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HESITANT ANTI-INTUITIONISTIC FUZZY SOFT COMMUTATIVE IDEALS OF BCK-ALGEBRAS

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ABSTRACT. In this paper, the notions of hesitant anti-intuitionistic fuzzy soft BCI-commutative ideals of BCI-algebras and hesitant anti-intuitionistic fuzzy soft sub-commutative ideals of BCK-algebras are introduced and their related properties are investigate. Relations between a hesitant anti-intuitionistic fuzzy soft ideals and hesitant anti-intuitionistic fuzzy soft BCI-commutative ideals are discussed. Conditions for a hesitant anti-intuitionistic fuzzy soft ideal to be a hesitant anti-intuitionistic fuzzy soft BCI-commutative ideal are provided. Finally, it is proved that a hesitant anti-intuitionistic fuzzy soft p-ideal is a hesitant anti-intuitionistic fuzzy soft sub-commutative ideal in a BCK-algebra.

1. INTRODUCTION

Zadeh in 1965, [40] put forward his idea of fuzzy set theory which is considered to be the most suitable tool in overcoming the uncertainties. The concept of fuzzy set was suggested to achieve a simplified modeling of complex systems. The application of basic operations as direct generalization of complement, intersection and union for characteristic function was also proposed as a result of this idea. This theory is considered as a substitute of probability theory and is widely used in solving decision making problems. Later this "Fuzziness" concept lead to the highly acclaimed theory of Fuzzy Logic. This theory has been applied with a good deal of success to many areas of engineering, economics, medical science etc.

After the invention of fuzzy sets many other hybrid concepts begun to develop. Atanassov [8], in 1983, generalized the fuzzy sets by presenting the idea of intuitionistic fuzzy sets, a set with each member having a degree of belongingness as well as a degree of non-belongingness. Although, fuzzy set theory is very successful in handling uncertainties arising from vagueness or partial belongingness of an element in a set, it cannot model all sorts of uncertainties prevailing in different real physical problems. Thus search for

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new theories has been continued. As a result two new theories; rough set theory and theory of interval mathematics were also introduced to deal with uncertainties. In daily life, conventional methods are not efficacious for solving difficult problems.

Imai and Iseki [12, 13], in 1966, considered the properties of set difference and presented the idea of a BCK-algebra. Iseki, in the same year, generalized BCK-algebras and presented the notion of BCI-algebras. BCK-algebras are inspired by BCK logic. Torra [39] introduced the concept of hesitant fuzzy set as one of the extensions of Zadehs fuzzy set allows the membership degree of an element to a set presented by several possible values, and it can express the hesitant information more comprehensively than other extensions of fuzzy set. Jun et al. [16] proposed hesitant fuzzy set theory applied to BCK/BCI-algebras. Jun [14] introduced doubt fuzzy BCK/BCI-algebras. Muhiuddin et al. [29, 32, 33, 36] introduced the various concepts are applied to BCK/BCI-algebras.

Molodtsov [28] pointed out that due to insufficiency of parametrization tool, the theories like, the probability theory, the fuzzy set theory, the theory of interval mathematics are difficult to apply. He solved this problem by presenting the idea of soft set theory. This theory is extensively used in many different fields. Soft set theory was primarily based on parametrization of tools. In dealing with uncertain situations, fuzzy set theory was perhaps the most appropriate theory till then. But the main difficulty with fuzzy sets is to frame a suitable membership function for a specific problem. The reason behind this is the inability of the parametrization tool of the theory. Soft set theory is considered to be the one of the most reliable method for dealing with uncertainties. This theory is a classification of elements of the universe with respect to some given set of parameters. It has been proved that soft set is more general in nature and has more capabilities in handling uncertain information.

Maji et al. [26] introduced the concept of soft set theory. Also, he defined some new operations. Maji et al. [24, 25, 27] proposed intuitionistic fuzzy soft sets as a generalization of the standard fuzzy soft sets. Jun et al. [19] generalized soft set theory applied to ideals in d-algebras. Jun [15] was the first who applied the idea of soft sets to BCK/BCI-algebras. He presented the idea of soft subalgebras and soft BCK/BCI-algebras. Jun et al. [20] introduced fuzzy soft set theory applied to BCK/BCI-algebras.

Babitha et al. [9] defined another important soft set, hesitant fuzzy soft set. They introduced basic operations such as union, intersection, complement, and De Morgans law was proven. Alsheri et al. [6] introduced new types of hesitant fuzzy soft ideals in BCK-algebras. Alsheri et al. [7] first come out with the idea of hesitant anti-fuzzy soft set in BCK-algebras. Balamurugan et al. [10] introduced translations of intuitionistic fuzzy soft structure of B-algebras. They defined an intuitionistic fuzzy soft subalgebras, intuitionistic fuzzy soft ideals, and intuitionistic fuzzy soft a-ideals of B-algebra. Balasubramanian et al. [11] studied generalizations of $(\in, \in \vee q)$-anti intuitionistic fuzzy soft subalgebras of BG-algebras. Some researchers used Doubt instead of Anti (see [4, 5]). Muhiuddin et al. [1, 30] introduced normal unisoft filters in R_0-algebras. Jun et al. [18] developed concave soft sets, critical soft points, and union-soft ideals of ordered semigroups. Muhiuddin et al. [31, 35] discussed the cubic soft sets with applications in BCK/BCI-algebras and subalgebras of BCK/BCI-algebras based on cubic soft sets. Muhiuddin et al. [34] introduced the concept of N-soft p-ideal of BCI-algebras.

For more details of BCK/BCI-algebras and related study with this topic, the reader is referred to [4, 5, 21, 22, 23, 37, 38].

The work carried out in this paper is organized as follows: section 2 summarizes some definitions and properties related to BCK/BCI-algebras which are needed to develop our main results. In section 3, the notion of hesitant anti-intuitionistic fuzzy soft BCI-commutative ideals of BCI-algebras and associated results are studied. In section 4, the concept of hesitant anti-intuitionistic fuzzy soft sub-commutative ideals of BCK-algebras and related properties are investigated.

2. PRELIMINARIES

An algebra (X; *, 0) of type (2, 0) is called a BCI-algebra if it satisfies the following axioms for all x, y, z ∈ X,
(C_1) $((x*y)*(x*z))*(z*y)=0$;
(C_2) $(x*(x*y))*y=0$;
(C_3) $x*x=0$;
(C_4) $x*y=0$ and $y*x=0 \Rightarrow x=y$;
(C_5) $x*0=0 \Rightarrow x=0$.
In a BCI-algebra, a partial ordering " $\leq$ is demarcated as $x \leq y \Rightarrow x*y=0$.

Moreover, in a BCI-algebra the succeeding axioms hold:
(C_6) $(x*y)*z=(x*z)*y$
(C_7) $x*0=x$
(C_8) $x \leq y \Rightarrow x*z \leq y*z$ and $z*y \leq z*x$
(C_9) $0*(x*y)=(0*x)*(0*y)$
(C_{10}) $0*(0*(x*y))=0*(y*x)$
(C_{11}) $(x*y)*(x*z) \leq (z*y)$
for all x, y, z ∈ X.
If a BCI-algebra X satisfies $0*x=0$ for every $x \in X$, then X is a BCK-algebra.

Let X be a BCI-algebra. A nonempty subset $I \subseteq X$ containing 0 is called

- an ideal of X if $x*y \in I$ and $y \in I$ whenever $x \in I$.
- a p-ideal of X if $(x*z)*(y*z) \in I$ and $y \in I$ whenever $x \in I$.
- a BCI-commutative ideal of X if $(x*y)*z \in I$ and $z \in I$ whenever $x*((y*(y*x))*(0*(0*(x*y)))) \in I$.
- a sub-commutative ideal of X if $(y*(y*(x*(x*y))))*z \in I$ and $z \in I$ whenever $(x*(x*y)) \in I$.

An ideal I of a BCI-algebra X is termed as closed if $0*x \in I$, for all $x \in I$.

Definition 2.1. A fuzzy set on a nonempty set X is a function $h: X \to [0,1]$.

Definition 2.2. Let E be a reference set. A hesitant fuzzy set on E is defined in terms of a function that when applied to E returns a subset of $[0,1]$ which can be seen as the following mathematical representation: $H_E=\{(e,h_E(e)) \mid e \in E\}$, where $h_E: E \to P([0,1])$.

Definition 2.3. Let X be a primary universe set and let E act as a set of factors. Let $\mathbb{F}(X)$ denotes the set of all fuzzy sets in X. Then $(\tilde{F},A)$ is called a fuzzy soft set over X and $A \subset E$, where $\tilde{F}: A \to \mathbb{F}(X)$.

Definition 2.4. Denote by $\mathbb{HF}(X)$ the set of all hesitant fuzzy sets. A couple $(\tilde{H},A)$ is called a hesitant fuzzy soft set over a reference set X, where $\tilde{H}: A \to \mathbb{HF}(X)$.

Definition 2.5. A $HFSS$ $(\tilde{H},A)$ is called a hesitant anti-fuzzy soft ideal $(HAFSI)$ of X if $\tilde{H}[\delta]=\{(x,h_{\tilde{H}[\delta]}(x) \mid x \in X \; and \; \delta \in A\}$ satisfies the following conditions:

(i) $h_{\tilde{H}[\delta]}(0) \leq h_{\tilde{H}[\delta]}(x)$,
(ii) $h_{\tilde{H}[\delta]}(x) \leq h_{\tilde{H}[\delta]}(x * y) \vee h_{\tilde{H}[\delta]}(y)$,
for every $x, y \in X$.

3. Hesitant anti-intuitionistic fuzzy soft BCI-commutative ideals

In this section, hesitant anti-intuitionistic fuzzy soft BCI-commutative ideals (briefly, $HAIFS_{BCI}CIs$) of BCI-algebras are defined. Throught this section, X will stand for a BCK-algebra.

Definition 3.1. Denote by $\mathbb{HIF}(X)$ the set of all hesitant intuitionistic fuzzy sets. A pair $(\tilde{H}, A)$ is called a hesitant intuitionistic fuzzy soft set over a reference set X, where $\tilde{H}$: $A \rightarrow \mathbb{HIF}(X)$.

Definition 3.2. A $HIFSS$ $(\tilde{H}, A)$ is called a hesitant anti-intutionistic fuzzy soft BCI-commutative ideal (briefly, $HAIFS_{BCI}CI$) of X if $\tilde{H}[\delta]$ $= \{(x, h_{1_{\tilde{H}[\delta]}}(x), h_{2_{\tilde{H}[\delta]}}(x)) | \, x \in X \; and \; \delta \in A\}$ satisfies the following conditions:

(i) $h_{1\tilde{H}[\delta]}(0) \leq h_{1\tilde{H}[\delta]}(x)$ and $h_{2\tilde{H}[\delta]}(0) \geq h_{2\tilde{H}[\delta]}(x)$,
(ii) $h_{1\tilde{H}[\delta]}(x * ((y * (y * x)) * (0 * (0 * (x * y))))) \leq h_{1\tilde{H}[\delta]}((x * y) * z) \vee h_{1\tilde{H}[\delta]}(z)$,
(iii) $h_{2\tilde{H}[\delta]}(x * ((y * (y * x)) * (0 * (0 * (x * y))))) \geq h_{2\tilde{H}[\delta]}((x * y) * z) \wedge h_{2\tilde{H}[\delta]}(z)$,
for every $x, y \in X$.

Example 3.3. Consider a BCI-algebra $X = \{0, 1, 2, 3\}$ with Cayley table.

$*$	0	1	2	3
0	0	0	0	0
1	1	0	0	1
2	2	1	0	2
3	3	2	3	0

Define a $HIFS$ $\tilde{H}[\delta] = (h_{1\tilde{H}[\delta]}(x), h_{2\tilde{H}[\delta]}(x))$ in X as follows:

$$h_{1\tilde{H}[\delta]}(0) = h_{1\tilde{H}[\delta]}(3) = 0, h_{1\tilde{H}[\delta]}(1) = h_{1\tilde{H}[\delta]}(2) = \chi$$

and

$$h_{2\tilde{H}[\delta]}(0) = h_{2\tilde{H}[\delta]}(3) = 1, h_{2\tilde{H}[\delta]}(1) = h_{2\tilde{H}[\delta]}(2) = \gamma,$$

where $\chi, \gamma \in (0, 1)$ and $\chi + \gamma \leq 1$.
By routine calculation it is easy to verify that $\tilde{H}[\delta]$ is a $HAIF_{BCI}CI$ of X.
Hence $(\tilde{H}, A)$ is a $HAIFS_{BCI}CI$ of X.

Proposition 3.1. *Any $HAIFS_{BCI}CI$ of X is order-preserving.*

Proof. Let $(\tilde{H}, A)$ be a $HAIFS_{BCI}CI$ of X. Let $\delta \in A$ and $x, y, z \in X$ be such that $x \leq y \;\; \Rightarrow \;\; x * y = 0$, then

$$\begin{aligned} & h_{1\tilde{H}[\delta]}(x * ((y * (y * x)) * (0 * (0 * (x * y))))) \\ & \leq h_{1\tilde{H}[\delta]}((x * y) * z) \vee h_{1\tilde{H}[\delta]}(z) \\ & = h_{1\tilde{H}[\delta]}(0 * z) \vee h_{1\tilde{H}[\delta]}(z) \\ & = h_{1\tilde{H}[\delta]}(0) \vee h_{1\tilde{H}[\delta]}(z) \\ & = h_{1\tilde{H}[\delta]}(z). \end{aligned}$$

Putting $y = 0$ and $z = y$, we get ${h_1}_{\tilde{H}[\delta]}(x) \leq {h_1}_{\tilde{H}[\delta]}(y)$. Also, we have

$$\begin{aligned}
&{h_2}_{\tilde{H}[\delta]}(x * ((y * (y * x)) * (0 * (0 * (x * y)))))\\
&\geq {h_2}_{\tilde{H}[\delta]}((x * y) * z) \wedge {h_2}_{\tilde{H}[\delta]}(z)\\
&= {h_2}_{\tilde{H}[\delta]}(0 * z) \wedge {h_2}_{\tilde{H}[\delta]}(z)\\
&= {h_2}_{\tilde{H}[\delta]}(0) \wedge {h_2}_{\tilde{H}[\delta]}(z)\\
&= {h_2}_{\tilde{H}[\delta]}(z).
\end{aligned}$$

Again, putting $y = 0$ and $z = y$, we get ${h_2}_{\tilde{H}[\delta]}(x) \geq {h_2}_{\tilde{H}[\delta]}(y)$. □

Lemma 3.2. *Let $(\tilde{H}, A)$ be a $HAIFSI$ of X. If $x * y \leq z$ holds in X, then ${h_1}_{\tilde{H}[\delta]}(x) \leq {h_1}_{\tilde{H}[\delta]}(y) \vee {h_1}_{\tilde{H}[\delta]}(z)$ and ${h_2}_{\tilde{H}[\delta]}(x) \geq {h_2}_{\tilde{H}[\delta]}(y) \wedge {h_2}_{\tilde{H}[\delta]}(z)$.*

Lemma 3.3. *Let $(\tilde{H}, A)$ be a $HAIFSI$ of X. If $x \leq y$ holds in X, then ${h_1}_{\tilde{H}[\delta]}(x) \leq {h_1}_{\tilde{H}[\delta]}(y)$ and ${h_2}_{\tilde{H}[\delta]}(x) \geq {h_2}_{\tilde{H}[\delta]}(y)$.*

Proposition 3.4. *If $(\tilde{H}, A)$ be a $HAIFS_{BCI}CI$ of X, then $(\tilde{H}, A)$ be a $HAIFSI$ of X.*

Proof. Let $(\tilde{H}, A)$ be a $HAIFS_{BCI}CI$ of X. Then for every $\delta \in A$ and $x, y, z \in X$, we have

$$\begin{aligned}
{h_1}_{\tilde{H}[\delta]}(x) &= {h_1}_{\tilde{H}[\delta]}(x * ((0 * (0 * x)) * (0 * (0 * (x * 0)))))\\
&\leq {h_1}_{\tilde{H}[\delta]}((x * 0) * z) \vee {h_1}_{\tilde{H}[\delta]}(z)\\
&\leq {h_1}_{\tilde{H}[\delta]}(x * z) \vee {h_1}_{\tilde{H}[\delta]}(z)
\end{aligned}$$

and

$$\begin{aligned}
{h_2}_{\tilde{H}[\delta]}(x) &= {h_2}_{\tilde{H}[\delta]}(x * ((0 * (0 * x)) * (0 * (0 * (x * 0)))))\\
&\geq {h_2}_{\tilde{H}[\delta]}((x * 0) * z) \vee {h_2}_{\tilde{H}[\delta]}(z)\\
&\geq {h_2}_{\tilde{H}[\delta]}(x * z) \vee {h_2}_{\tilde{H}[\delta]}(z).
\end{aligned}$$

Hence $(\tilde{H}, A)$ is a $HAIFSI$ of X. □

Theorem 3.5. *Let $(\tilde{H}, A)$ be a $HAIFSI$ of X. Then the following are equivalent:*

(1) $(\tilde{H}, A)$ *is an* $HAIFS_{BCI}CI$ *of* X.

(2) ${h_1}_{\tilde{H}[\delta]}(x * ((y * (y * x)) * (0 * (0 * (x * y))))) \leq {h_1}_{\tilde{H}[\delta]}((x * y) * z) \vee {h_1}_{\tilde{H}[\delta]}(z)$ *and* ${h_2}_{\tilde{H}[\delta]}(x * ((y * (y * x)) * (0 * (0 * (x * y))))) \geq {h_2}_{\tilde{H}[\delta]}((x * y) * z) \wedge {h_2}_{\tilde{H}[\delta]}(z)$.

(3) If $x \leq y$, *then* ${h_1}_{\tilde{H}[\delta]}(x * ((y * (y * x)) * (0 * (0 * (x * y))))) = {h_1}_{\tilde{H}[\delta]}((x * y) * z) \vee {h_1}_{\tilde{H}[\delta]}(z)$ *and* ${h_2}_{\tilde{H}[\delta]}(x * ((y * (y * x)) * (0 * (0 * (x * y))))) = {h_2}_{\tilde{H}[\delta]}((x * y) * z) \wedge {h_2}_{\tilde{H}[\delta]}(z)$, *for all* $x, y, z \in X$ *and* $\delta \in A$.

Proof. $(1) \Rightarrow (2)$. Let $(\tilde{H}, A)$ be an $HAIFS_{BCI}CI$ of X. Then $x, y, z \in X$ and $\delta \in A, {h_1}_{\tilde{H}[\delta]}(x * ((y * (y * x)) * (0 * (0 * (x * y))))) \leq {h_1}_{\tilde{H}[\delta]}((x * y) * z) \vee {h_1}_{\tilde{H}[\delta]}(z)$ and ${h_2}_{\tilde{H}[\delta]}(x * ((y * (y * x)) * (0 * (0 * (x * y))))) \geq {h_2}_{\tilde{H}[\delta]}((x * y) * z) \wedge {h_2}_{\tilde{H}[\delta]}(z)$. By putting $z = 0$, we get

$${h_1}_{\tilde{H}[\delta]}(x * ((y * (y * x)) * (0 * (0 * (x * y))))) \leq {h_1}_{\tilde{H}[\delta]}((x * y) * 0) \vee {h_1}_{\tilde{H}[\delta]}(0)$$

and

$${h_2}_{\tilde{H}[\delta]}(x * ((y * (y * x)) * (0 * (0 * (x * y))))) \geq {h_2}_{\tilde{H}[\delta]}((x * y) * 0) \wedge {h_2}_{\tilde{H}[\delta]}(0),$$

$$i.e., {h_1}_{\tilde{H}[\delta]}(x * ((y * (y * x)) * (0 * (0 * (x * y))))) \leq {h_1}_{\tilde{H}[\delta]}(x * y)$$

and

$$h_{2\tilde{H}[\delta]}(x*((y*(y*x))*(0*(0*(x*y))))) \geq h_{2\tilde{H}[\delta]}(x*y).$$

(2) ⇒ (3). Let

$$h_{1\tilde{H}[\delta]}(x*((y*(y*x))*(0*(0*(x*y))))) \leq h_{1\tilde{H}[\delta]}(x*y) \tag{3.1}$$

and

$$h_{2\tilde{H}[\delta]}(x*((y*(y*x))*(0*(0*(x*y))))) \geq h_{2\tilde{H}[\delta]}(x*y). \tag{3.2}$$

Since $(y*(y*x))*(0*(0*(x*y))) = (y*(y*x))*(0*(x*y)) \leq y$ (by C_{10} and C_{11}) implies $x*y \leq x*((y*(y*x))*(0*(0*(x*y))))$. Then by Lemma 3.4, we have

$$h_{1\tilde{H}[\delta]}(x*((y*(y*x))*(0*(0*(x*y))))) \geq h_{1\tilde{H}[\delta]}(x*y) \tag{3.3}$$

and

$$h_{2\tilde{H}[\delta]}(x*((y*(y*x))*(0*(0*(x*y))))) \leq h_{2\tilde{H}[\delta]}(x*y). \tag{3.4}$$

From (3.1), (3.3), (3.2) and (3.4), we get
$h_{1\tilde{H}[\delta]}(x*((y*(y*x))*(0*(0*(x*y))))) = h_{1\tilde{H}[\delta]}((x*y)*z) \vee h_{1\tilde{H}[\delta]}(z)$
and $h_{2\tilde{H}[\delta]}(x*((y*(y*x))*(0*(0*(x*y))))) = h_{2\tilde{H}[\delta]}((x*y)*z) \wedge h_{2\tilde{H}[\delta]}(z)$.

(3) ⇒ (1). Let

$$h_{1\tilde{H}[\delta]}(x*((y*(y*x))*(0*(0*(x*y))))) = h_{1\tilde{H}[\delta]}((x*y)*z) \vee h_{1\tilde{H}[\delta]}(z) \tag{3.5}$$

and

$$h_{2\tilde{H}[\delta]}(x*((y*(y*x))*(0*(0*(x*y))))) = h_{2\tilde{H}[\delta]}((x*y)*z) \wedge h_{2\tilde{H}[\delta]}(z), \tag{3.6}$$

for all $x, y, z \in X$ and $\delta \in A$.
Since $(x*y)*((x*y)*z) \leq z$, therefore by Lemma 3.5,

$$h_{1\tilde{H}[\delta]}(x*y) \leq h_{1\tilde{H}[\delta]}((x*y)*z) \vee h_{1\tilde{H}[\delta]}(z) \tag{3.7}$$

and

$$h_{2\tilde{H}[\delta]}(x*y) \geq h_{2\tilde{H}[\delta]}((x*y)*z) \wedge h_{2\tilde{H}[\delta]}(z). \tag{3.8}$$

From (3.1), (3.7),(3.2) and (3.8), we have

$$h_{1\tilde{H}[\delta]}(x*((y*(y*x))*(0*(0*(x*y))))) \leq h_{1\tilde{H}[\delta]}((x*y)*z) \vee h_{1\tilde{H}[\delta]}(z)$$

and

$$h_{2\tilde{H}[\delta]}(x*((y*(y*x))*(0*(0*(x*y))))) \geq h_{2\tilde{H}[\delta]}((x*y)*z) \wedge h_{2\tilde{H}[\delta]}(z).$$

Therefore, $\tilde{H}[\delta]$ is a $HAIF_{BCI}CI$ of X. Hence $(\tilde{H}, A)$ is a $HAIFS_{BCI}CI$ of X. □

Definition 3.4. A $HIFSS$ $(\tilde{H}, A)$ is called a hesitant anti-intutionistic fuzzy soft closed BCI-commutative ideal (briefly, $HAIFSC_{BCI}CI$) of X if $\tilde{H}[\delta] = \{(x, h_{1_{\tilde{H}[\delta]}}(x), h_{2_{\tilde{H}[\delta]}}(x)) \mid x \in X \text{ and } \delta \in A\}$ satisfies the following condtions:

(i) $h_{1\tilde{H}[\delta]}(x*((y*(y*x))*(0*(0*(x*y))))) \leq h_{1\tilde{H}[\delta]}((x*y)*z) \vee h_{1\tilde{H}[\delta]}(z)$,
(ii) $h_{2\tilde{H}[\delta]}(x*((y*(y*x))*(0*(0*(x*y))))) \geq h_{2\tilde{H}[\delta]}((x*y)*z) \wedge h_{2\tilde{H}[\delta]}(z)$,
(iii) $h_{1\tilde{H}[\delta]}(0*x) \leq h_{1\tilde{H}[\delta]}(x)$,
(iv) $h_{2\tilde{H}[\delta]}(0*x) \geq h_{2\tilde{H}[\delta]}(x)$,

for every $x, y, z \in X$.

Theorem 3.6. *Let $(\tilde{H}, A)$ be a $HAIFS$ closed ideal of X. Then $(\tilde{H}, A)$ is an $HAIFS_{BCI}CI$ of X if and only if*

(a) $h_{1\tilde{H}[\delta]}(x * (y * (y * x))) \leq h_{1\tilde{H}[\delta]}(x * y)$,
(b) $h_{2\tilde{H}[\delta]}(x * (y * (y * x))) \geq h_{2\tilde{H}[\delta]}(x * y)$,

for all $x, y \in X$ and $\delta \in A$.

Proof. Let $(\tilde{H}, A)$ be an $HAIFS_{BCI}CI$ of X. Since $(\tilde{H}, A)$ is an $HAIFS$ closed ideal of X, so for any $x, y \in X$ and $\delta \in A$,
(a) $h_{1\tilde{H}[\delta]}(0 * (x * y)) \leq h_{1\tilde{H}[\delta]}(x * y)$.
Since by C_1, C_6 and C_{10},

$$(x * (y * (y * x))) * (x * ((y * (y * x)) * (0 * (0 * (x * y))))) \leq (0 * (x * y)).$$

Hence by Lemma 3.5,

$$h_{1\tilde{H}[\delta]}(x*(y*(y*x))) \leq h_{1\tilde{H}[\delta]}(x*((y*(y*x))*(0*(0*(x*y))))) \vee h_{1\tilde{H}[\delta]}(0*(x*y))$$

Now by using (3.1),

$$h_{1\tilde{H}[\delta]}(x * (y * (y * x))) \leq h_{1\tilde{H}[\delta]}(x * y) \vee h_{1\tilde{H}[\delta]}(0 * (x * y))$$
$$h_{1\tilde{H}[\delta]}(x * (y * (y * x))) \leq h_{1\tilde{H}[\delta]}(x * y).$$

(b) $H_{2\tilde{H}[\delta]}(0 * (x * y)) \geq h_{2\tilde{H}[\delta]}(x * y)$.
Since by C_1, C_6 and C_{10},

$$(x * (y * (y * x))) * (x * ((y * (y * x)) * (0 * (0 * (x * y))))) \leq (0 * (x * y)).$$

Hence by Lemma 3.5,

$$h_{2\tilde{H}[\delta]}(x*(y*(y*x))) \geq h_{2\tilde{H}[\delta]}(x*((y*(y*x))*(0*(0*(x*y))))) \wedge h_{2\tilde{H}[\delta]}(0*(x*y)).$$

Now by using (3.2),

$$\begin{aligned} h_{2\tilde{H}[\delta]}(x * (y * (y * x))) &\geq h_{2\tilde{H}[\delta]}(x * y) \wedge h_{2\tilde{H}[\delta]}(0 * (x * y)) \\ &\geq h_{2\tilde{H}[\delta]}(x * y). \end{aligned}$$

Conversely, let $(\tilde{H}, A)$ is an HAIFS closed ideal of X satisfying the conditions: $h_{1\tilde{H}[\delta]}(x * (y * (y * x))) \leq h_{1\tilde{H}[\delta]}(x * y)$ and $h_{2\tilde{H}[\delta]}(x * (y * (y * x))) \geq h_{2\tilde{H}[\delta]}(x * y)$ for all x, y, z $\in X$. By C_1 and C_2, we have

$$(x * (y * (y * x))) * (x * ((y * (y * x)) * (0 * (0 * (x * y))))) \leq 0 * (0 * (x * y)).$$

Therefore by Lemma 3.5,

$$\begin{aligned} &h_{1\tilde{H}[\delta]}(x * ((y * (y * x)) * (0 * (0 * (x * y))))) \\ &\leq h_{1\tilde{H}[\delta]}(x * (y * (y * x))) \vee h_{1\tilde{H}[\delta]}(0 * (0 * (x * y))) \\ &= h_{1\tilde{H}[\delta]}(x * y) \vee h_{1\tilde{H}[\delta]}(0 * (0 * (x * y))) \\ &= h_{1\tilde{H}[\delta]}(x * y). \end{aligned}$$

Thus, $h_{1\,\tilde{H}[\delta]}(x*((y*(y*x))*(0*(0*(x*y))))) \leq h_{1\,\tilde{H}[\delta]}(x*y)$.
Again by Lemma 3.5, we have

$$\begin{aligned} & h_{2\,\tilde{H}[\delta]}(x*((y*(y*x))*(0*(0*(x*y))))) \\ & \geq h_{2\,\tilde{H}[\delta]}(x*(y*(y*x))) \wedge h_{2\,\tilde{H}[\delta]}(0*(0*(x*y))) \\ & = h_{2\,\tilde{H}[\delta]}(x*y) \wedge h_{2\,\tilde{H}[\delta]}(0*(0*(x*y))) \\ & = h_{2\,\tilde{H}[\delta]}(x*y). \end{aligned}$$

Thus, $h_{2\,\tilde{H}[\delta]}(x*((y*(y*x))*(0*(0*(x*y))))) \geq h_{2\,\tilde{H}[\delta]}(x*y)$. Hence $(\tilde{H}, A)$ is an $HAIFS_{BCI}CI$ of X. □

Definition 3.5. Let $(\tilde{H}, A)$ be a hesitant intuitionistic fuzzy soft set over X. For each $\varepsilon \in \mathbb{HIF}(X)$, the set $(\tilde{H}, A)^{\varepsilon} = (\tilde{H}^{\varepsilon}, A)$ is called a hesitant anti-intuitionistic fuzzy ε-level soft set of $(\tilde{H}, A)$, where $\tilde{H}[\delta]^{\varepsilon} = \{x \in X : h_{1\,\tilde{H}[\delta]}(x) \leq \varepsilon \quad \text{and} \quad h_{2\,\tilde{H}[\delta]}(x) \geq \varepsilon\}$ for every $x \in X$ and $\delta \in A$.

Theorem 3.7. *Let $(\tilde{H}, A)$ be a $HAIFS_{BCI}CI$ of X. Then $(\tilde{H}, A)^{\varepsilon}$ is a BCI-commutative ideal of X.*

Proof. Suppose that $(\tilde{H}, A)$ is a $HAIFS_{BCI}CI$ of X. Define $\tilde{H}[\delta]^{\varepsilon} = \{x \in X : h_{1\,\tilde{H}[\delta]}(x) \leq \varepsilon \ \ and \ \ h_{2\,\tilde{H}[\delta]}(x) \geq \varepsilon\}$ for every $x \in X$ and $\delta \in A$. Since $\tilde{H}[\delta]^{\varepsilon} \neq \Phi$, let $x \in \tilde{H}[\delta]^{\varepsilon} \Rightarrow h_{1\,\tilde{H}[\delta]}(x) \leq \varepsilon$ and $h_{2\,\tilde{H}[\delta]}(x) \geq \varepsilon$. By definition, we have

$$h_{1\,\tilde{H}[\delta]}(0) \leq h_{1\,\tilde{H}[\delta]}(x) \Rightarrow 0 \in \tilde{H}[\delta]^{\varepsilon}$$

and

$$h_{2\,\tilde{H}[\delta]}(0) \Rightarrow h_{2\,\tilde{H}[\delta]}(x) \Rightarrow 0 \in \tilde{H}[\delta]^{\varepsilon}.$$

Let $x, y, z \in X$ and $\delta \in A$ be such that $(x*y)*z \in \tilde{H}[\delta]^{\varepsilon} \ \ and \ \ z \in \tilde{H}[\delta]^{\varepsilon}$. Then $h_{1\,\tilde{H}[\delta]}((x*y)*z) \leq \varepsilon, h_{1\,\tilde{H}[\delta]}(z) \leq \varepsilon, h_{2\,\tilde{H}[\delta]}((x*y)*z) \geq \varepsilon$ and $h_{2\,\tilde{H}[\delta]}(z) \geq \varepsilon$. Since $(\tilde{H}, A)$ is a $HAIFS_{BCI}CI$ of X,

$$h_{1\,\tilde{H}[\delta]}(x*((y*(y*x))*(0*(0*(x*y))))) \leq h_{1\,\tilde{H}[\delta]}((x*y)*z) \vee h_{1\,\tilde{H}[\delta]}(z) \leq \varepsilon$$

and

$$h_{2\,\tilde{H}[\delta]}(x*((y*(y*x))*(0*(0*(x*y))))) \geq h_{2\,\tilde{H}[\delta]}((x*y)*z) \wedge h_{2\,\tilde{H}[\delta]}(z) \geq \varepsilon.$$

Therefore $x*((y*(y*x))*(0*(0*(x*y)))) \in \tilde{H}[\delta]^{\varepsilon}$. Hence $(\tilde{H}, A)^{\varepsilon}$ is a BCI-commutative ideal of X.

Conversely, suppose that $(\tilde{H}, A)^{\varepsilon}$ is a BCI-commutative ideal of X.
Put $h_{1\,\tilde{H}[\delta]}(x) = \varepsilon \quad \text{and} \quad h_{2\,\tilde{H}[\delta]}(x) = \varepsilon$ for every $x \in X$ and $\delta \in A$. So $0 \in \tilde{H}[\delta]^{\varepsilon} \Rightarrow h_{1\,\tilde{H}[\delta]}(0) \leq \varepsilon = h_{1\,\tilde{H}[\delta]}(x)$ and $h_{2\,\tilde{H}[\delta]}(0) \geq \varepsilon = h_{2\,\tilde{H}[\delta]}(x)$. Now we prove that $(\tilde{H}, A)$ is a $HAIFS_{BCI}CI$ of X. In contrast, there exist $x_0, y_0, z_0 \in X$ such that
(i) $h_{1\,\tilde{H}[\delta]}(x_0*((y_0*(y_0*x_0))*(0*(0*(x_0*y_0))))) \leq h_{1\,\tilde{H}[\delta]}((x_0*y_0)*z_0) \vee h_{1\,\tilde{H}[\delta]}(z_0)$.
Taking $\varepsilon_0 = 1/2$, we get

$$\{h_{1\,\tilde{H}[\delta]}(x_0*((y_0*(y_0*x_0))*(0*(0*(x_0*y_0))))) + \{h_{1\,\tilde{H}[\delta]}((x_0*y_0)*z_0) \vee h_{1\,\tilde{H}[\delta]}(z_0)\}\}.$$

It follows that $h_{1\,\tilde{H}[\delta]}((x_0*y_0)*z_0) \leq \varepsilon_0$ and $h_{1\,\tilde{H}[\delta]}(z_0) \leq \varepsilon_0$, implies $((x_0*y_0)*z_0) \in \tilde{H}[\delta]^{\varepsilon_0}$ and $z_0 \in \tilde{H}[\delta]^{\varepsilon_0}$. As $(\tilde{H}, A)^{\varepsilon_0}$ is a BCI-commutative ideal of X, $h_{1\,\tilde{H}[\delta]}(x_0*((y_0*(y_0*x_0))*(0*(0*(x_0*y_0))))) \leq \varepsilon_0$. This is a contradiction.

(ii) $h_{2\tilde{H}[\delta]}(x_0*((y_0*(y_0*x_0))*(0*(0*(x_0*y_0))))) \geq h_{2\tilde{H}[\delta]}((x_0*y_0)*z_0)\wedge h_{2\tilde{H}[\delta]}(z_0)$. Taking $\varepsilon_0 = 1/2$, we get

$\{h_{2\tilde{H}[\delta]}(x_0*((y_0*(y_0*x_0))*(0*(0*(x_0*y_0)))))+\{h_{2\tilde{H}[\delta]}((x_0*y_0)*z_0)\wedge h_{2\tilde{H}[\delta]}(z_0)\}\}$.

It follows that $h_{2\tilde{H}[\delta]}((x_0 * y_0) * z_0) \geq \varepsilon_0$ and $h_{2\tilde{H}[\delta]}(z_0) \geq \varepsilon_0$, which implies that $((x_0 * y_0) * z_0) \in \tilde{H}[\delta]^{\varepsilon_0}$ and $z_0 \in \tilde{H}[\delta]^{\varepsilon_0}$. Since $(\tilde{H}, A)^{\varepsilon_0}$ is a commutative ideal of X. So, $h_{2\tilde{H}[\delta]}(x_0 * ((y_0 * (y_0 * x_0)) * (0 * (0 * (x_0 * y_0))))) \geq \varepsilon_0$. This is a contradiction. Thus from (i) and (ii), it follows that $(\tilde{H}, A)$ is a $HAIFS_{BCI}CI$ of X. □

4. Hesitant anti-intuitionistic fuzzy soft sub-commutative ideals

In this section, hesitant anti-intuitionistic fuzzy soft sub-commutative ideals $(HAIFSSCIs)$ and hesitant anti-intuitionistic fuzzy soft p-ideals $(HAIFSPIDs)$ of BCK-algebras are studied. Throught this section, X denote a BCK-algebra.

Definition 4.1. A $HIFSS$ $(\tilde{H}, A)$ is called a hesitant anti-intutionistic fuzzy soft sub-commutative ideal (briefly, $HAIFSSCI$) of X if $\tilde{H}[\delta] = \{(x, h_{1\tilde{H}[\delta]}, h_{2\tilde{H}[\delta]}(x)) : x \in X \ \ and \ \ \delta \in A\}$ satisfies the following conditions:

(i) $h_{1\tilde{H}[\delta]}(0) \leq h_{1\tilde{H}[\delta]}(x)$ and $h_{2\tilde{H}[\delta]}(0) \geq h_{2\tilde{H}[\delta]}(x)$,
(ii) $h_{1\tilde{H}[\delta]}(x * (x * y)) \leq h_{1\tilde{H}[\delta]}((y * (y * (x * (x * y)))) * z) \vee h_{1\tilde{H}[\delta]}(z)$,
(iii) $h_{2\tilde{H}[\delta]}(x * (x * y)) \geq h_{2\tilde{H}[\delta]}((y * (y * (x * (x * y)))) * z) \wedge h_{2\tilde{H}[\delta]}(z)$,

for every $x, y, z \in X$.

Example 4.2. Consider a BCK-algebra $X = \{0, 1, 2, 3\}$ with Cayley table.

$*$	0	1	2	3
0	0	0	0	0
1	1	0	0	1
2	2	1	0	2
3	3	3	3	0

Define a $HIFS$ $\tilde{H}[\delta] = (h_{1\tilde{H}[\delta]}(x), h_{2\tilde{H}[\delta]}(x))$ in X as follows: Then $\tilde{H}[\delta]$ is a $HAIFSCI$

X	0	1	2	3
$\tilde{H}[\delta]$	[0.1, 0.8]	[0.7, 0.2]	[0.7, 0.2]	[0.1, 0.8]

of X. Hence $(\tilde{H}, A)$ is a $HAIFSSCI$ of X.

Theorem 4.1. *Let $(\tilde{H}, A)$ be a $HAIFSI$ of X. Then the following are equivalent for all $x, y, z \in X$ and $\delta \in A$:*

(1) $(\tilde{H}, A)$ *is a* $HAIFSSCI$ *of* X.
(2) $h_{1\tilde{H}[\delta]}(x * (x * y)) \leq h_{1\tilde{H}[\delta]}(y * (y * (x * (x * y))))$ *and* $h_{2\tilde{H}[\delta]}(x * (x * y)) \geq h_{2\tilde{H}[\delta]}(y * (y * (x * (x * y))))$.
(3) $h_{1\tilde{H}[\delta]}(x * (x * y)) = h_{1\tilde{H}[\delta]}(y * (y * (x * (x * y))))$ *and* $h_{2\tilde{H}[\delta]}(x * (x * y)) = h_{2\tilde{H}[\delta]}(y * (y * (x * (x * y))))$.
(4) *If* $x \leq y$, *then* $h_{1\tilde{H}[\delta]}(x) = h_{1\tilde{H}[\delta]}(y*(y*x))$ *and* $h_{2\tilde{H}[\delta]}(x) = h_{2\tilde{H}[\delta]}(y*(y*x))$.
(5) *If* $x \leq y$, *then* $h_{1\tilde{H}[\delta]}(x) \leq h_{1\tilde{H}[\delta]}(y*(y*x))$ *and* $h_{2\tilde{H}[\delta]}(x) \geq h_{2\tilde{H}[\delta]}(y*(y*x))$.

Proof. $(1) \Rightarrow (2)$. Suppose that $(\widetilde{H}, A)$ is a $HAIFSSCI$ of X. Then

$$h_{1\tilde{H}[\delta]}(x*(x*y)) \leq h_{1\tilde{H}[\delta]}((y*(y*(x*(x*y))))*z) \vee h_{1\tilde{H}[\delta]}(z)$$

and

$$h_{2\tilde{H}[\delta]}(x*(x*y)) \geq h_{2\tilde{H}[\delta]}((y*(y*(x*(x*y))))*z) \wedge h_{2\tilde{H}[\delta]}(z).$$

Putting $z = 0$, we get

$$\begin{aligned} h_{1\tilde{H}[\delta]}(x*(x*y)) &\leq h_{1\tilde{H}[\delta]}((y*(y*(x*(x*y))))*0) \vee h_{1\tilde{H}[\delta]}(0) \\ &\leq h_{1\tilde{H}[\delta]}(y*(y*(x*(x*y)))) \end{aligned}$$

and

$$\begin{aligned} h_{2\tilde{H}[\delta]}(x*(x*y)) &\geq h_{2\tilde{H}[\delta]}((y*(y*(x*(x*y))))*0) \wedge h_{2\tilde{H}[\delta]}(0) \\ &\geq h_{2\tilde{H}[\delta]}(y*(y*(x*(x*y)))). \end{aligned}$$

$(2) \Rightarrow (3)$. Since $y*(y*(x*(x*y))) \leq x*(x*y)$, we have

$$h_{1\tilde{H}[\delta]}(y*(y*(x*(x*y)))) \leq h_{1\tilde{H}[\delta]}(x*(x*y))$$

and

$$h_{2\tilde{H}[\delta]}(y*(y*(x*(x*y)))) \geq h_{2\tilde{H}[\delta]}(x*(x*y)).$$

Combining 2, we get

$$h_{1\tilde{H}[\delta]}(x*(x*y)) = h_{1\tilde{H}[\delta]}(y*(y*(x*(x*y))))$$

and

$$h_{2\tilde{H}[\delta]}(x*(x*y)) = h_{2\tilde{H}[\delta]}(y*(y*(x*(x*y))))$$

for all $x, y \in X$ and $\delta \in A$.

$(3) \Rightarrow (4)$. If $x \leq y$, then $x*y = 0$. We have

$$h_{1\tilde{H}[\delta]}(x*(x*y)) = h_{1\tilde{H}[\delta]}(y*(y*(x*(x*y))))$$

and

$$h_{2\tilde{H}[\delta]}(x*(x*y)) = h_{2\tilde{H}[\delta]}(y*(y*(x*(x*y))))$$

for all $x, y \in X$ and $\alpha \in A$.
$(4) \Rightarrow (5)$. Obvious.

$(5) \Rightarrow (1)$. Since $x*(x*y) \leq y$, by condition 5, we have

$$h_{1\tilde{H}[\delta]}(x*(x*y)) \leq h_{1\tilde{H}[\delta]}((y*(y*(x*(x*y))))*z) \vee h_{1\tilde{H}[\delta]}(z)$$

and

$$h_{2\tilde{H}[\delta]}(x*(x*y)) \geq h_{2\tilde{H}[\delta]}((y*(y*(x*(x*y))))*z) \wedge h_{2\tilde{H}[\delta]}(z)$$

for all $x, y, z \in X$ and $\delta \in A$. Hence $(\tilde{H}, A)$ is a $HAIFSSCI$ of X. □

Definition 4.3. A $HIFSS$ $(\tilde{H}, A)$ is called a hesitant anti-intutionistic fuzzy soft p-ideal (briefly $HAIFSPID$) of X if $\tilde{H}[\delta] = \{(x, h_{1\tilde{H}[\delta]}, h_{2\tilde{H}[\delta]}(x)) : x \in X \ \ and \ \ \delta \in A\}$ satisfies the following conditions:

(i) $h_{1\tilde{H}[\delta]}(0) \leq h_{1\tilde{H}[\delta]}(x)$ and $h_{2\tilde{H}[\delta]}(0) \geq h_{2\tilde{H}[\delta]}(x)$,
(ii) $h_{1\tilde{H}[\delta]}(x) \leq h_{1\tilde{H}[\delta]}((x*z)*(y*z)) \vee h_{1\tilde{H}[\delta]}(y)$,
(iii) $h_{2\tilde{H}[\delta]}(x*(x*y)) \geq h_{2\tilde{H}[\delta]}((x*z)*(y*z)) \wedge h_{2\tilde{H}[\delta]}(y)$,

for every $x, y \in X$.

Example 4.4. Consider a BCK-algebra $X = \{0, 1, 2, 3\}$ with Cayley table.

$*$	0	1	2	3
0	0	0	0	0
1	1	0	0	1
2	2	1	0	2
3	3	3	3	0

Define a $HIFS\tilde{H}[\delta] = (h_{1\tilde{H}[\delta]}(x), h_{2\tilde{H}[\delta]}(x))$ in X as follows:

X	0	1	2	3
$\widetilde{H}[\delta]$	[0.2, 0.7]	[0.6, 0.1]	[0.6, 0.1]	[0.2, 0.7]

Then $\tilde{H}[\delta]$ is a $HAIFSCI$ of X. Therefore $(\tilde{H}, A)$ is a $HAIFSSCI$ of X. But it is not a $HAIFSPID$ of X because $h_{1\tilde{H}[\delta]}(1) \leq h_{1\tilde{H}[\delta]}((0*0)*2) \Rightarrow 0.6 \nleq 0.2$ and $h_{2\tilde{H}[\delta]}(1) \geq h_{1\tilde{H}[\delta]}((0*0)*2) \Rightarrow 0.1 \ngeq 0.7$.

Lemma 4.2. *If $(\tilde{H}, A)$ is a $HAIFSI$ of X, then $(\tilde{H}, A)$ is a $HAIFSPID$ of X if and only if for every $x, y \in X$ and $\delta \in A$ satisfies the inequalities: $h_{1\tilde{H}[\delta]}(x) \leq h_{1\tilde{H}[\delta]}((0*0)*x)$ and $h_{2\tilde{H}[\delta]}(x) \geq h_{2\tilde{H}[\delta]}((0*0)*x)$.*

Theorem 4.3. *If $(\tilde{H}, A)$ be a $HAIFSPID$ of X, then $(\tilde{H}, A)$ is a $HAIFSSCI$ of X.*

Proof. Let $(\tilde{H}, A)$ be a $HAIFSPID$ of X. Then $(\tilde{H}, A)$ is a $HAIFSI$ of X. As

$$\begin{aligned}
&[0*(0*(x*(x*y)))]*[y*(y*(x*(x*y)))] \\
&= [0*(y*(y*(x*(x*y))))]*[0*(x*(x*y))] \\
&= [(0*y)*((0*y)*(0*(x*(x*y))))*[0*(x*(x*y))] \\
&\leq [0*(x*(x*y))]*[0*(x*(x*y))] = 0,
\end{aligned}$$

we have, $0*(0*(x*(x*y))) \leq y*(y*(x*(x*y)))$. Therefore

$$h_{1\tilde{H}[\delta]}(0*(0*(x*(x*y)))) \leq h_{1\tilde{H}[\delta]}(y*(y*(x*(x*y))))$$

and

$$h_{2\tilde{H}[\delta]}(0*(0*(x*(x*y)))) \geq h_{2\tilde{H}[\delta]}(y*(y*(x*(x*y)))).$$

It follows that

$$h_{1\tilde{H}[\delta]}(x*(x*y)) \leq h_{1\tilde{H}[\delta]}(y*(y*(x*(x*y))))$$

and

$$h_{2\tilde{H}[\delta]}(x*(x*y)) \geq h_{2\tilde{H}[\delta]}(y*(y*(x*(x*y)))).$$

Hence $(\tilde{H}, A)$ is called a $HAIFSSCI$ of X. □

5. Conclusion

In this paper, we brought the notions of hesitant anti-fuzzy soft BCI-commutative ideals of BCI-algebras and hesitant anti-fuzzy soft sub-commutative ideals of BCK-algebras. We have shown that the hesitant anti-intuitionistic fuzzy soft ideal is a hesitant anti-intuitionistic fuzzy soft BCI-commutative ideal of BCI-algebras. Also, we have proved that hesitant anti-intuitionistic fuzzy soft p-ideal is hesitant anti-intuitionistic fuzzy soft sub-commutative ideal of BCK-algebras. In our future work, the idea of present research work will be extended to different algebras such as BL-algebras, MTL-algebras, R0-algebras, MV-algebras, EQ-algebras and lattice implication algebras etc.

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