quasi-open set is fuzzy quasi δ-semi open sets.

(iii) Every fuzzy quasi-open set is fuzzy quasi δ-pre open sets.
(iv) Every fuzzy quasi δ-semi open and quasi δ-pre open set is fuzzy quasi e-open sets.
(v) Every fuzzy quasi a-open set is fuzzy quasi e-open sets.
(vi) Every fuzzy quasi a-open set is fuzzy quasi β-open sets.
(vii) Every fuzzy quasi e-open set is fuzzy quasi β-open sets.
(viii) Every fuzzy quasi e^*-open set is fuzzy quasi β-open sets.

But the converse is not true as shown in the following Examples.

Example 3.2. Let $X = \{a,\ b,\ c,\ d\}$, $\tau_1 = \{0,\ 1,\ \mu_1,\ \mu_2,\ \mu_3,\ \mu_4\}$ and $\tau_2 = \{0,\ 1,\ \eta_1,\ \eta_2,\ \eta_3\}$ where $\mu_1,\ \mu_2,\ \mu_3,\ \mu_4,\ \eta_1,\ \eta_2,\ \eta_3 : X \to [0,1]$ are defined as follows: $\mu_1 = \frac{0}{a} + \frac{0.1}{b} + \frac{1}{c} + \frac{1}{d}$, $\mu_2 = \frac{0}{a} + \frac{0.4}{b} + \frac{0}{c} + \frac{0}{d}$, $\mu_3 = \frac{0}{a} + \frac{0.4}{b} + \frac{1}{c} + \frac{1}{d}$, $\mu_4 = \frac{0}{a} + \frac{0.1}{b} + \frac{0}{c} + \frac{0}{d}$, $\eta_1 = \frac{0}{a} + \frac{0.3}{b} + \frac{0}{c} + \frac{0}{d}$, $\eta_2 = \frac{0}{a} + \frac{0}{b} + \frac{1}{c} + \frac{1}{d}$, $\eta_3 = \frac{0}{a} + \frac{0.3}{b} + \frac{1}{c} + \frac{1}{d}$. Then (X, τ_1, τ_2) is a fbts with fuzzy topologies τ_1 and τ_2. Let λ be a fuzzy set in X defined as $\lambda : X \to [0,\ 1]$ such that (i) $\lambda = \frac{0}{a} + \frac{0.7}{b} + \frac{1}{c} + \frac{1}{d}$. But $\lambda = \mu_5 \vee \eta_5$ where $\mu_5 = \frac{0}{a} + \frac{0.5}{b} + \frac{1}{c} + \frac{1}{d} \in \tau_1^{fqeo}$ and $\eta_5 = \frac{0}{a} + \frac{0.7}{b} + \frac{0}{c} + \frac{0}{d} \in \tau_2^{fqeo}$, therefore, λ is $fqeo$, but it is not fuzzy quasi-open and fuzzy quasi a-open. (ii) $\lambda = \frac{1}{a} + \frac{0.4}{b} + \frac{0.4}{c} + \frac{0.4}{d}$. But $\lambda = \mu \vee \eta$ where $\mu = \frac{1}{a} + \frac{0.4}{b} + \frac{0}{c} + \frac{0}{d} \in \tau_1^{fq\beta o}$ and $\eta = \frac{0}{a} + \frac{0}{b} + \frac{0.4}{c} + \frac{0.4}{d} \in \tau_2^{fq\beta o}$, therefore, λ is $fq\beta o$, but it is not fqe^*o and $fqao$. (iii) $\lambda = \frac{0.7}{a} + \frac{0.7}{b} + \frac{1}{c} + \frac{1}{d}$. But $\lambda = \mu \vee \eta$ where $\mu = \frac{0}{a} + \frac{0.5}{b} + \frac{1}{c} + \frac{1}{d} \in \tau_1^{fqeo}$ and $\eta = \frac{0.7}{a} + \frac{0.7}{b} + \frac{0}{c} + \frac{0}{d} \in \tau_2^{fqeo}$, therefore, λ is $fqeo$, but it is not $fq\delta po$ and $fq\delta so$. (iv) $\lambda = \frac{0}{a} + \frac{0.4}{b} + \frac{1}{c} + \frac{1}{d}$. But $\lambda = \mu \vee \eta$ where $\mu = \frac{0}{a} + \frac{0.4}{b} + \frac{1}{c} + \frac{1}{d} \in \tau_1^{fq\delta po}$ (resp. $\in \tau_1^{fq\delta so}$) and $\eta = \frac{0}{a} + \frac{0.2}{b} + \frac{0}{c} + \frac{0}{d} \in \tau_2^{fq\delta po}$ (resp. $\in \tau_1^{fq\delta so}$) therefore, λ is $fq\delta po$ and $fq\delta so$, but it is not fqo.

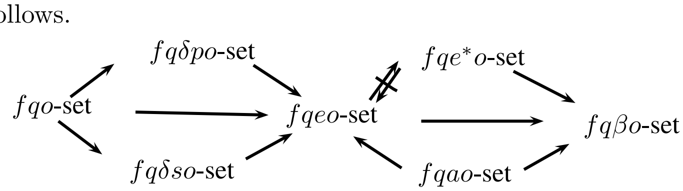
Example 3.3. Let $X = \{a,\ b,\ c\}$, $\tau_1 = \{0,\ 1,\ \mu_1,\ \mu_2,\ \mu_3,\ \mu_4\}$ and $\tau_2 = \{0,\ 1,\ \eta_1,\ \eta_2,\ \eta_3\}$ where $\mu_1,\ \mu_2,\ \mu_3,\ \mu_4,\ \eta_1,\ \eta_2,\ \eta_3 : X \to [0,1]$ are defined as follows: $\mu_1 = \frac{0.7}{a} + \frac{1}{b} + \frac{0}{c}$, $\mu_2 = \frac{0.2}{a} + \frac{0}{b} + \frac{1}{c}$, $\mu_3 = \frac{0.7}{a} + \frac{1}{b} + \frac{1}{c}$, $\mu_4 = \frac{0.2}{a} + \frac{0}{b} + \frac{0}{c}$, $\eta_1 = \frac{0}{a} + \frac{0.3}{b} + \frac{0}{c}$, $\eta_2 = \frac{0}{a} + \frac{0}{b} + \frac{1}{c}$, $\eta_3 = \frac{0}{a} + \frac{0.3}{b} + \frac{1}{c}$. Then (X, τ_1, τ_2) is a fbts with fuzzy topologies τ_1 and τ_2. Let λ be a fuzzy set in X defined as $\lambda : X \to [0,\ 1]$ such that $\lambda = \frac{0.3}{a} + \frac{0}{b} + \frac{0.2}{c}$. But $\lambda = \mu \vee \eta$ where $\mu = \frac{0.2}{a} + \frac{0}{b} + \frac{0.1}{c} \in \tau_1^{fq\beta o}$ and $\eta = \frac{0.3}{a} + \frac{0}{b} + \frac{0.2}{c} \in \tau_2^{fq\beta o}$, therefore, λ is $fq\beta o$, but it is not $fqeo$.

Remark. Fuzzy quasi e-open set and fuzzy quasi e^*-open sets are independent.

Example 3.4. In Example 3.2, $\lambda = \frac{1}{a} + \frac{0.4}{b} + \frac{0.4}{c} + \frac{0.4}{d}$. But $\lambda = \mu \vee \eta$ where $\mu = \frac{1}{a} + \frac{0.4}{b} + \frac{0}{c} + \frac{0}{d} \in \tau_1^{fqeo}$ and $\eta = \frac{0}{a} + \frac{0}{b} + \frac{0.4}{c} + \frac{0.4}{d} \in \tau_2^{fqeo}$, therefore, λ is $fqeo$, but it is not fqe^*o.

Example 3.5. In Example 3.3, $\lambda = \frac{0.3}{a} + \frac{0}{b} + \frac{0.2}{c}$. But $\lambda = \mu \vee \eta$ where $\mu = \frac{0.2}{a} + \frac{0}{b} + \frac{0.1}{c} \in \tau_1^{fe^*o}$ and $\eta = \frac{0.3}{a} + \frac{0}{b} + \frac{0.2}{c} \in \tau_2^{fe^*o}$, therefore, λ is fqe^*o, but it is not $fqeo$.

All above discussed interrelation can be put together in an arrow diagram is given as follows.



Proposition 3.1. A fuzzy set λ in a fbts (X, τ_1, τ_2) is

(i) $fqeo$ iff $\lambda = eInt_{\tau_1}(\lambda) \vee eInt_{\tau_2}(\lambda)$.
(ii) fqe^*o iff $\lambda = e^*Int_{\tau_1}(\lambda) \vee e^*Int_{\tau_2}(\lambda)$.
(iii) $fqao$ iff $\lambda = aInt_{\tau_1}(\lambda) \vee aInt_{\tau_2}(\lambda)$.
(iv) $fq\beta o$ iff $\lambda = \beta Int_{\tau_1}(\lambda) \vee \beta Int_{\tau_2}(\lambda)$.
(v) $fq\delta so$ iff $\lambda = \delta sInt_{\tau_1}(\lambda) \vee \delta sInt_{\tau_2}(\lambda)$.
(vi) $fq\delta po$ iff $\lambda = \delta pInt_{\tau_1}(\lambda) \vee \delta pInt_{\tau_2}(\lambda)$.

Proof. Prove the first part only the other cases are similar.

(i) Suppose λ is $fqeo$ set in fbts (X, τ_1, τ_2). Then, by definition, we have

$$\lambda = \mu \vee \eta \tag{3.1}$$

for some $\mu \in \tau_1^{feo}$ and $\eta \in \tau_2^{feo}$.

From (3.1), $\mu = eInt_{\tau_1}(\lambda), \eta = eInt_{\tau_2}(\lambda)$ and so $\lambda = eInt_{\tau_1}(\lambda) \vee eInt_{\tau_2}(\lambda)$.

Conversely, suppose that $\lambda = eInt_{\tau_1}(\lambda) \vee eInt_{\tau_2}(\lambda) = \lambda_1 \vee \lambda_2$ (say) where $\lambda_1 = eInt_{\tau_1}(\lambda)$ and $\lambda_2 = eInt_{\tau_2}(\lambda)$. Clearly λ_1 and λ_2 are τ_1-fuzzy e-open and τ_2-fuzzy e-open sets respectively. Therefore λ is $fqeo$ set. □

Remark.

(i) Every τ_1-fuzzy e-open (or τ_2-fuzzy e-open) sets is $fqeo$ set.
(ii) Every τ_1-fuzzy a-open (or τ_2-fuzzy a-open) sets is $fqao$ set.
(iii) Every τ_1-fuzzy e^*-open (or τ_2-fuzzy e^*-open) sets is fqe^*o set.
(iv) Every τ_1-fuzzy β-open (or τ_2-fuzzy β-open) sets is $fq\beta o$ set.
(v) Every τ_1-fuzzy δp-open (or τ_2-fuzzy δp-open) sets is $fq\delta po$ set.
(vi) Every τ_1-fuzzy δs-open (or τ_2-fuzzy δs-open) sets is $fq\delta so$ set.

But the converse is not true as shown in the following Example ??.

Example 3.6. In Example 3.2, let λ be a fuzzy set in X defined as $\lambda : X \to [0,\ 1]$ such that $\lambda = \frac{0}{a} + \frac{0.7}{b} + \frac{1}{c} + \frac{1}{d}$. But $\lambda = \mu_5 \vee \eta_4$ where $\mu_5 \in \tau_1^{feo}$ and $\eta_4 \in \tau_2^{feo}$, therefore, λ is $fqeo$, but it is not τ_2-fuzzy e-open set.

Example 3.7. In Examples 3.3 and 3.9, let λ be a fuzzy set in X defined as $\lambda : X \to [0,\ 1]$ such that (i) $\lambda = \frac{0.6}{a} + \frac{1}{b} + \frac{1}{c}$. But $\lambda = \mu_5 \vee \eta_4$ where $\mu_5 \in \tau_1^{fe^*o}$ and $\eta_4 \in \tau_2^{fe^*o}$, therefore, λ is fqe^*o, but it is not τ_2-fuzzy e^*-open set.
(ii) $\lambda = \frac{0.5}{a} + \frac{1}{b} + \frac{1}{c}$. But $\lambda = \mu_6 \vee \eta_5$ where $\mu_6 \in \tau_1^{f\beta o}$ and $\eta_5 \in \tau_2^{f\beta o}$, therefore, λ is $fq\beta o$, but it is not τ_2-fuzzy β-open set.
(iii) $\lambda = \frac{0.1}{a} + \frac{0}{b} + \frac{1}{c}$. But $\lambda = \mu_7 \vee \eta_2$ where $\mu_7 \in \tau_1^{fao}$ and $\eta_2 \in \tau_2^{fao}$, therefore, λ is $fqao$, but it is not τ_2-fuzzy a-open set.
(iv) $\lambda = \frac{0.5}{a} + \frac{1}{b} + \frac{1}{c}$. But $\lambda = \mu_6 \vee \eta_6$ where $\mu_6 \in \tau_1^{f\delta po}$ and $\eta_6 \in \tau_2^{f\delta po}$, therefore, λ is $fq\delta po$, but it is not τ_2-fuzzy δp-open set.
(v) $\lambda = \frac{0.1}{a} + \frac{0}{b} + \frac{1}{c}$. But $\lambda = \mu_6 \vee \eta_2$ where $\mu_6 \in \tau_1^{f\delta so}$ and $\eta_2 \in \tau_2^{f\delta so}$, therefore, λ is $fq\delta so$, but it is not τ_2-fuzzy δs-open set.

Proposition 3.2. Arbitrary union of $fqeo$ (resp. fqe^*o, $fqao$, $fq\beta o$, $fq\delta so$, $fq\delta po$) sets is a $fqeo$ (resp. fqe^*o, $fqao$, $fq\beta o$, $fq\delta so$, $fq\delta po$) set.

Proof. Let $\lambda_i,\ i \in I$ be fqeo sets in a fts X. To prove that $\bigvee_{i\in I} \lambda_i$ is fqeo set. Let $\bigvee_{i\in I} \lambda_i = \lambda_1 \vee \lambda_2 \vee \cdots = (\mu_1 \vee \eta_1) \vee (\mu_2 \vee \eta_2) \cdots = \bigvee_{i\in I} (\mu_i \vee \eta_i)$ as $\lambda_1,\ \lambda_2, \ldots$ are fqeo set. Then $\bigvee_{i\in I} \lambda_i = \bigvee_{i\in I} (\mu_i \vee \eta_i) = (\bigvee_{i\in I} \mu_i) \vee (\bigvee_{i\in I} \eta_i)$ for some $\bigvee_{i\in I} \mu_i \in T_1^{feo}$ and

$\bigvee_{i\in I} \eta_i \in T_2^{feo}$. Hence $\bigvee_{i\in I} \lambda_i$ is $fqeo$ set. Thus arbitrary union of $fqeo$ sets is a fqeo set. □

The proof of the other cases are similar.

Remark. If λ_1 and λ_2 are two $fqeo$ (resp. fqe^*o, $fqao$, $fq\beta o$, $fq\delta so$, $fq\delta po$) sets in a fbts (X, τ_1, τ_2) then $\lambda_1 \wedge \lambda_2$ need not be $fqeo$ (resp. fqe^*o, $fqao$, $fq\beta o$, $fq\delta so$, $fq\delta po$) as shown in the following Examples 3.8 and 3.9.

Example 3.8. In Example 3.2, let λ_1, λ_2, μ_5, μ_6, η_4 and η_5 be two fuzzy sets on X and are defined as λ_1, λ_2 , μ_5, μ_6, η_4, $\eta_5 : X \to [0,\ 1]$ such that $\lambda_1 = \frac{0}{a} + \frac{0.7}{b} + \frac{1}{c} + \frac{1}{d}$, $\lambda_2 = \mu_6 = \frac{0.1}{a} + \frac{0.5}{b} + \frac{1}{c} + \frac{1}{d}$, $\mu_5 = \frac{0}{a} + \frac{0.5}{b} + \frac{1}{c} + \frac{1}{d}$, $\eta_4 = \frac{0}{a} + \frac{0.7}{b} + \frac{0}{c} + \frac{0}{d}$ and $\eta_5 = \frac{0}{a} + \frac{0.4}{b} + \frac{0}{c} + \frac{1}{d}$. As $\lambda_1 = \mu_5 \vee \eta_4$, $\mu_5 \in \tau_1^{feo}$ and $\eta_4 \in \tau_2^{feo}$, and $\lambda_2 = \mu_6 \vee \eta_5$, $\mu_6 \in \tau_1^{feo}$, and $\eta_5 \in \tau_2^{feo}$, λ_1 and λ_2 are fuzzy quasi e-open ($fqeo$) sets in X. However $\lambda_1 \wedge \lambda_2 = \lambda$ (say) where $\lambda = \frac{0}{a} + \frac{0.5}{b} + \frac{1}{c} + \frac{1}{d}$, is not equal to $\mu \vee \eta$ for some $\mu \in \tau_1^{feo}$ and $\eta \in \tau_2^{feo}$, which shows that $\lambda_1 \wedge \lambda_2$ is not $fqeo$ set in X. Hence the Remark 3

Example 3.9. In Example 3.3, let $\lambda_1 = \mu_3$, $\lambda_2 = \mu_5$, μ_6, μ_7, μ_8,μ_9, η_4, η_5 and η_6 be two fuzzy sets on X and are defined as λ_1, λ_2, $\eta_4 : X \to [0,\ 1]$ such that $\lambda_2 = \mu_5 = \frac{0.6}{a} + \frac{1}{b} + \frac{1}{c}$, $\mu_6 = \frac{0.5}{a} + \frac{1}{b} + \frac{1}{c}$, $\mu_7 = \frac{0.1}{a} + \frac{0}{b} + \frac{0}{c}$, $\mu_8 = \frac{0.9}{a} + \frac{1}{b} + \frac{0}{c}$, $\mu_9 = \frac{0.1}{a} + \frac{0}{b} + \frac{1}{c}$, $\eta_4 = \frac{0.2}{a} + \frac{0.1}{b} + \frac{0}{c}$, $\eta_5 = \frac{0.3}{a} + \frac{0.2}{b} + \frac{0}{c}$ and $\eta_6 = \frac{0}{a} + \frac{0.2}{b} + \frac{0}{c}$.

(i) As $\lambda_1 = \mu_3 \vee \eta_4$, $\mu_3 \in \tau_1^{fe^*o}$ and $\eta_4 \in \tau_2^{fe^*o}$, and $\lambda_2 = \mu_5 \vee \eta_4$, $\mu_5 \in \tau_1^{fe^*o}$, and $\eta_4 \in \tau_2^{fe^*o}$, λ_1 and λ_2 are fuzzy quasi e^*-open (fqe^*o) sets in X. However $\lambda_1 \wedge \lambda_2 = \lambda$ (say) where $\lambda = \frac{0.6}{a} + \frac{1}{b} + \frac{1}{c}$, is not equal to $\mu \vee \eta$ for some $\mu \in \tau_1^{fe^*o}$ and $\eta \in \tau_2^{fe^*o}$, which shows that $\lambda_1 \wedge \lambda_2$ is not fqe^*o set in X.

(ii) As $\lambda_1 = \mu_3 \vee \eta_5$, $\mu_3 \in \tau_1^{f\beta o}$ and $\eta_5 \in \tau_2^{f\beta o}$, and $\lambda_2 = \mu_6 \vee \eta_5$, $\mu_6 \in \tau_1^{f\beta o}$, and $\eta_5 \in \tau_2^{f\beta o}$, λ_1 and λ_2 are fuzzy quasi β-open ($fq\beta o$) sets in X. However $\lambda_1 \wedge \lambda_2 = \lambda$ (say) where $\lambda = \frac{0.5}{a} + \frac{1}{b} + \frac{1}{c}$, is not equal to $\mu \vee \eta$ for some $\mu \in \tau_1^{f\beta o}$ and $\eta \in \tau_2^{f\beta o}$, which shows that $\lambda_1 \wedge \lambda_2$ is not $fq\beta o$ set in X.

(iii) As $\lambda_1 = \mu_7 \vee \eta_2$, $\mu_7 \in \tau_1^{fao}$ and $\eta_2 \in \tau_2^{fao}$, and $\lambda_2 = \mu_2 \vee \eta_2$, $\mu_2 \in \tau_1^{fao}$, and $\eta_2 \in \tau_2^{fao}$, λ_1 and λ_2 are fuzzy quasi a-open ($fqao$) sets in X. However $\lambda_1 \wedge \lambda_2 = \lambda$ (say) where $\lambda = \frac{0.1}{a} + \frac{0}{b} + \frac{1}{c}$, is not equal to $\mu \vee \eta$ for some $\mu \in \tau_1^{fao}$ and $\eta \in \tau_2^{fao}$, which shows that $\lambda_1 \wedge \lambda_2$ is not $fqao$ set in X.

(iv) As $\lambda_1 = \mu_3 \vee \eta_6$, $\mu_3 \in \tau_1^{f\delta po}$ and $\eta_6 \in \tau_2^{f\delta po}$, and $\lambda_2 = \mu_6 \vee \eta_6$, $\mu_6 \in \tau_1^{f\delta po}$, and $\eta_6 \in \tau_2^{f\delta po}$, λ_1 and λ_2 are fuzzy quasi δp-open ($fq\delta po$) sets in X. However $\lambda_1 \wedge \lambda_2 = \lambda$ (say) where $\lambda = \frac{0.5}{a} + \frac{1}{b} + \frac{1}{c}$, is not equal to $\mu \vee \eta$ for some $\mu \in \tau_1^{f\delta po}$ and $\eta \in \tau_2^{f\delta po}$, which shows that $\lambda_1 \wedge \lambda_2$ is not $fq\delta po$ set in X.

(v) As $\lambda_1 = \mu_8 \vee \eta_2$, $\mu_8 \in \tau_1^{f\delta so}$ and $\eta_2 \in \tau_2^{f\delta so}$, and $\lambda_2 = \mu_9 \vee \eta_2$, $\mu_9 \in \tau_1^{f\delta so}$, and $\eta_2 \in \tau_2^{f\delta so}$, λ_1 and λ_2 are fuzzy quasi δs-open ($fq\delta so$) sets in X. However $\lambda_1 \wedge \lambda_2 = \lambda$ (say) where $\lambda = \frac{0.1}{a} + \frac{0}{b} + \frac{1}{c}$, is not equal to $\mu \vee \eta$ for some $\mu \in \tau_1^{f\delta so}$ and $\eta \in \tau_2^{f\delta so}$, which shows that $\lambda_1 \wedge \lambda_2$ is not $fq\delta so$ set in X. Hence the Remark 3

Definition 3.10. A fuzzy set λ in a fbts (X, τ_1, τ_2) is said to be fuzzy quasi e (resp. e^*, a, β, δp and δs)-closed (in short $fqec$) (resp. in short, fqe^*c, $fq\beta c$, $fqac$, $fq\delta sc$, $fq\delta pc$) iff its complement $1 - \lambda$ or $\lambda^{'}$ is $fqeo$ (in short fqe^*o, $fq\beta o$, $fqao$, $fq\delta so$, $fq\delta po$) set.

Remark. Every τ_1-fuzzy e (resp. e^*, a, β, δp and δs)-closed (or τ_2-fuzzy e (resp. e^*, a, β, δp and δs)-closed) set in any fbts (X, τ_1, τ_2) is fuzzy quasi e (resp. e^*, a, β, δp and δs)-closed.

Remark. Finite intersection of $fqec$ (resp. fqe^*c, $fqac$, $fq\beta c$, $fq\delta pc$ and $fq\delta sc$) sets is also a $fqec$ (resp. fqe^*c, $fqac$, $fq\beta c$, $fq\delta pc$ and $fq\delta sc$) set.

Definition 3.11. Let λ be a fuzzy set in fbts (X, τ_1, τ_2). Then fuzzy quasi e-closure and fuzzy quasi e-interior of λ, denoted by $fqeCl(\lambda)$ and $fqeInt(\lambda)$ respectively, are defined as follows:

(1) $fqeCl(\lambda) = \wedge\{\delta : \delta \text{ is } fqec \text{ set and } \delta \geq \lambda\}$
(2) $fqeInt(\lambda) = \vee\{\eta : \eta \text{ is } fqeo \text{ set and } \eta \leq \lambda\}$

In fact a fuzzy set λ in a fbts (X, τ_1, τ_2) is

(1) $fqec$ iff $\lambda = fqeCl(\lambda)$,
(2) $fqeo$ iff $\lambda = fqeInt(\lambda)$.

In a similar way we can define fqe^*Cl, $fqaCl$, $fq\beta Cl$, $fq\delta pCl$ and $fq\delta sCl$ (resp. fqe^*Int, $fqaInt$, $fq\beta Int$, $fq\delta pInt$ and $fq\delta sInt$).

Remark. The interrelation between $fqeCl(\lambda)$ and $fqeInt(\lambda)$ are given as follows:
(i) $1 - fqeCl(\lambda) = 1 - \wedge\{\delta : \delta \text{ is } fqec \text{ set and } \delta \geq \lambda\}$
$= \vee\{\delta^{'} : \delta^{'} \text{ is } fqeo \text{ set and } \delta^{'} \leq \lambda^{'}\}$.
Thus $1 - fqeCl(\lambda) = fqeInt(\lambda^{'})$.
(ii) $1 - fqeInt(\lambda) = 1 - \vee\{\eta : \eta \text{ is } fqeo \text{ set and } \eta \leq \lambda\}$
$= \wedge\{\eta^{'} : \eta^{'} \text{ is } fqec \text{ set and } \eta^{'} \geq \lambda^{'}\}$.
Thus $1 - fqInt(\lambda) = fqeCl(\lambda^{'})$. In a similar way we relate the interrelation between other generalized sets also.

Definition 3.12. A fuzzy set λ in a fbts (X, τ_1, τ_2) is said to be (τ_1, τ_2)-fuzzy e (resp. e^*, a, β, δp and δs)-clopen (resp. (τ_2, τ_1)-fuzzy e (resp. e^*, a, β, δp and δs)-clopen) set if it is τ_1-fuzzy e (resp. e^*, a, β, δp and δs)-closed and τ_2-fuzzy e (resp. e^*, a, β, δp and δs)-open (resp. τ_2-fuzzy e (resp. e^*, a, β, δp and δs)-closed and τ_1-fuzzy e (resp. e^*, a, β, δp and δs)-open).

Definition 3.13. A fuzzy set λ in a fbts (X, τ_1, τ_2) is said to be fuzzy e (resp. e^*, a, β, δp and δs)-biclopen if it is both (τ_1, τ_2)-fuzzy e (resp. e^*, a, β, δp and δs)-clopen as well as (τ_2, τ_1)-fuzzy e (resp. e^*, a, β, δp and δs)-clopen set.

Definition 3.14. In a fbts (X, τ_1, τ_2) a fuzzy quasi e (resp. e^*, a, β, δp and δs)-clopen set means a set which is both fuzzy quasi e (resp. e^*, a, β, δp and δs)-closed as well as fuzzy quasi e (resp. e^*, a, β, δp and δs)-open.

Remark. Every (τ_1, τ_2)-fuzzy e (resp. e^*, a, β, δp and δs)-clopen (resp., (τ_2, τ_1)-fuzzy e (resp. e^*, a, β, δp and δs)-clopen) set is fuzzy quasi e (resp. e^*, a, β, δp and δs)-clopen. But the converse is not true as shown by Example ??

Example 3.15. In Example 3.2, let μ be a fuzzy set on X defined as $\mu : X \to [0,\ 1]$ such that $\mu = \frac{0}{a} + \frac{0.7}{b} + \frac{1}{c} + \frac{1}{d}$. Then μ is fuzzy quasi e-clopen in the fbts (X, τ_1, τ_2). But it is not (τ_1, τ_2)-fuzzy e-clopen.

Example 3.16. In Example 3.3, let μ be a fuzzy set on X defined as $\mu : X \to [0,\ 1]$ such that (i) $\mu = \frac{0.6}{a} + \frac{1}{b} + \frac{1}{c}$. Then μ is fuzzy quasi e^*-clopen in the fbts (X, τ_1, τ_2). But it is not (τ_1, τ_2)-fuzzy e^*-clopen.

(ii) $\mu = \frac{0.5}{a} + \frac{1}{b} + \frac{1}{c}$. Then μ is fuzzy quasi β (resp. δp)-clopen in the fbts (X, τ_1, τ_2). But it is not (τ_1, τ_2)-fuzzy β (resp. δp)-clopen.
(iii) $\mu = \frac{0.1}{a} + \frac{0}{b} + \frac{1}{c}$. Then μ is fuzzy quasi a (resp. δs)-clopen in the fbts (X, τ_1, τ_2). But it is not (τ_1, τ_2)-fuzzy a(resp. δs)-clopen.

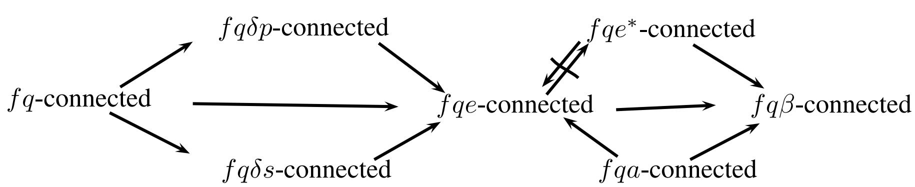
4. FUZZY QUASI e (RESP. e^*, a, β, δp AND δs)-CONNECTEDNESS BETWEEN FUZZY SETS

Definition 4.1. A fbts (X, τ_1, τ_2) is said to be (τ_1, τ_2)-fuzzy e (resp. e^*, a, β, δp and δs)-connected between some fuzzy sets λ_1 and λ_2 if there exists no (τ_1, τ_2)-fuzzy e (resp. e^*, a, β, δp and δs)-clopen set μ such that $\lambda_1 \leq \mu \leq 1 - \lambda_2$. Further (X, τ_1, τ_2) is said to be pairwise fuzzy e (resp. e^*, a, β, δp and δs) -connected between the fuzzy sets λ_1 and λ_2 if it is $(\tau_1,\ \tau_2)$-fuzzy e (resp. e^*, a, β, δp and δs) -connected as well as (τ_2, τ_1)-fuzzy e (resp. e^*, a, β, δp and δs) -connected between λ_1 and λ_2.

Example 4.2. In Examples 3.8 and 3.9 is an example of pairwise fuzzy e-connected.

Definition 4.3. A fbts $(X,\ \tau_1,\ \tau_2)$ is said to be fuzzy quasi e (resp. e^*, a, β, δp and δs) -connected between its fuzzy sets λ_1 and λ_2, if it has no fuzzy quasi e (resp. e^*, a, β, δp and δs) -clopen set μ such that $\lambda_1 \leq \mu \leq 1 - \lambda_2$.

Remark. The implications contained in the following diagram are true and the reverse implications need not be true as shown in the following examples.



Example 4.4. In Example 3.3, let λ_1 and λ_2 be two fuzzy sets on X defined as $\lambda_1,\ \lambda_2 : X \to [0,\ 1]$ such that (i) $\lambda_1(x) = \frac{0.7}{a} + \frac{0.8}{b} + \frac{0}{c}$ and $\lambda_2(x) = \frac{0.2}{a} + \frac{0}{b} + \frac{1}{c}$. We note that $\mu = \frac{0.7}{a} + \frac{0.9}{b} + \frac{0}{c}$ is fuzzy quasi e (resp. δs and δp)-connected but not fuzzy quasi (resp. e^*)-connected. Since $\lambda_1 \leq \mu \leq 1 - \lambda_2$, we can conclude that $(X,\ \tau_1,\ \tau_2)$ is not $(\tau_1,\ \tau_2)$ fuzzy quasi connected between λ_1 and λ_2. But, as there is no $(\tau_1,\ \tau_2)$-fuzzy clopen set μ satisfying $\lambda_1 \leq \mu \leq 1 - \lambda_2$, $(X,\ \tau_1,\ \tau_2)$ is not fuzzy quasi (resp. e^*)-connected.
(ii) $\lambda_1(x) = \frac{0.4}{a} + \frac{0}{b} + \frac{0.6}{c}$ and $\lambda_2(x) = \frac{0.3}{a} + \frac{0}{b} + \frac{0.2}{c}$. We note that $\mu = \frac{0.6}{a} + \frac{0}{b} + \frac{0.8}{c}$ is fuzzy quasi e^* (resp. β)-connected but not fuzzy quasi e-connected. Since $\lambda_1 \leq \mu \leq 1 - \lambda_2$, we can conclude that $(X,\ \tau_1,\ \tau_2)$ is not $(\tau_1,\ \tau_2)$ fuzzy quasi e-connected between λ_1 and λ_2.
(iii) $\lambda_1(x) = \frac{0.7}{a} + \frac{0.5}{b} + \frac{0}{c}$ and $\lambda_2(x) = \frac{0.2}{a} + \frac{0.3}{b} + \frac{0.9}{c}$. We note that $\mu = \frac{0.7}{a} + \frac{0.6}{b} + \frac{0}{c}$ is fuzzy quasi e-connected but not fuzzy quasi δp-connected. Since $\lambda_1 \leq \mu \leq 1 - \lambda_2$, we can conclude that $(X,\ \tau_1,\ \tau_2)$ is not $(\tau_1,\ \tau_2)$ fuzzy quasi δp-connected between λ_1 and λ_2.
(iv) $\lambda_1(x) = \frac{0.5}{a} + \frac{0.2}{b} + \frac{0.3}{c}$ and $\lambda_2(x) = \frac{0.3}{a} + \frac{0.5}{b} + \frac{0.5}{c}$. We note that $\mu = \frac{0.5}{a} + \frac{0.4}{b} + \frac{0.3}{c}$ is fuzzy quasi β-connected but not fuzzy quasi e^*-connected. Since $\lambda_1 \leq \mu \leq 1 - \lambda_2$, we can conclude that $(X,\ \tau_1,\ \tau_2)$ is not $(\tau_1,\ \tau_2)$ fuzzy quasi e^*-connected between λ_1 and λ_2.

Example 4.5. Let $X = \{a,\ b,\ c,\ d\}$, $\tau_1 = \{0,\ 1,\ \mu_1,\ \mu_2,\ \mu_3,\ \mu_4\}$ and $\tau_2 = \{0,\ 1,\ \eta_1,\ \eta_2,\ \eta_3\}$ where $\mu_1,\ \mu_2,\ \mu_3,\ \mu_4,\ \eta_1,\ \eta_2,\ \eta_3 : X \to [0,1]$ are defined as follows: $\mu_1 = \frac{0}{a} + \frac{0.1}{b} + \frac{1}{c} + \frac{1}{d}$, $\mu_2 = \frac{0}{a} + \frac{0.4}{b} + \frac{0}{c} + \frac{0}{d}$, $\mu_3 = \frac{0}{a} + \frac{0.4}{b} + \frac{1}{c} + \frac{1}{d}$, $\mu_4 = \frac{0}{a} + \frac{0.1}{b} + \frac{0}{c} + \frac{0}{d}$, $\eta_1 = \frac{1}{a} + \frac{0.7}{b} + \frac{1}{c} + \frac{1}{d}$, $\eta_2 = \frac{1}{a} + \frac{1}{b} + \frac{0}{c} + \frac{0}{d}$, $\eta_3 = \frac{1}{a} + \frac{0.7}{b} + \frac{0}{c} + \frac{0}{d}$. Then (X, τ_1, τ_2) is a fbts with fuzzy topologies τ_1 and τ_2. Let λ_1 and λ_2 be two fuzzy sets on X defined as $\lambda_1,\ \lambda_2 : X \to [0,\ 1]$ such that $\lambda_1(x) = \frac{0}{a} + \frac{0.4}{b} + \frac{0}{c} + \frac{0}{d}$ and $\lambda_2(x) = \frac{0}{a} + \frac{0.2}{b} + \frac{0.8}{c} + \frac{0.8}{d}$. We note that $\mu = \frac{1}{a} + \frac{0.6}{b} + \frac{0}{c} + \frac{0}{d}$ is fuzzy quasi e (resp. β)-connected but not fuzzy quasi a-connected. Since $\lambda_1 \leq \mu \leq 1 - \lambda_2$, we can conclude that $(X,\ \tau_1,\ \tau_2)$ is not $(\tau_1,\ \tau_2)$ fuzzy quasi e (resp. β)-connected between λ_1 and λ_2.

Example 4.6. Let $X = \{a,\ b\}$, $\tau_1 = \{0,\ 1,\ \mu_1,\ \mu_2\}$ and $\tau_2 = \{0,\ 1,\ \eta_1,\ \eta_2\}$ where $\mu_1,\ \mu_2,\ \eta_1,\ \eta_2 : X \to [0,1]$ are defined as follows: $\mu_1 = \frac{0.2}{a} + \frac{0.1}{b}$, $\mu_2 = \frac{0}{a} + \frac{0.1}{b}$, $\eta_1 = \frac{0.3}{a} + \frac{0.1}{b}$, $\eta_2 = \frac{0.2}{a} + \frac{0.1}{b}$. Then (X, τ_1, τ_2) is a fbts with fuzzy topologies τ_1 and τ_2. Let λ_1 and λ_2 be two fuzzy sets on X defined as $\lambda_1,\ \lambda_2 : X \to [0,\ 1]$ such that $\lambda_1(x) = \frac{0.4}{a} + \frac{0.4}{b}$ and $\lambda_2(x) = \frac{0.2}{a} + \frac{0.3}{b}$. We note that $\mu = \frac{0.6}{a} + \frac{0.7}{b}$ is fuzzy quasi e-connected but not fuzzy quasi δs-connected.

Proposition 4.1. If a fbts $(X,\ \tau_1,\ \tau_2)$ is fuzzy quasi e-connected between its fuzzy sets λ_1 and λ_2 and if $\lambda_1 \leq \eta_1$ and $\lambda_2 \leq \eta_2$, then $(X,\ \tau_1,\ \tau_2)$ is fuzzy quasi e-connected between fuzzy sets η_1 and η_2.

Proof. Suppose (X, τ_1, τ_2) is not fuzzy quasi e-connected between fuzzy sets η_1 and η_2. Then it has fuzzy quasi e-clopen set μ such that

$$\eta_1 \leq \mu \leq 1 - \eta_2. \tag{4.1}$$

By construction we have, $\lambda_1 \leq \eta_1$ and $\lambda_2 \leq \eta_2$. From (1) we get $\lambda_1 \leq \eta_1$ implies

$$\lambda_1 \leq \mu. \tag{4.2}$$

And also $\lambda_2 \leq \eta_2$ gives

$$1 - \lambda_2 \geq 1 - \eta_2 \geq \mu \Rightarrow 1 - \lambda_2 \geq \mu$$

$$(i.e)\ \mu \leq 1 - \lambda_2. \tag{4.3}$$

From (4.1) and (4.2) we get $\lambda_1 \leq \mu \leq 1-\lambda_2$. This shows (X, τ_1, τ_2) is not fuzzy quasi e-connected between fuzzy sets λ_1 and λ_2, a contradiction. Hence the proposition is proved. □

Proposition 4.2. If a fbts $(X,\ \tau_1,\ \tau_2)$ is fuzzy quasi e^* (resp. a, β, δp and δs)-connected between its fuzzy sets λ_1 and λ_2 and if $\lambda_1 \leq \eta_1$ and $\lambda_2 \leq \eta_2$, then $(X,\ \tau_1,\ \tau_2)$ is fuzzy quasi e (resp. a, β, δp and δs) -connected between fuzzy sets η_1 and η_2.

Proof. Follows from Proposition 4.1 □

Proposition 4.3. A fbts $(X,\ \tau_1,\ \tau_2)$ is fuzzy quasi e-connected between fuzzy sets λ_1 and λ_2 iff it is fuzzy quasi e-connected between $fqeCl(\lambda_1)$ and $fqeCl(\lambda_2)$.

Proof. Sufficiency: Suppose $(X,\ \tau_1,\ \tau_2)$ is not fuzzy quasi e-connected between fuzzy sets λ_1 and λ_2. Then X has fuzzy quasi e-clopen set μ such that $\lambda_1 \leq \mu \leq 1 - \lambda_2$.

Thus we have, $\lambda_1 \leq \mu$ and $\mu \leq 1-\lambda_2$. Now $\lambda_1 \leq \mu$ implies $fqeCl(\lambda_1) \leq fqeCl(\mu) = \mu$, as μ is fuzzy quasi e-closed.

$$i.e\ fqeCl(\lambda_1) \leq \mu. \tag{4.4}$$

Also $\mu \leq 1-\lambda_2$ implies $\mu = fqeInt(\mu) \leq fqeInt(1-\lambda_2) = 1 - fqeCl(\lambda_2)$ as μ is fuzzy quasi e-open.

$$i.e\ \mu \leq 1 - fqeCl(\lambda_2). \tag{4.5}$$

From 4.4 and 4.5 we get $fqeCl(\lambda_1) \leq \mu \leq 1 - fqeCl(\lambda_2)$. This shows that (X, T_1, T_2) is not fuzzy quasi e-connected between fuzzy sets $fqeCl(\lambda_1)$ and $fqCl(\lambda_2)$.

Necessity: Suppose (X, τ_1, τ_2) is not fuzzy quasi e-connected between $fqeCl(\lambda_1)$ and $fqeCl(\lambda_2)$. Then X has fuzzy quasi e-clopen set μ such that $fqeCl(\lambda_1) \leq \mu \leq 1 - fqeCl(\lambda_2)$

Now $\lambda_1 \leq fqeCl(\lambda_1) \leq \mu$ implies

$$\lambda_1 \leq \mu. \tag{4.6}$$

Also, $\mu \leq 1 - fqeCl(\lambda_2)$. But $fqeCl(\lambda_2) \geq \lambda_2$ which implies $1 - fqeCl(\lambda_2) \leq 1 - \lambda_2$.

Therefore, $\mu \leq 1 - fqeCl(\lambda_2) \leq 1 - \lambda_2$

$$i.e\ \mu \leq 1 - \lambda_2 \tag{4.7}$$

Combining (4.6) and (4.7) we get $\lambda_1 \leq \mu \leq 1 - \lambda_2$. This shows that $(X,\ \tau_1,\ \tau_2)$ is not fuzzy quasi e-connected between λ_1 and λ_2. □

Proposition 4.4. A fbts $(X,\ \tau_1,\ \tau_2)$ is fuzzy quasi e^* (resp. a, β, δp and δs) -connected between fuzzy sets λ_1 and λ_2 iff it is fuzzy quasi e^* (resp. a, β, δp and δs) -connected between $fqe^*Cl(\lambda_1)$ and $fqe^*Cl(\lambda_2)$ (resp. $fqaCl(\lambda_1)$ and $fqaCl(\lambda_2)$, $fq\beta Cl(\lambda_1)$ and $fq\beta Cl(\lambda_2)$, $fq\delta pCl(\lambda_1)$ and $fq\delta pCl(\lambda_2)$, $fq\delta sCl(\lambda_1)$ and $fq\delta sCl(\lambda_2)$).

Proof. Follows from Proposition 4.3 □

Proposition 4.5. If a fbts $(X,\ \tau_1,\ \tau_2)$ is fuzzy quasi e-connected neither between λ_1 and η_1 nor between λ_1 and η_2 then it is not fuzzy quasi e-connected between λ_1 and $\eta_1 \vee \eta_2$.

Proof. Suppose (X, τ_1, τ_2) is fuzzy quasi e-connected neither between λ_1 and η_1 nor between λ_1 and η_2. Then it has fuzzy quasi e-clopen sets μ_1, μ_2 such that

$$\lambda_1 \leq \mu_1 \leq 1 - \eta_1 \ and \ \lambda_1 \leq \mu_2 \leq 1 - \eta_2. \tag{4.8}$$

Now, put $\mu_1 \wedge \mu_2 = \mu$. Then, $\lambda_1 \wedge \lambda_1 \leq \mu_1 \wedge \mu_2 \leq (1-\eta_1) \wedge (1-\eta_2), \Rightarrow \lambda_1 \leq \mu \leq 1 - (\eta_1 \vee \eta_2)$, which shows that (X, τ_1, τ_2) is not fuzzy quasi e-connected between λ_1 and $\eta_1 \vee \eta_2$. □

Proposition 4.6. If a fbts $(X,\ \tau_1,\ \tau_2)$ is fuzzy quasi e^* (resp. a, β, δp and δs) -connected neither between λ_1 and η_1 nor between λ_1 and η_2 then it is not fuzzy quasi e^* (resp. a, β, δp and δs) -connected between λ_1 and $\eta_1 \vee \eta_2$.

Proof. Follows from Proposition 4.5 □

Definition 4.7. A fbts (X, τ_1, τ_2) is said to be fuzzy quasi e (resp. e^*, a, β, δp and δs) -connected iff it has no proper fuzzy set which is both fuzzy quasi e (resp. e^*, a, β, δp and δs) -open and fuzzy quasi e (resp. e^*, a, β, δp and δs)-closed.

Equivalently, a fbts $(X,\ \tau_1,\ \tau_2)$ is said to be fuzzy quasi e (resp. e^*, a, β, δp and δs)-connected iff it has no proper fuzzy quasi e (resp. e^*, a, β, δp and δs)-clopen set.

Proposition 4.7. A fbts $(X,\ \tau_1,\ \tau_2)$ is fuzzy quasi e- connected iff X has no proper fuzzy sets λ_1 and λ_2 , which are fuzzy quasi e-open sets such that $\lambda_1 + \lambda_2 = 1$.

Proof. Necessity: Suppose $(X,\ \tau_1,\ \tau_2)$ is not fuzzy quasi e-connected. Then X has proper fuzzy set λ_1, which is both fuzzy quasi e-open and fuzzy quasi e-closed. Take $1 - \lambda_1 = \lambda_2$. Since λ_1 is fuzzy quasi e-closed λ_2 is $fqeo$ set. Then $\lambda_1 + \lambda_2 = 1$.

Sufficiency: Suppose that fbts $(X,\ \tau_1,\ \tau_2)$ has $fqeo$ sets λ_1 and λ_2, such that $\lambda_1 + \lambda_2 = 1$. Then $\lambda_1 = 1 - \lambda_2$ is fuzzy quasi e-closed set. Similarly, λ_2 is also fuzzy quasi e-closed. Hence $(X,\ \tau_1,\ \tau_2)$ is not fuzzy quasi e-connected. □

Proposition 4.8. A fbts $(X,\ \tau_1,\ \tau_2)$ is fuzzy quasi e^* (resp. a, β, δp and δs) - connected iff X has no proper fuzzy sets λ_1 and λ_2 , which are fuzzy quasi e^* (resp. a, β, δp and δs)-open sets such that $\lambda_1 + \lambda_2 = 1$.

Proof. Follows from Proposition 4.7 □

Proposition 4.9. A fbts (X, τ_1, τ_2) is said to be fuzzy quasi e-connected iff it is fuzzy quasi e-connected between each pair of its non-zero fuzzy sets λ_1 and λ_2.

Proof. Necessity: Let λ_1 and λ_2 be any a pair of non-zero fuzzy sets on X. Suppose (X, τ_1, τ_2) is not fuzzy quasi e-connected between λ_1 and λ_2. Then it has fuzzy quasi e-clopen set μ such that $\lambda_1 \leq \mu \leq 1 - \lambda_2$. Since $\lambda_1,\ \lambda_2 \neq 0$ then $\mu \neq 0,\ 1$ is a proper fuzzy quasi e-clopen set of X. This shows that X has a proper fuzzy quasi e-clopen set. Therefore (X, τ_1, τ_2) is not fuzzy quasi e-connected.

Sufficiency: Suppose that (X, τ_1, τ_2) is not fuzzy quasi e-connected. Then there exists a proper fuzzy set μ, (say) in X such that μ is fuzzy quasi e-clopen. Since μ is proper it is easy to find non-zero fuzzy sets λ_1, λ_2 in X such that $\lambda_1 \leq \mu \leq 1 - \lambda_2$. This implies that (X, τ_1, τ_2) is not fuzzy quasi e-connected between λ_1 and λ_2. □

Proposition 4.10. A fbts (X, τ_1, τ_2) is said to be fuzzy quasi e^* (resp. a, β, δp and δs)-connected iff it is fuzzy quasi e^* (resp. a, β, δp and δs)-connected between each pair of its non-zero fuzzy sets λ_1 and λ_2.

Proof. Follows from Proposition 4.9 □

Proposition 4.11. For a non-empty subset Y of X let $(Y, \tau_1/Y, \tau_2/Y)$ be a subspace of the fbts (X, τ_1, τ_2) and let λ_1 and λ_2 be fuzzy sets of Y. If $(Y, \tau_1/Y, \tau_2/Y)$ is fuzzy quasi e (resp. e^*, a, β, δp and δs) -connected between the fuzzy sets λ_1 and λ_2 then (X, τ_1, τ_2) is also fuzzy quasi e (resp. e^*, a, β, δp and δs) -connected between its fuzzy sets λ_1^* and λ_2^*, where $\lambda_1^* : X \to [0,\ 1]$ is such that

$$\begin{aligned}\lambda_1^*(x) &= \lambda_1(x) \text{ if } x \in Y\\ &= 0 \text{ if } x \in X \setminus Y\end{aligned}$$

and $\lambda_2^* : X \to [0,\ 1]$ is such that

$$\begin{aligned}\lambda_2^*(x) &= \lambda_2(x) \text{ if } x \in Y\\ &= 0 \text{ if } x \in X \setminus Y.\end{aligned}$$

Proposition 4.12. In a fbts (X, τ_1, τ_2) let $(Y, \tau_1/Y, \tau_2/Y)$ be a fuzzy subspace for a non-empty subset Y of X such that χ_Y is fuzzy e-clopen set in (X, τ_1, τ_2) and let δ_1 and δ_2 be fuzzy sets of Y. If $(Y, \tau_1/Y, \tau_2/Y)$ is fuzzy quasi e-connected between δ_1 and δ_2 then (X, τ_1, τ_2) is also fuzzy quasi e-connected between δ_1 and δ_2.

Proof. Suppose (X, τ_1, τ_2) is not fuzzy quasi e-connected between δ_1 and δ_2. By construction χ_Y is fuzzy e-clopen in (X, τ_1, τ_2) and hence it is fuzzy quasi e-clopen in (X, τ_1, τ_2). Then by our supposition on X we have

$$\delta_1 \leq \chi_Y \leq 1 - \delta_2 \tag{4.9}$$

holds true. Obviously, χ_Y is fuzzy e-clopen in $(Y, \tau_1/Y, \tau_2/Y)$, and it is fuzzy quasi e-clopen in $(Y, \tau_1/Y, \tau_2/Y)$.

As χ_Y in fuzzy quasi e-clopen in $(Y, \tau_1/Y, \tau_2/Y)$, the existence of the inequality (4.9) for fuzzy sets δ_1 and δ_2 in Y implies that $(Y, \tau_1/Y, \tau_2/Y)$ is not fuzzy quasi e-connected between δ_1 and δ_2. □

Proposition 4.13. In a fbts (X, τ_1, τ_2) let $(Y, \tau_1/Y, \tau_2/Y)$ be a fuzzy subspace for a non-empty subset Y of X such that χ_Y is fuzzy e^* (resp. a, β, δp and δs)-clopen set in (X, τ_1, τ_2) and let δ_1 and δ_2 be fuzzy sets of Y. If $(Y, \tau_1/Y, \tau_2/Y)$ is fuzzy quasi e^* (resp. a, β, δp and δs)-connected between δ_1 and δ_2 then (X, τ_1, τ_2) is also fuzzy quasi e^* (resp. a, β, δp and δs)-connected between δ_1 and δ_2.

Proof. Follows from Proposition 4.12 □

Proposition 4.14. Let (X, τ_1, τ_2) be a fuzzy bitopological space and let $Y \subset X$ be a non-empty subset of X such that χ_Y is fuzzy e (resp. e^*, a, β, δp and δs) -biopen set in (X, τ_1, τ_2). Suppose that $(Y, \tau_1/Y, \tau_2/Y)$ be a fuzzy subspace of (X, τ_1, τ_2). If (X, τ_1, τ_2) is fuzzy quasi e (resp. e^*, a, β, δp and δs) -connected between its fuzzy sets λ_1 and λ_2, then $(Y, \tau_1/Y, \tau_2/Y)$ is also fuzzy quasi e (resp. e^*, a, β, δp and δs)-connected between λ_1/Y and λ_2/Y.

5. FUZZY QUASI e (RESP. e^*, a, β, δp, δs)-SEPARATED SETS IN FUZZY BITOPOLOGICAL SPACES

Definition 5.1. In a fbts (X, τ_1, τ_2) two fuzzy sets λ_1 and λ_2 are termed as

(i) fuzzy quasi e-separated if $\lambda_1 \wedge fqeC1(\lambda_2) = 0 = fqeC1(\lambda_1) \wedge \lambda_2$.
(ii) fuzzy quasi e^*-separated if $\lambda_1 \wedge fqe^*C1(\lambda_2) = 0 = fqe^*C1(\lambda_1) \wedge \lambda_2$.
(iii) fuzzy quasi a-separated if $\lambda_1 \wedge fqaC1(\lambda_2) = 0 = fqaC1(\lambda_1) \wedge \lambda_2$.
(iv) fuzzy quasi β-separated if $\lambda_1 \wedge fq\beta C1(\lambda_2) = 0 = fq\beta C1(\lambda_1) \wedge \lambda_2$.
(v) fuzzy quasi δp-separated if $\lambda_1 \wedge fq\delta pC1(\lambda_2) = 0 = fq\delta pC1(\lambda_1) \wedge \lambda_2$.
(vi) fuzzy quasi δs-separated if $\lambda_1 \wedge fq\delta sC1(\lambda_2) = 0 = fq\delta sC1(\lambda_1) \wedge \lambda_2$.

Proposition 5.1. Let λ_1 and λ_2 be $fqeo$ sets in a fbts (X, τ_1, τ_2). If λ_1 and λ_2 are fuzzy quasi e-separated then $\lambda_1 \wedge \lambda_2 = 0$.

Proof. Suppose $\lambda_1 \wedge \lambda_2 \neq 0$. Then $\lambda_1 \neq 0$ and $\lambda_2 \neq 0$, and so $fqeC1(\lambda_1) \neq 0$, $fqeC1(\lambda_2) \neq 0$.

Therefore $\lambda_1 \wedge fqeC1(\lambda_2) \neq 0$ and $fqeC1(\lambda_1) \wedge \lambda_2 \neq 0$.

Thus we get $\lambda_1 \wedge fqeC1(\lambda_2) \neq 0 \neq fqeC1(\lambda_1) \wedge \lambda_2$ which shows that λ_1 and λ_2 are not fuzzy quasi e-separated. □

Proposition 5.2. Let λ_1 and λ_2 be fqe^*o (resp. a, β, δp and δs) sets in a fbts (X, τ_1, τ_2). If λ_1 and λ_2 are fuzzy quasi e^* (resp. a, β, δp and δs)-separated then $\lambda_1 \wedge \lambda_2 = 0$.

Proof. Follows from Proposition 5.1 □

Proposition 5.3. If λ and μ, are fuzzy quasi e-separated sets and $\lambda \vee \mu$ is fuzzy biopen in a fbts $(X,\ \tau_1,\ \tau_2)$, then λ and μ, are fuzzy quasi e-open sets in $(X,\ \tau_1,\ \tau_2)$.

Proof. Since λ and μ are fuzzy quasi e-separated sets, $\lambda \wedge \mu = 0$. Then

$$\lambda = (\lambda \vee \mu) \wedge (1 - fqeC1(\lambda)). \quad (5.1)$$

Since $\lambda \vee \mu$ is fuzzy biopen set and $(1-fqeCl(\lambda))$ is $fqeo$ set then, by Proposition 5.1, $(\lambda \vee \mu) \wedge (1 - fqeCl(\lambda))$ is fuzzy quasi e-open set and equivalently, λ is fuzzy quasi e-open set.

Similarly, from $\mu = (\lambda \vee \mu) \wedge (1 - fqeCl(\mu))$, we can prove that μ, is $fqeo$ set. Thus we get λ and μ, are $fqeo$ sets in (X, τ_1, τ_2). □

Proposition 5.4. If λ and μ, are fuzzy quasi e^* (resp. a, β, δp and δs)-separated sets and $\lambda \vee \mu$ is fuzzy biopen in a fbts $(X,\ \tau_1,\ \tau_2)$, then λ and μ, are fuzzy quasi e^* (resp. a, β, δp and δs)-open sets in $(X,\ \tau_1,\ \tau_2)$.

Proof. Follows from Proposition 5.3 □

Proposition 5.5. Let $Y \subset X$ and ψ_Y be fuzzy biopen in $(X,\ \tau_1,\ \tau_2)$. If λ is fuzzy quasi e (resp. e^*, a, β, δp and δs)-open in $(X,\ \tau_1,\ \tau_2)$ then $\lambda \wedge \psi_Y$ is fuzzy quasi e (resp. e^*, a, β, δp and δs)-open in the fuzzy subspace $(Y, \tau_1/Y, \tau_2/Y)$ of (X, τ_1, τ_2).

Proposition 5.6. Let $(Y, \tau_1/Y, \tau_2/Y)$ be a fuzzy subspace of a fbts (X, τ_1, τ_2). If λ is fuzzy quasi e (resp. e^*, a, β, δp and δs)-open in $(Y, \tau_1/Y, \tau_2/Y)$ and ψ_Y is fuzzy biopen in (X, τ_1, τ_2) then λ is fuzzy quasi e (resp. e^*, a, β, δp and δs)-open in (X, τ_1, τ_2).

Proposition 5.7. Let $Y \subset X$ such that ψ_Y is fuzzy biopen in fbts $(X,\ \tau_1,\ \tau_2)$ and $(Y, \tau_1/Y, \tau_2/Y)$ be fuzzy subspace of $(X,\ \tau_1,\ \tau_2)$ and let λ_1 and λ_1 are two fuzzy sets on Y. Then λ_1 and λ_2 are fuzzy quasi e (resp. e^*, a, β, δp and δs)-separated in $(X,\ \tau_1,\ \tau_2)$ iff they are fuzzy quasi e (resp. e^*, a, β, δp and δs)-separated in the subspace $(Y, \tau_1/Y, \tau_2/Y)$.

6. CONCLUSION

In this paper, we have introduced and studied fuzzy quasi e (resp. e^*, a, β, δs and δp)-open sets, fuzzy quasi e (resp. e^*, a, β, δs and δp)-closed sets, fuzzy quasi e (resp. e^*, a, β, δs and δp)-connectedness between fuzzy sets fuzzy quasi e (resp. e^*, a, β, δs and δp)-separated sets in fuzzy bitopological spaces and some properties and characterizations of them are investigated.

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G. Saravanakumar
Department of Mathematics, M.Kumarasamy College of Engineering (Autonomous) Karur, Tamilnadu -639 113, India.
Email address: saravananguru2612@gmail.com

A. Vadivel
Department of Mathematics, Govt Arts College (Autonomous), Karur, Tamil Nadu-639005,India
Department of Mathematics, Annamalai University, Annamalainagar 608 002, India.
Email address: avmaths@gmail.com

S. Murugambigai
Department of Mathematics, Govt Arts College (Autonomous), Karur, Tamil Nadu-639005,India.
Email address: muruga.jb@gmail.com

M. Kamaraj
Department of Mathematics, Govt. Arts and Science College, Sivakasi, Virudhunagar, Tamil Nadu-626124,India.
Email address: kamarajan17366@rediffmail.com