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SYNOPSIS OF THE NOTIONS OF MULTISETS AND FUZZY MULTISETS

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ABSTRACT. This paper is an attempt to summarize the basic concepts of the theories of multiset and fuzzy multiset. We begin by describing multisets and the operations between them with some related results. In the same vein, the basic concepts of fuzzy multiset theory as well as the operations between fuzzy multisets are buttressed in relation to multiset theory. Finally, we present some properties of fuzzy multisets with some related results.

1. INTRODUCTION

In classical set theory, the concept of "well-definedness" is key. This implies that the collection of objects must be distinct and definite. Zadeh [36] violated the fact that the elements/members of a set must be definite to propose the concept of fuzzy sets with degree of membership. By relaxing the restriction on the distinctiveness of the elements in classical set, the notion of multiset sprung out. The term multiset was first suggested by De Bruijn [5] to Knuth in a private correspondence as a generalisation of classical set theory as noted in [10]. Because of the appropriateness of the term multiset, it has replaced terms like bag, heap, bunch, sample, etc. which were hitherto used in different literature [29].

Multisets are very important structures applicable in real-life situations such as in database queries, information retrieval on the web, multicriteria decision making, knowledge representation in database systems, biological systems, membrane computing, musical note, frequency, chemical compositions, processes in an operating system, etc [21, 23, 28]. In mathematics, the prime factorisation of an integer $n > 0$ is a multiset whose elements are primes [31]. In fact, the relevance of multisets can not be over emphasised. A complete account on the theory of multiset and its categorical models can be found in [3, 4, 7, 8, 9, 11, 18, 20, 22, 28, 29, 30, 31, 32, 34].

The concept of fuzzy multisets or fuzzy bags proposed by Yager [35] combines both the relaxations captured in fuzzy sets and multisets. A fuzzy multiset is a collection which simultaneously deals with quantities and degrees of membership of the elements it contains [25]. Thus, it is meet to say that a fuzzy multiset is a fuzzy set in multiset framework. In fact, fuzzy multiset generalises fuzzy set in such a way that every fuzzy set is a fuzzy multiset but the converse is not true. Some basic operations between fuzzy multisets were discussed in [12]. An outline on the development of the concept of fuzzy multisets can be found in [27]. More studies on the theory of fuzzy multisets have been carried out in literature as seen in [6, 13, 14, 19, 33]. The idea of fuzzy mutisets is very applicable in

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real-life problems, like in data analysis, decision making, clustering, information retrieval, flexible querying, etc [1, 2, 15, 16, 17, 24, 23, 25, 26].

Many works have been done in multisets and fuzzy multisets in literature. Notwithstanding, the aim of this paper is to provide a handy and an abridge document that succinctly discusses the concepts of multisets and fuzzy multisets at glance. This paper is a survey of the literature on multisets and fuzzy multisets which extends through time, with clarity. This account is certainly not a history of the concepts, but it is more than simply an annotated bibliography. It is most certainly appropriate to write a comprehensive account of multisets and fuzzy multisets due to the applicative important of the notions. In fact, the relevant and interplay of the notions suggest that the time is right to consolidate the concepts (as much as that is possible) into a single survey that permits access to the literature from a single source. In a nut shell, this paper presents some fundamentals of the theories of multiset and fuzzy multiset in details. The basic operations between multisets and fuzzy multisets are explicitly discussed and exemplified. Some algebraic properties of multisets and fuzzy multisets are explicated with some related results.

2. Concept of multisets

This section explicitly presents some basic definitions in multiset theory, its representations and operations. We review multiset theory [3, 7, 9, 18, 29, 31], exemplify and deduce some relevant results.

2.1. **Some basic definitions in multiset theory.** The definitions in this subsection are either taken from [3, 7, 8, 9, 18, 28, 29, 30, 31] or adapted with more expository note.

Definition 2.1. Let a set $X = \{x_1, ..., x_n\}$ for simplicity. A multiset $\tilde{A}$ of X is characterized by a function $C_{\tilde{A}}(x_i)$ defined by

$$C_{\tilde{A}} : X \to \mathcal{N}$$

such that the nonnegative integer $\mathcal{N} = \{0, 1, ...\}$ corresponds to each $x_i \in X$, for $i = 1, ..., n.$ whereby an element, say x_i of X may appear more than once in $\tilde{A}$.

The number of appearance of an element, say x_i in $\tilde{A}$ is called the count or multiplicity of x_i in $\tilde{A}$.

We denote the set of all multisets drawn from X by $MS(X)$ unless otherwise stated.

Definition 2.2. Let $\tilde{A} \in MS(X)$. We say $\tilde{A}$ is an empty or a null multiset if and only if $C_{\tilde{A}}(x_i) = 0 \, \forall x_i \in X$.

Example 2.3. Let $\tilde{A} \in MS(X)$. Suppose $X = \{a, b, c, d\}$ such that

$$C_{\tilde{A}}(a) = 2,\ C_{\tilde{A}}(b) = 1,\ C_{\tilde{A}}(c) = 3,\ C_{\tilde{A}}(d) = 0.$$

Thus, $\tilde{A} = \{a, a, b, c, c, c\}$ impying a, b, c and d appear 2, 1, 3 and 0 times, respectively in $\tilde{A}$.

Definition 2.4. If $\tilde{A}$ and $\tilde{B}$ are two multisets over X. Then $\tilde{A}$ and $\tilde{B}$ are equal if and only if $C_{\tilde{A}}(x) = C_{\tilde{B}}(x) \, \forall x \in X$.

Definition 2.5. Let $\tilde{A}$ and $\tilde{B}$ be two multisets over X. We say $\tilde{A}$ is a submultiset of $\tilde{B}$ denoted by $\tilde{A} \subseteq \tilde{B}$ if $C_{\tilde{A}}(x) \leq C_{\tilde{B}}(x) \, \forall x \in X$. Also, if $\tilde{A} \subseteq \tilde{B}$ and $\tilde{A} \neq \tilde{B}$, then $\tilde{A}$ is called a proper submultiset of $\tilde{B}$ and denoted by $\tilde{A} \subset \tilde{B}$. A multiset is called the parent in relation to its submultiset. $\tilde{A}$ and $\tilde{B}$ are comparable to each other or equal if $\tilde{A} \subseteq \tilde{B}$ or $\tilde{B} \subseteq \tilde{A}$.

Example 2.6. Let $X = \{a, b, c, d\}$. Suppose $\tilde{A} = \{a, a, a, b, b, c, c, c, d, d\}$ and $\tilde{B} = \{a, a, a, a, b, b, c, c, c, d, d, d\}$ are multisets drawn from X. Clearly, $\tilde{A} \subseteq \tilde{B}$. In fact, $\tilde{A} \subset \tilde{B}$ since $\tilde{A} \neq \tilde{B}$.

Definition 2.7. A multiset $\tilde{A}$ over X is a regular multiset if all of its objects occur with the same multiplicity, and such common multiplicity is called its height. $\tilde{A}$ is irregular if otherwise.

Example 2.8. If $X = \{a, b, c, \}$. Then $\tilde{A} = \{a^3, b^3, c^3\}$ is a regular multiset of height 3.

Definition 2.9. Suppose $\tilde{A} \in MS(X)$, the subset $\tilde{A}_*$ of X is called the support or root of $\tilde{A}$ if for every $x \in X$ such that $C_{\tilde{A}}(x) > 0$, $\exists\, x \in \tilde{A}$, and if for every $x \in X$ such that $C_{\tilde{A}}(x) = 0$, $\exists\, x \notin \tilde{A}$. That is,

$$\tilde{A}_* = \{x \in X | C_{\tilde{A}}(x) > 0\}.$$

A multiset is called finite if its root set is finite and the multiplicity of each of its object is finite; infinite otherwise.

Example 2.10. From Example 2.3, $\tilde{A}_* = \{a, b, c\}$.

Definition 2.11. A multiset $\tilde{A}$ over a set X is called simple if all its elements are the same. That is, $[a^3]$ is a simple multiset. It follows that the root set of a simple multiset is a singleton.

A submultiset of a given multiset is called whole if it contains all multiplicities of the common objects. That is, if $X = \{a, b, c\}$, then $[a^2, b^3]$ is a whole submultiset of $[a^2, b^3, c^4]$. Whereas, a submultiset of a given multiset is called full if it contains all objects of the parent multiset. For example, $[a, b^2, c^3]$ is a full submultiset of $[a^2, b^3, c^4]$ over $X = \{a, b, c\}$.

Definition 2.12. Let $\tilde{A}$ be a multiset over X. The cardinality of $\tilde{A}$ denoted by $|\tilde{A}|$ or $card(\tilde{A})$ is defined by

$$|\tilde{A}| = \sum_{x \in X} C_{\tilde{A}}(x).$$

If $\tilde{A}$ is a submultiset of $\tilde{B}$, then $|\tilde{B}| \geq |\tilde{A}|$. The cardinality of the root set of a multiset is called its dimension.

Example 2.13. From Example 2.3, $|\tilde{A}| = 6$.

Remark. Two multisets $\tilde{A}$ and $\tilde{B}$ drawn from a nonempty set X are said to be equivalent if and only if $|\tilde{A}| = |\tilde{B}|$.

2.2. **Representations of multiset.** We have been making use of a particular form of multiset representation so far. Here, we enumerate other forms of multiset representations as found in literature.

Let $X = \{x_1, x_2, x_3 ..., x_n\}$. Suppose $\tilde{A}$ is a multiset over X such that

$$C_{\tilde{A}}(x_1) = k_1,\ C_{\tilde{A}}(x_2) = k_2,\ C_{\tilde{A}}(x_3) = k_3, ...,\ C_{\tilde{A}}(x_n) = k_n.$$

Thus, $\tilde{A}$ can be represented as follows;

$$\tilde{A} = \{\frac{k_1}{x_1}, \frac{k_2}{x_2}, \frac{k_3}{x_3}, ..., \frac{k_n}{x_n}\},$$

$$\tilde{A} = \{\frac{x_1}{k_1}, \frac{x_2}{k_2}, \frac{x_3}{k_3}, ..., \frac{x_n}{k_n}\}$$

$$\tilde{A} = [x_1^{k_1}, x_2^{k_2}, x_3^{k_3}, ..., x_n^{k_n}],$$
$$\tilde{A} = [x_1, x_2, x_3, ..., x_n]_{k_1, k_2, k_3, ..., k_n}$$

and

$$\tilde{A} = [x_1 k_1, x_2 k_2, x_3 k_3, ..., x_n k_n].$$

Example 2.14. Let $X = \{w, x, y, z\}$. Suppose $\tilde{A}$ is a multiset over X such that

$$C_{\tilde{A}}(w) = 5,\ C_{\tilde{A}}(x) = 4,\ C_{\tilde{A}}(y) = 7,\ C_{\tilde{A}}(z) = 3.$$

Then $\tilde{A}$ can be represented by

$$\tilde{A} = \{\frac{5}{w}, \frac{4}{x}, \frac{7}{y}, \frac{3}{z}\},$$
$$\tilde{A} = \{\frac{w}{5}, \frac{x}{4}, \frac{y}{7}, \frac{z}{3}\}$$
$$\tilde{A} = [w^5, x^4, y^7, z^3],$$
$$\tilde{A} = [w, x, y, z]_{5,4,7,3}$$

and

$$\tilde{A} = [w5, x4, y7, z3].$$

2.3. **Operations between multisets.** This subsecion deals with operations between multisets and exemplifies the operations in a tabular form.

2.3.1. *Union and intersection.*

Definition 2.15. Let $\tilde{A}, \tilde{B} \in MS(X)$. Then, their union is a multiset $\tilde{A} \cup \tilde{B}$ such that $\forall$ $x \in X$,

$$C_{\tilde{A}\cup\tilde{B}}(x) = C_{\tilde{A}}(x) \vee C_{\tilde{B}}(x),$$

where $\vee$ is a maximum operation.

Definition 2.16. Let $\tilde{A}, \tilde{B} \in MS(X)$. Then, their intersection is a multiset $\tilde{A} \cap \tilde{B}$ such that $\forall$ $x \in X$,

$$C_{\tilde{A}\cap\tilde{B}}(x) = C_{\tilde{A}}(x) \wedge C_{\tilde{B}}(x),$$

where $\wedge$ is a minimum operation.

2.3.2. *Sum and difference.*

Definition 2.17. Let $\tilde{A}, \tilde{B} \in MS(X)$. Then, their sum is a multiset $\tilde{A} \oplus \tilde{B}$ such that $\forall$ $x \in X$,

$$C_{\tilde{A}\oplus\tilde{B}}(x) = C_{\tilde{A}}(x) + C_{\tilde{B}}(x).$$

Definition 2.18. Let $\tilde{A}, \tilde{B} \in MS(X)$. Then, the difference of $\tilde{B}$ from $\tilde{A}$ is a multiset $\tilde{A} \ominus \tilde{B}$ such that $\forall$ $x \in X$,

$$C_{\tilde{A}\ominus\tilde{B}}(x) = C_{\tilde{A}}(x) - C_{\tilde{B}}(x) \vee 0.$$

2.3.3. *Scalar multiplication and complementation.*

Definition 2.19. If $\tilde{A} \in MS(X)$. Then, the scalar multiplication of $\tilde{A}$ is a multiset $\alpha\tilde{A}$ such that $\forall$ $x \in X$,

$$C_{\alpha\tilde{A}}(x) = \alpha C_{\tilde{A}}(x),$$

where α is a positive integer.

We shall present the notion of complement in multiset setting in two perspectives.

Definition 2.20. Let $\tilde{A}, \tilde{B} \in MS(X)$. Then, the complement of $\tilde{B}$ with respect to $\tilde{A}$ is a multiset $\tilde{B}'$ such that $\forall\, x \in X$,

$$C_{\tilde{B}'}(x) = C_{\tilde{A}}(x) - C_{\tilde{B}}(x) \vee 0.$$

Before we consider the second perspective of complementation, the following statements are helpful.

Let X be the set from which mulisets are constructed. The multiset X^k is the set of all multisets of X such that no element occurs more than k times. Likewise, the multiset X^∞ is the set of all multisets of X such that there is no limit on the number of occurrences of an element.

If $X = \{x_1, x_2, ..., x_n\}$, then we define

$$X^k = \{\frac{k_1}{x_1}, \frac{k_2}{x_2}, ..., \frac{k_n}{x_n}\}$$

for $i = 1, 2, ..., n$, $k_i \in \mathcal{N} = \{0, 1, ...\}$, and

$$X^\infty = \{\frac{k_1}{x_1}, \frac{k_2}{x_2}, ..., \frac{k_n}{x_n}, ...\}$$

for $i = 1, 2, ..., n, ...$, $k_i \in \mathcal{N} = \{0, 1, ...\}$.

Definition 2.21. Let X be a nonempty set and X^k be the multiset space defined over X. Then for any $\tilde{A} \in X^k$, the complement of $\tilde{A}$ in X^k denoted by $\tilde{A}'$ is a multiset such that $\forall$ $x \in X$,

$$C_{\tilde{A}'}(x) = k - C_{\tilde{A}}(x).$$

Example 2.22. Let $X = \{a, b, c, d\}$ and X^k be a multiset space defined over X, where multisets $\tilde{A}$ and $\tilde{B}$ are drawn from. Suppose $k = 10$, $\alpha = 2$,

$$\tilde{A} = \{\frac{4}{a}, \frac{3}{b}, \frac{5}{c}, \frac{4}{d}\}$$

and

$$\tilde{B} = \{\frac{7}{a}, \frac{4}{b}, \frac{2}{c}, \frac{10}{d}\}.$$

We verify the aforesaid operations with this example in a tabular form below.

TABLE 1. Demonstration of the operations on multisets

Operations	Multisets of X^{10}	Multisets of X^∞
$\tilde{A}'$	$\{\frac{6}{a}, \frac{7}{b}, \frac{5}{c}, \frac{6}{d}\}$	$\{\frac{3}{a}, \frac{1}{b}, \frac{6}{d}\}$ wrt $\tilde{B}$
$\tilde{B}'$	$\{\frac{3}{a}, \frac{6}{b}, \frac{8}{c}\}$	$\{\frac{3}{c}\}$ wrt $\tilde{A}$
$2\tilde{A}$	$\{\frac{8}{a}, \frac{6}{b}, \frac{10}{c}, \frac{8}{d}\}$	same
$2\tilde{B}$	$\{\frac{8}{b}, \frac{4}{c}\}$	$\{\frac{14}{a}, \frac{8}{b}, \frac{4}{c}, \frac{20}{d}\}$
$\tilde{A} \cap \tilde{B}$	$\{\frac{4}{a}, \frac{3}{b}, \frac{2}{c}, \frac{4}{d}\}$	same
$\tilde{A} \cup \tilde{B}$	$\{\frac{7}{a}, \frac{4}{b}, \frac{5}{c}, \frac{10}{d}\}$	same
$\tilde{A} \ominus \tilde{B}$	$\{\frac{3}{c}\}$	same
$\tilde{B} \ominus \tilde{A}$	$\{\frac{3}{a}, \frac{1}{b}, \frac{6}{d}\}$	same
$\tilde{A} \oplus \tilde{B}$	$\{\frac{7}{b}, \frac{7}{c}\}$	$\{\frac{11}{a}, \frac{7}{b}, \frac{7}{c}, \frac{14}{d}\}$

We recall some properties of multisets with respect to their operations.

Proposition 2.1. *Let $\tilde{A}, \tilde{B}, \tilde{C} \in MS(X)$. Then, the following properties hold:*

(i) $\tilde{A} \cap \tilde{B} = \tilde{B} \cap \tilde{A}$,

(ii) $(\tilde{A} \cap \tilde{B}) \cap \tilde{C} = \tilde{A} \cap (\tilde{B} \cap \tilde{C})$,

(iii) $\tilde{A} \cap \tilde{A} = \tilde{A}$,
(iv) $A \cap \emptyset = \emptyset$.

Proof. Straightforward. ☐

Proposition 2.2. *Let $\tilde{A}, \tilde{B}, \tilde{C} \in MS(X)$. Then, the following properties hold:*

(i) $\tilde{A} \cup \tilde{B} = \tilde{B} \cup \tilde{A}$,
(ii) $(\tilde{A} \cup \tilde{B}) \cup \tilde{C} = \tilde{A} \cup (\tilde{B} \cup \tilde{C})$,
(iii) $\tilde{A} \cup \tilde{A} = \tilde{A}$,
(iv) $A \cup \emptyset = A$.

Proof. Straightforward. ☐

Proposition 2.3. *Let $\tilde{A}, \tilde{B}, \tilde{C} \in MS(X)$. Then, the following properties hold:*

(i) $\tilde{A} \oplus \tilde{B} = \tilde{B} \oplus \tilde{A}$,
(ii) $(\tilde{A} \oplus \tilde{B}) \oplus \tilde{C} = \tilde{A} \oplus (\tilde{B} \oplus \tilde{C})$,
(iii) $\tilde{A} \oplus \tilde{A} \neq \tilde{A}$,
(iv) $A \oplus \emptyset = A$.

Proof. Straightforward. ☐

Proposition 2.4. *Let $\tilde{A}, \tilde{B}, \tilde{C} \in MS(X)$. Then, the following properties hold:*

(i) $\tilde{A} \cup (\tilde{B} \cap \tilde{C}) = (\tilde{A} \cup \tilde{B}) \cap (\tilde{A} \cup \tilde{C})$,
(ii) $\tilde{A} \cap (\tilde{B} \cup \tilde{C}) = (\tilde{A} \cap \tilde{B}) \cup (\tilde{A} \cap \tilde{C})$,
(iii) $\tilde{A} \oplus (\tilde{B} \cup \tilde{C}) = (\tilde{A} \oplus \tilde{B}) \cup (\tilde{A} \oplus \tilde{C})$,
(iv) $\tilde{A} \oplus (\tilde{B} \cap \tilde{C}) = (\tilde{A} \oplus \tilde{B}) \cap (\tilde{A} \oplus \tilde{C})$.

Proof. Straightforward. ☐

Proposition 2.5. *Let $\tilde{A}, \tilde{B} \in MS(X)$ such that $\tilde{B} \subseteq \tilde{A}$. Then, the following properties hold:*

(i) $\tilde{A} \ominus \tilde{B} = \tilde{A} \ominus (\tilde{A} \cap \tilde{B})$,
(ii) $\tilde{A} \ominus \tilde{A} = \emptyset$.

Proof. Straightforward. ☐

Having gone through the above stated properties of multisets' operations, we deduce the following results.

Proposition 2.6. *Let $\tilde{A}, \tilde{B}, \tilde{C} \in MS(X)$. Then, the following properties hold:*

(i) $\tilde{A} \cap (\tilde{A} \oplus \tilde{B}) = \tilde{A}$,
(ii) $\tilde{A} \cup (\tilde{A} \oplus \tilde{B}) = \tilde{A} \oplus \tilde{B}$,
(iii) $\tilde{A} \oplus \tilde{B} = (\tilde{A} \cup \tilde{B}) \oplus (\tilde{A} \cap \tilde{B})$.

Proof. For all $x \in X$, we have

(i)

$$\begin{aligned} C_{\tilde{A} \cap (\tilde{A} \oplus \tilde{B})}(x) &= C_{\tilde{A}}(x) \wedge C_{\tilde{A} \oplus \tilde{B}}(x) \\ &= C_{\tilde{A}}(x) \wedge [C_{\tilde{A}}(x) + C_{\tilde{B}}(x)] \\ &= C_{\tilde{A}}(x). \end{aligned}$$

Hence, $\tilde{A} \cap (\tilde{A} \oplus \tilde{B}) = \tilde{A}$.

(ii)

$$\begin{aligned} C_{\tilde{A}\cup(\tilde{A}\oplus\tilde{B})}(x) &= C_{\tilde{A}}(x) \vee C_{\tilde{A}\oplus\tilde{B}}(x) \\ &= C_{\tilde{A}}(x) \vee [C_{\tilde{A}}(x) + C_{\tilde{B}}(x)] \\ &= C_{\tilde{A}}(x) + C_{\tilde{B}}(x). \end{aligned}$$

Hence, $\tilde{A} \cup (\tilde{A} \oplus \tilde{B}) = \tilde{A} \oplus \tilde{B}$.

(iii)

$$\begin{aligned} C_{\tilde{A}\oplus\tilde{B}}(x) &= C_{\tilde{A}}(x) + C_{\tilde{B}}(x) \\ &= [C_{\tilde{A}}(x) \vee C_{\tilde{B}}(x)] + [C_{\tilde{A}}(x) \wedge C_{\tilde{B}}(x)] \\ &= C_{\tilde{A}\cup\tilde{B}}(x) + C_{\tilde{A}\cap\tilde{B}}(x). \end{aligned}$$

Hence, $\tilde{A} \oplus \tilde{B} = (\tilde{A} \cup \tilde{B}) \oplus (\tilde{A} \cap \tilde{B})$. □

Proposition 2.7. *Suppose $\tilde{A}$ and $\tilde{B}$ are drawn from X^k. Then $(\tilde{A}')' = \tilde{A}$.*

Proof. Given that $\tilde{A}, \tilde{B} \in X^k$. Then, for all $x \in X$, it follows that

$$C_{\tilde{A}'}(x) = k - C_{\tilde{A}}(x).$$

Certainly,

$$C_{(\tilde{A}')'}(x) = k - [k - C_{\tilde{A}}(x)] = C_{\tilde{A}}(x).$$

Thus $(\tilde{A}')' = \tilde{A}$. □

Remark. Let $\tilde{A}, \tilde{B} \in MS(X)$ such that $\tilde{A} \subseteq \tilde{B}$. It still follows that $(\tilde{A}')' = \tilde{A}$.

Proposition 2.8. *Suppose $\tilde{A}$ and $\tilde{B}$ are drawn from X^k. Then*

(i) $(\tilde{A} \cap \tilde{B})' = \tilde{A}' \cup \tilde{B}'$.
(ii) $(\tilde{A} \cup \tilde{B})' = \tilde{A}' \cap \tilde{B}'$.

Proof.

(i) Given that $\tilde{A}, \tilde{B} \in X^k$. For all $x \in X$, we have

$$\begin{aligned} C_{(\tilde{A}\cap\tilde{B})'}(x) &= k - C_{\tilde{A}\cap\tilde{B}}(x) \\ &= k - [C_{\tilde{A}}(x) \wedge C_{\tilde{B}}(x)] \\ &= [k - C_{\tilde{A}}(x)] \vee [k - C_{\tilde{B}}(x)] \\ &= C_{(\tilde{A}'\cup\tilde{B}')}(x). \end{aligned}$$

Hence $(\tilde{A} \cap \tilde{B})' = \tilde{A}' \cup \tilde{B}'$.

(ii) Straightforward from (i). □

Remark. Let $\tilde{A}, \tilde{B} \in MS(X)$ such that $\tilde{A} \subseteq \tilde{B}$. Then, it still follow that

(i) $(\tilde{A} \cap \tilde{B})' = \tilde{A}' \cup \tilde{B}'$.
(ii) $(\tilde{A} \cup \tilde{B})' = \tilde{A}' \cap \tilde{B}'$.

Proposition 2.9. *Let $\tilde{A}, \tilde{B} \in MS(X)$. Then $\tilde{A} \cap (\tilde{A} \cup \tilde{B}) = \tilde{A} \cup (\tilde{A} \cap \tilde{B})$.*

Proof. For all $x \in X$, we get

$$\begin{aligned}
C_{\tilde{A}\cap(\tilde{A}\cup\tilde{B})}(x) &= C_{\tilde{A}}(x) \wedge C_{\tilde{A}\cup\tilde{B}}(x) \\
&= C_{\tilde{A}}(x) \wedge [C_{\tilde{A}}(x) \vee C_{\tilde{B}}(x)] \\
&= [C_{\tilde{A}}(x) \wedge C_{\tilde{A}}(x)] \vee [C_{\tilde{A}}(x) \wedge C_{\tilde{B}}(x)] \\
&= C_{\tilde{A}}(x) \vee C_{\tilde{A}\cap\tilde{B}}(x) \\
&= C_{\tilde{A}\cup(\tilde{A}\cap\tilde{B})}(x)
\end{aligned}$$

$\Rightarrow \tilde{A}\cap(\tilde{A}\cup\tilde{B}) \subseteq \tilde{A}\cup(\tilde{A}\cap\tilde{B})$.

Again,

$$\begin{aligned}
C_{\tilde{A}\cup(\tilde{A}\cap\tilde{B})}(x) &= C_{\tilde{A}}(x) \vee C_{\tilde{A}\cap\tilde{B}}(x) \\
&= C_{\tilde{A}}(x) \vee [C_{\tilde{A}}(x) \wedge C_{\tilde{B}}(x)] \\
&= [C_{\tilde{A}}(x) \vee C_{\tilde{A}}(x)] \wedge [C_{\tilde{A}}(x) \vee C_{\tilde{B}}(x)] \\
&= C_{\tilde{A}}(x) \wedge C_{\tilde{A}\cup\tilde{B}}(x) \\
&= C_{\tilde{A}\cap(\tilde{A}\cup\tilde{B})}(x)
\end{aligned}$$

$\Rightarrow \tilde{A}\cup(\tilde{A}\cap\tilde{B}) \subseteq \tilde{A}\cap(\tilde{A}\cup\tilde{B})$. These complete the result. □

Theorem 2.10. *Suppose $\tilde{A}$ and $\tilde{B}$ are submultisets of $\tilde{C} \in MS(X)$ such that $\tilde{A} = \tilde{B}'$ and $\tilde{B} = \tilde{A}'$. Then*

(i) $(\tilde{A}'\cup\tilde{B})\cap(\tilde{A}\cup\tilde{B}') = (\tilde{A}'\cap\tilde{B}')\cup(\tilde{A}\cap\tilde{B})$.
(ii) $(\tilde{A}'\cap\tilde{B})\cup(\tilde{A}\cap\tilde{B}') = (\tilde{A}'\cup\tilde{B}')\cap(\tilde{A}\cup\tilde{B})$.

Proof. Given that $\tilde{A} = \tilde{B}'$ and $\tilde{B} = \tilde{A}'$.

(i) For all $x \in X$, we have
$C_{\tilde{A}'\cup\tilde{B}}(x) = C_{\tilde{B}\cup\tilde{B}}(x) = C_{\tilde{B}}(x)$ and $C_{\tilde{A}\cup\tilde{B}'}(x) = C_{\tilde{A}\cup\tilde{A}}(x) = C_{\tilde{A}}(x)$.
Thus,
$$C_{(\tilde{A}'\cup\tilde{B})\cap(\tilde{A}\cup\tilde{B}')}(x) = C_{\tilde{A}\cup\tilde{B}}(x).$$
Also,
$$C_{\tilde{A}'\cap\tilde{B}'}(x) = C_{\tilde{A}\cap\tilde{B}}(x).$$
Thus,
$$C_{(\tilde{A}'\cap\tilde{B}')\cup(\tilde{A}\cap\tilde{B})}(x) = C_{\tilde{A}\cap\tilde{B}}(x).$$
Hence $(\tilde{A}'\cup\tilde{B})\cap(\tilde{A}\cup\tilde{B}') = (\tilde{A}'\cap\tilde{B}')\cup(\tilde{A}\cap\tilde{B})$.

(ii) Using the same logic in (i), we get
$$C_{(\tilde{A}'\cap\tilde{B})\cup(\tilde{A}\cap\tilde{B}')}(x) = C_{(\tilde{B}\cap\tilde{B})\cup(\tilde{A}\cap\tilde{A})}(x) = C_{(\tilde{A}\cup\tilde{B})}(x)$$
and similarly,
$$C_{(\tilde{A}'\cup\tilde{B}')\cap(\tilde{A}\cup\tilde{B})}(x) = C_{(\tilde{A}\cup\tilde{B})\cap(\tilde{A}\cup\tilde{B})}(x) = C_{\tilde{A}\cup\tilde{B}}(x).$$
Hence $(\tilde{A}'\cap\tilde{B})\cup(\tilde{A}\cap\tilde{B}') = (\tilde{A}'\cup\tilde{B}')\cap(\tilde{A}\cup\tilde{B})$.

□

Theorem 2.11. *Let* $\tilde{A}_1, \tilde{A}_2, \tilde{A}_3, \tilde{A}_4 \in MS(X)$. *Then*

$$\begin{aligned}
|\tilde{A}_1 \cup \tilde{A}_2 \cup \tilde{A}_3 \cup \tilde{A}_4| &= |\tilde{A}_1| + |\tilde{A}_2| + |\tilde{A}_3| + |\tilde{A}_4| - |\tilde{A}_1 \cap \tilde{A}_2| - |\tilde{A}_1 \cap \tilde{A}_3| \\
&- |\tilde{A}_1 \cap \tilde{A}_4| - |\tilde{A}_2 \cap \tilde{A}_3| - |\tilde{A}_2 \cap \tilde{A}_4| - |\tilde{A}_3 \cap \tilde{A}_4| \\
&+ |\tilde{A}_1 \cap \tilde{A}_3 \cap \tilde{A}_4| + |\tilde{A}_2 \cap \tilde{A}_3 \cap \tilde{A}_4| \\
&+ |\tilde{A}_1 \cap \tilde{A}_2 \cap (\tilde{A}_3 \cup \tilde{A}_4)|.
\end{aligned}$$

Proof. Firstly, we show that

$$|\tilde{A}_1 \cup \tilde{A}_2| = |\tilde{A}_1| + |\tilde{A}_2| - |\tilde{A}_1 \cap \tilde{A}_2|$$

and

$$|\tilde{A}_1 \cup \tilde{A}_2 \cup \tilde{A}_3| = |\tilde{A}_1| + |\tilde{A}_2| + |\tilde{A}_3| - |\tilde{A}_1 \cap \tilde{A}_2| - |\tilde{A}_2 \cap \tilde{A}_3| - |\tilde{A}_1 \cap \tilde{A}_3| + |\tilde{A}_1 \cap \tilde{A}_2 \cap \tilde{A}_3|.$$

Thus,

$$\begin{aligned}
|\tilde{A}_1 \cup \tilde{A}_2| + |\tilde{A}_1 \cap \tilde{A}_2| &= \Sigma_{x \in X} C_{\tilde{A}_1 \cup \tilde{A}_2}(x) + \Sigma_{x \in X} C_{\tilde{A}_1 \cap \tilde{A}_2}(x) \; \forall x \in X \\
&= \Sigma_{x \in X} C_{\tilde{A}_1}(x) \vee C_{\tilde{A}_2}(x) + \Sigma_{x \in X} C_{\tilde{A}_1}(x) \wedge C_{\tilde{A}_2}(x) \\
&= \Sigma_{x \in X} [C_{\tilde{A}_1}(x) \vee C_{\tilde{A}_2}(x) + C_{\tilde{A}_1}(x) \wedge C_{\tilde{A}_2}(x)] \\
&= \Sigma_{x \in X} [C_{\tilde{A}_1}(x) + C_{\tilde{A}_2}(x)] \\
&= \Sigma_{x \in X} C_{\tilde{A}_1}(x) + \Sigma_{x \in X} C_{\tilde{A}_2}(x) \\
&= |\tilde{A}_1| + |\tilde{A}_2|.
\end{aligned}$$

Hence, $|\tilde{A}_1 \cup \tilde{A}_2| = |\tilde{A}_1| + |\tilde{A}_2| - |\tilde{A}_1 \cap \tilde{A}_2|$.

Again,

$$\begin{aligned}
|\tilde{A}_1 \cup \tilde{A}_2 \cup \tilde{A}_3| &= |\tilde{A}_1| + |\tilde{A}_2 \cup \tilde{A}_3| - |\tilde{A}_1 \cap (\tilde{A}_2 \cup \tilde{A}_3)| \\
&= |\tilde{A}_1| + |\tilde{A}_2| + |\tilde{A}_3| \\
&- |\tilde{A}_2 \cap \tilde{A}_3| - |(\tilde{A}_1 \cap \tilde{A}_2) \cup (\tilde{A}_1 \cap \tilde{A}_3)| \\
&= |\tilde{A}_1| + |\tilde{A}_2| + |\tilde{A}_3| - |\tilde{A}_2 \cap \tilde{A}_3| \\
&- (|\tilde{A}_1 \cap \tilde{A}_2| + |\tilde{A}_1 \cap \tilde{A}_3| - |(\tilde{A}_1 \cap \tilde{A}_2) \cap (\tilde{A}_1 \cap \tilde{A}_3)|) \\
&= |\tilde{A}_1| + |\tilde{A}_2| + |\tilde{A}_3| - |\tilde{A}_2 \cap \tilde{A}_3| \\
&- |\tilde{A}_1 \cap \tilde{A}_2| - |\tilde{A}_1 \cap \tilde{A}_3| + |\tilde{A}_1 \cap (\tilde{A}_2 \cap \tilde{A}_3)| \\
&= |\tilde{A}_1| + |\tilde{A}_2| + |\tilde{A}_3| - |\tilde{A}_1 \cap \tilde{A}_2| \\
&- |\tilde{A}_2 \cap \tilde{A}_3| - |\tilde{A}_1 \cap \tilde{A}_3| + |\tilde{A}_1 \cap \tilde{A}_2 \cap \tilde{A}_3|.
\end{aligned}$$

Now,

$$\begin{aligned}
|\tilde{A}_1 \cup \tilde{A}_2 \cup \tilde{A}_3 \cup \tilde{A}_4| &= |\tilde{A}_1| + |\tilde{A}_2 \cup \tilde{A}_3 \cup \tilde{A}_4| - |\tilde{A}_1 \cap (\tilde{A}_2 \cup \tilde{A}_3 \cup \tilde{A}_4)| \\
&= |\tilde{A}_1| + |\tilde{A}_2| + |\tilde{A}_3 \cup \tilde{A}_4| \\
&- |\tilde{A}_2 \cap (\tilde{A}_3 \cup \tilde{A}_4)| - |\tilde{A}_1 \cap (\tilde{A}_2 \cup \tilde{A}_3 \cup \tilde{A}_4)| \\
&= |\tilde{A}_1| + |\tilde{A}_2| + |\tilde{A}_3| + |\tilde{A}_4| - |\tilde{A}_3 \cap \tilde{A}_4| \\
&- |(\tilde{A}_2 \cap \tilde{A}_3) \cup (\tilde{A}_2 \cap \tilde{A}_4)| - |\tilde{A}_1 \cap (\tilde{A}_2 \cup \tilde{A}_3 \cup \tilde{A}_4)| \\
&= |\tilde{A}_1| + |\tilde{A}_2| + |\tilde{A}_3| + |\tilde{A}_4| - |\tilde{A}_3 \cap \tilde{A}_4| \\
&- (|\tilde{A}_2 \cap \tilde{A}_3| + |\tilde{A}_2 \cap \tilde{A}_4| - |(\tilde{A}_2 \cap \tilde{A}_3) \cap (\tilde{A}_2 \cap \tilde{A}_4)|) \\
&- |\tilde{A}_1 \cap (\tilde{A}_2 \cup \tilde{A}_3 \cup \tilde{A}_4)| \\
&= |\tilde{A}_1| + |\tilde{A}_2| + |\tilde{A}_3| + |\tilde{A}_4| - |\tilde{A}_3 \cap \tilde{A}_4| \\
&- |\tilde{A}_2 \cap \tilde{A}_3| - |\tilde{A}_2 \cap \tilde{A}_4| + |\tilde{A}_2 \cap \tilde{A}_3 \cap \tilde{A}_4| \\
&- |\tilde{A}_1 \cap (\tilde{A}_2 \cup \tilde{A}_3 \cup \tilde{A}_4)|.
\end{aligned}$$

By simplifying $|\tilde{A}_1 \cap (\tilde{A}_2 \cup \tilde{A}_3 \cup \tilde{A}_4)|$, we get

$$\begin{aligned}
|\tilde{A}_1 \cap (\tilde{A}_2 \cup \tilde{A}_3 \cup \tilde{A}_4)| &= |(\tilde{A}_1 \cap \tilde{A}_2) \cup (\tilde{A}_1 \cap \tilde{A}_3) \cup (\tilde{A}_1 \cap \tilde{A}_4)| \\
&= |\tilde{A}_1 \cap \tilde{A}_2| + |(\tilde{A}_1 \cap \tilde{A}_3) \cup (\tilde{A}_1 \cap \tilde{A}_4)| \\
&- |(\tilde{A}_1 \cap \tilde{A}_2) \cap ((\tilde{A}_1 \cap \tilde{A}_3) \cup (\tilde{A}_1 \cap \tilde{A}_4))| \\
&= |\tilde{A}_1 \cap \tilde{A}_2| + |\tilde{A}_1 \cap \tilde{A}_3| + |\tilde{A}_1 \cap \tilde{A}_4| \\
&- |(\tilde{A}_1 \cap \tilde{A}_3) \cap (\tilde{A}_1 \cap \tilde{A}_4)| \\
&- |(\tilde{A}_1 \cap \tilde{A}_2) \cap ((\tilde{A}_1 \cap \tilde{A}_3) \cup (\tilde{A}_1 \cap \tilde{A}_4))| \\
&= |\tilde{A}_1 \cap \tilde{A}_2| + |\tilde{A}_1 \cap \tilde{A}_3| + |\tilde{A}_1 \cap \tilde{A}_4| \\
&- |\tilde{A}_1 \cap \tilde{A}_3 \cap \tilde{A}_4| - |(\tilde{A}_1 \cap \tilde{A}_2) \cap (\tilde{A}_1 \cap (\tilde{A}_3 \cup \tilde{A}_4))| \\
&= |\tilde{A}_1 \cap \tilde{A}_2| + |\tilde{A}_1 \cap \tilde{A}_3| + |\tilde{A}_1 \cap \tilde{A}_4| \\
&- |\tilde{A}_1 \cap \tilde{A}_3 \cap \tilde{A}_4| - |\tilde{A}_1 \cap \tilde{A}_2 \cap (\tilde{A}_3 \cup \tilde{A}_4)|.
\end{aligned}$$

Hence,

$$\begin{aligned}
|\tilde{A}_1 \cup \tilde{A}_2 \cup \tilde{A}_3 \cup \tilde{A}_4| &= |\tilde{A}_1| + |\tilde{A}_2| + |\tilde{A}_3| + |\tilde{A}_4| - |\tilde{A}_1 \cap \tilde{A}_2| - |\tilde{A}_1 \cap \tilde{A}_3| \\
&- |\tilde{A}_1 \cap \tilde{A}_4| - |\tilde{A}_2 \cap \tilde{A}_3| - |\tilde{A}_2 \cap \tilde{A}_4| - |\tilde{A}_3 \cap \tilde{A}_4| \\
&+ |\tilde{A}_1 \cap \tilde{A}_3 \cap \tilde{A}_4| + |\tilde{A}_2 \cap \tilde{A}_3 \cap \tilde{A}_4| \\
&+ |\tilde{A}_1 \cap \tilde{A}_2 \cap (\tilde{A}_3 \cup \tilde{A}_4)|
\end{aligned}$$

as required. □

3. CONCEPT OF FUZZY MULTISETS

This section discusses some basic definitions in fuzzy multiset theory, its representations and operations. We review fuzzy multiset theory [2, 24, 23, 12, 33, 35] and thereby deduce some relevant results.

3.1. **Some fundamentals of fuzzy multiset theory.**

The ideas in this subsection are taken from [2, 24, 23, 12, 33, 35] with lucid explanations.

Definition 3.1. Assume X is a set of elements. Then, a fuzzy bag/multiset A drwan from X can be characterised by a count membership function CM_A such that

$$CM_A : X \to Q,$$

where Q is the set of all crisp bags or multisets from the unit interval $I = [0, 1]$.

A fuzzy multiset can also be characterised by a high-order function. In particular, a fuzzy multiset A can be characterised by a function

$$CM_A : X \to N^I \text{ or } CM_A : X \to [0, 1] \to N,$$

where $I = [0, 1]$ and $N = \mathbb{N} \cup \{0\}$.

It implies that $CM_A(x)$ for $x \in X$ is given as

$$CM_A(x) = \{\mu_A^1(x), \mu_A^2(x), ..., \mu_A^n(x), ...\},$$

where $\mu_A^1(x), \mu_A^2(x), ..., \mu_A^n(x), ... \in [0, 1]$ such that $\mu_A^1(x) \geq \mu_A^2(x) \geq ... \geq \mu_A^n(x) \geq ...$, whereas in a finite case, we write

$$CM_A(x) = \{\mu_A^1(x), \mu_A^2(x), ..., \mu_A^n(x)\},$$

for $\mu_A^1(x) \geq \mu_A^2(x) \geq ... \geq \mu_A^n(x)$.

A fuzzy multiset A can be represented in the form

$$A = \{\langle \frac{CM_A(x)}{x} \rangle \mid x \in X\} \text{ or } A = \{\langle x, CM_A(x) \rangle \mid x \in X\}.$$

In a simple term, a fuzzy multiset A of X is characterised by the count membership function $CM_A(x)$ for $x \in X$, that takes the value of a multiset of a unit interval $I = [0, 1]$. We denote the set of all fuzzy multisets by $FMS(X)$.

Example 3.2. Assume that $X = \{a, b, c\}$ is a set. Then for $CM_A(a) = \{1, 0.5, 0.5\}$, $CM_A(b) = \{0.9, 0.7, 0\}$, $CM_A(c) = \{0, 0, 0\}$, A is a fuzzy multiset of X written as

$$A = \{\langle \frac{1, 0.5, 0.5}{a} \rangle, \langle \frac{0.9, 0.7, 0}{b} \rangle, \langle \frac{0, 0, 0}{c} \rangle\}.$$

Definition 3.3. Let $A, B \in FMS(X)$. Then, A is called a fuzzy submultiset of B written as $A \subseteq B$ if $CM_A(x) \leq CM_B(x) \forall x \in X$. Also, if $A \subseteq B$ and $A \neq B$, then A is called a proper fuzzy submultiset of B and denoted as $A \subset B$.

Suppose for $X = \{a, b, c\}$,

$$A = \{\langle \frac{0.5, 0.4, 0.3}{a} \rangle, \langle \frac{0.6, 0.4, 0.4}{b} \rangle, \langle \frac{0.7, 0.4, 0.2}{c} \rangle\}$$

and

$$B = \{\langle \frac{0.6, 0.6, 0.4}{a} \rangle, \langle \frac{0.6, 0.5, 0.45}{b} \rangle, \langle \frac{0.7, 0.5, 0.4}{c} \rangle\}$$

are fuzzy multisets of X. Then it is easy to see that $A \subseteq B$.

Definition 3.4. Let $A, B \in FMS(X)$. A and B are comparable to each other if and only if $A \subseteq B$ or $B \subseteq A$, and $A = B \Leftrightarrow CM_A(x) = CM_B(x) \forall x \in X$.

Definition 3.5. Let $A \in FMS(X)$. Then, the cardinality of A denoted by $|A|$ is the length of the membership sequence $CM_A(x) = \mu_A^1(x), \mu_A^2(x), ..., \mu_A^m(x)$. We define the length $L(x; A)$, that is, the length of $\mu_A^i(x), i = 1, ..., m$ as

$$L(x; A) = \vee\{i| \ \mu_A^i(x) \neq 0\},$$

where $\vee$ stands for maximum.

The cardinality between two fuzzy multisets, say A and B of X, is the lengths of the membership sequences

$$CM_A(x) = \mu_A^1(x), \mu_A^2(x), ..., \mu_A^m(x)$$

and

$$CM_B(x) = \mu_B^1(x), \mu_B^2(x), ..., \mu_B^n(x)$$

defined as $L(x; A, B) = \vee\{L(x; A), L(x; B)\}$. Where no ambiguity arises, we write $L(x) = L(x; A, B)$ for simplicity.

For example, let

$$A = \{\langle\frac{0.3, 0.2}{x}\rangle, \langle\frac{1, 0.5, 0.5}{y}\rangle\} \text{ and } B = \{\langle\frac{0.7, 0.1}{w}\rangle, \langle\frac{0.6}{x}\rangle, \langle\frac{0.8, 0.6}{y}\rangle\}.$$

Then $L(w; A) = 0$, $L(x; A) = 2$, $L(y; A) = 3$, $L(w; B) = 2$, $L(x; B) = 1$ and $L(y; B) = 2$. Also, $L(w) = 2$, $L(x) = 2$ and $L(y) = 3$. Then $|A| = 3$ and $|B| = 2$.

We can rewrite A and B as

$$A = \{\langle\frac{0, 0, 0}{w}\rangle, \langle\frac{0.3, 0.2, 0}{x}\rangle, \langle\frac{1, 0.5, 0.5}{y}\rangle\} \text{ and } B = \{\langle\frac{0.7, 0.1}{w}\rangle, \langle\frac{0.6, 0}{x}\rangle, \langle\frac{0.8, 0.6}{y}\rangle\}$$

by completing the membership sequences.

Definition 3.6. Let $A \in FMS(X)$. Then, the set A_* defined by

$$A_* = \{x \in X \mid CM_A(x) > 0\}$$

is called the support or root of A.

Definition 3.7. Let $A \in FMS(X)$. Then, for $\alpha \in [0, 1]$, the sets $A_{[\alpha]}$ and $A_{(\alpha)}$ defined by

$$A_{[\alpha]} = \{x \in X \mid CM_A(x) \geq \alpha\}$$

and

$$A_{(\alpha)} = \{x \in X \mid CM_A(x) > \alpha\}$$

are called strong and weak upper α–cuts of A.

Whenever the count membership values of x is greater than or equal to α, that is,

$$CM_A(x) = \{\mu^1, \mu^2, ..., \mu^n\} \geq \alpha,$$

the strong upper α–cut of A exist for such $x \in X$. Likewise the weak upper α–cut of A can be listed.

For example, let $X = \{a, b, c, d\}$ be a set. Then,

$$A = \{\langle\frac{1, 0.8}{a}\rangle, \langle\frac{0.7, 0.6}{b}\rangle, \langle\frac{0.6, 0.5}{c}\rangle, \langle\frac{0.6, 0.5}{d}\rangle\}$$

is a fuzzy multiset of X. Let $\alpha = 0.4, 0.5, 0.6, 0.7, 0.8, 0.9$, then

$$A_{[0.4]} = \{a, b, c, d\}$$
$$A_{[0.5]} = \{a, b, c, d\}$$
$$A_{[0.6]} = \{a, b\}$$
$$A_{[0.7]} = \{a\}$$
$$A_{[0.8]} = \{a\}$$
$$A_{[0.9]} = \emptyset$$

and

$$A_{(0.4)} = \{a, b, c, d\}$$

$$A_{(0.5)} = \{a, b\}$$
$$A_{(0.6)} = \{a\}$$
$$A_{(0.7)} = \{a\}$$
$$A_{(0.8)} = \emptyset$$
$$A_{(0.9)} = \emptyset.$$

Definition 3.8. Let $A \in FMS(X)$. Then, for $\alpha \in [0, 1]$, the sets $A^{[\alpha]}$ and $A^{(\alpha)}$ defined by
$$A^{[\alpha]} = \{x \in X \mid CM_A(x) \leq \alpha\}$$
and
$$A^{(\alpha)} = \{x \in X \mid CM_A(x) < \alpha\}$$
are called strong and weak lower α–cuts of A.

The strong and weak lower α–cuts of A can be constructed similarly as in the case of strong and weak upper α–cuts of A.

Remark. *Let $A \in FMS(X)$ and take any $\alpha \in [0, 1]$ such that $A_{[\alpha]}$ and $A^{[\alpha]}$ exist. Then, it follows that*

(i) $A_{(\alpha)} \subseteq A_{[\alpha]}$ *and* $A^{(\alpha)} \subseteq A^{[\alpha]}$.
(ii) $A_{[\alpha]} = B_{[\alpha]}, A_{(\alpha)} = B_{(\alpha)}, A^{[\alpha]} = B^{[\alpha]}$ *and* $A^{(\alpha)} = B^{(\alpha)}$ *iff* $A = B$.

3.2. **Representations of fuzzy multiset.** Here, we enumerate some forms of fuzzy multiset representations to enhance the study of its algebraic properties.

Let $X = \{y_1, y_2, y_3..., y_n\}$. Suppose A is a fuzzy multiset over X such that
$$\begin{aligned} CM_A(y_1) &= \mu^1(y_1), \mu^2(y_1), ..., \mu^n(y_1), \\ CM_A(y_2) &= \mu^1(y_2), \mu^2(y_2), ..., \mu^n(y_2), \\ CM_A(y_3) &= \mu^1(y_3), \mu^2(y_3), ..., \mu^n(y_3), \\ \vdots \quad & \quad \vdots \qquad \vdots \qquad \vdots \\ CM_A(y_n) &= \mu^1(y_n), \mu^2(y_n), ..., \mu^n(y_n). \end{aligned}$$
Thus, A can be represented as follows;
$$A = \{\langle \frac{\mu^1(y_1), ..., \mu^n(y_1)}{y_1} \rangle, \langle \frac{\mu^1(y_2), ..., \mu^n(y_2)}{y_2} \rangle, ..., \langle \frac{\mu^1(y_n), ..., \mu^n(y_n)}{y_n} \rangle\},$$
$$A = \{\langle \frac{y_1}{\mu^1(y_1), ..., \mu^n(y_1)} \rangle, \langle \frac{y_2}{\mu^1(y_2), ..., \mu^n(y_2)} \rangle, ..., \langle \frac{y_n}{\mu^1(y_n), ..., \mu^n(y_n)} \rangle\}$$
and
$$A = \{\langle y_1^{\mu^1(y_1), ..., \mu^n(y_1)} \rangle, \langle y_2^{\mu^1(y_2), ..., \mu^n(y_2)} \rangle, ..., \langle y_n^{\mu^1(y_n), ..., \mu^n(y_n)} \rangle\}.$$

Example 3.9. Let $X = \{a, b, c, d\}$. Suppose A is a fuzzy multiset over X such that
$$\begin{aligned} CM_A(a) &= 0.7, 0.6, 0.6, \\ CM_A(b) &= 0.8, 0.5, 0.4, \\ CM_A(c) &= 0.9, 0.7, 0.6, \\ CM_A(d) &= 1.0, 0.8, 0.5. \end{aligned}$$
Then, A can be represented by either
$$A = \{\langle \frac{0.7, 0.6, 0.6}{a} \rangle, \langle \frac{0.8, 0.5, 0.4}{b} \rangle, \langle \frac{0.9, 0.7, 0.6}{c} \rangle, \langle \frac{1.0, 0.8, 0.5}{d} \rangle\},$$

$$A = \{\langle \frac{a}{0.7, 0.6, 0.6} \rangle, \langle \frac{b}{0.8, 0.5, 0.4} \rangle, \langle \frac{c}{0.9, 0.7, 0.6} \rangle, \langle \frac{d}{1.0, 0.8, 0.5} \rangle\}$$

or

$$A = \{\langle a^{0.7,0.6,0.6} \rangle, \langle b^{0.8,0.5,0.4} \rangle, \langle c^{0.9,0.7,0.6} \rangle, \langle d^{1.0,0.8,0.5} \rangle\}.$$

3.3. **Operations between fuzzy multisets.** This subsecion deals with some operations between fuzzy multisets with their verifications.

3.3.1. *Union and intersection.*

Definition 3.10. Let $A, B \in FMS(X)$. Then, the intersection and union of A and B, denoted by $A \cap B$ and $A \cup B$, are defined by the rules that for any object $x \in X$,

(i) $CM_{A \cap B}(x) = CM_A(x) \wedge CM_B(x)$,
(ii) $CM_{A \cup B}(x) = CM_A(x) \vee CM_B(x)$,

where $\wedge$ and $\vee$ denote minimum and maximum operations.

3.3.2. *Sum and difference.*

Definition 3.11. Let $A, B \in FMS(X)$. Then, the sum of A and B denoted as $A \oplus B$, is defined by the addition operation in $X \times [0, 1]$ for crisp multiset. That is,

$$CM_{A \oplus B}(x) = CM_A(x) + CM_B(x) \forall x \in X.$$

The meaning of the addition operation here is not as in the case of crisp multiset, it is by merging the membership degrees in a decreasing order. For example, if

$$A = \{\langle \frac{0.7, 0.5}{x} \rangle, \langle \frac{1, 0.5}{y} \rangle, \langle \frac{0.5, 0.4}{z} \rangle\}$$

and

$$B = \{\langle \frac{0.8, 0.6}{x} \rangle, \langle \frac{0.9, 0.3}{y} \rangle, \langle \frac{1, 0.7}{z} \rangle\}$$

for $X = \{x, y, z\}$. Then,

$$A \oplus B = \{\langle \frac{0.8, 0.7, 0.6, 0.5}{x} \rangle, \langle \frac{1, 0.9, 0.5, 0.3}{y} \rangle, \langle \frac{1, 0.7, 0.5, 0.4}{z} \rangle\}.$$

Definition 3.12. Let $A, B \in FMS(X)$. Then, the difference of B from A is a multiset $A \ominus B$ such that $\forall\ x \in X$,

$$CM_{A \ominus B}(x) = CM_A(x) - CM_B(x) \vee 0.$$

3.3.3. *Complementation.*

Definition 3.13. Let $A \in FMS(X)$. Then, the complement of A is a fuzzy multiset A' such that $\forall\ x \in X$,

$$CM_{A'}(x) = 1 - CM_A(x).$$

It follows from Definition 3.1 that, $CM_{A'}(x)$ for $x \in X$ is given as

$$CM_{A'}(x) = \{\mu^1_{A'}(x), \mu^2_{A'}(x), ..., \mu^n_{A'}(x)\},$$

where $\mu^1_{A'}(x), \mu^2_{A'}(x), ..., \mu^n_{A'}(x) \in [0, 1]$ such that

$$\mu^1_{A'}(x) \leq \mu^2_{A'}(x) \leq ... \leq \mu^n_{A'}(x).$$

Example 3.14. Let $X = \{x, y, z\}$. Suppose A and B are fuzzy multisets over X such that

$$A = \{\langle \frac{0.7, 0.6, 0.6}{x} \rangle, \langle \frac{0.8, 0.5, 0.4}{y} \rangle, \langle \frac{0.9, 0.7, 0.6}{z} \rangle\}$$

and

$$B = \{\langle\frac{1.0, 0.8, 0.5}{x}\rangle, \frac{0.8, 0.6, 0.6}{y}\rangle, \langle\frac{1.0, 0.5, 0.4}{z}\rangle\}.$$

We verify the aforesaid operations with this example in a tabular form below.

TABLE 2. Demonstration of the operations on fuzzy multisets

Operations	Verifications
A'	$\{\langle\frac{0.3,0.4,0.4}{x}\rangle, \langle\frac{0.2,0.5,0.6}{y}\rangle, \langle\frac{0.1,0.3,0.4}{z}\rangle\}$
B'	$\{\langle\frac{0.0,0.2,0.5}{x}\rangle, \langle\frac{0.2,0.4,0.0}{y}\rangle, \langle\frac{0.0,0.5,0.6}{z}\rangle\}$
$A \cap B$	$\{\langle\frac{0.7,0.6,0.5}{x}\rangle, \langle\frac{0.8,0.5,0.4}{y}\rangle, \langle\frac{0.9,0.5,0.4}{z}\rangle\}$
$A \cup B$	$\{\langle\frac{1.0,0.8,0.6}{x}\rangle, \langle\frac{0.8,0.6,0.6}{y}\rangle, \langle\frac{1.0,0.7,0.6}{z}\rangle\}$
$A \ominus B$	$\{\langle\frac{0.0,0.0,0.1}{x}\rangle, \langle\frac{0.0,0.0,0.0}{y}\rangle, \langle\frac{0.0,0.2,0.2}{z}\rangle\}$
$B \ominus A$	$\{\langle\frac{0.3,0.2,0.0}{x}\rangle, \langle\frac{0.0,0.1,0.2}{y}\rangle, \langle\frac{0.1,0.0,0.0}{z}\rangle\}$
$A \oplus B$	$\{\langle\frac{1.0,0.8,0.7,0.6,0.6,0.5}{x}\rangle, \langle\frac{0.8,0.8,0.6,0.6,0.5,0.4}{y}\rangle, \langle\frac{1.0,0.9,0.7,0.6,0.5,0.4}{z}\rangle\}$

We reveiw some properties of fuzzy multisets with respect to their operations and deduce some new results.

Proposition 3.1. *Let $A \in FMS(X)$. Then, the following properties hold:*

(i) $A \cap \emptyset = \emptyset$,
(ii) $A \cup \emptyset = A$,
(iii) $A \oplus \emptyset = A$,
(iv) $A \ominus \emptyset = A$,
(v) $\emptyset \ominus A = \emptyset$.

Proof. Straightforward. □

Proposition 3.2. *Let $A \in FMS(X)$. Then, the following properties hold:*

(i) $A \cap A = A$,
(ii) $A \cup A = A$,
(iii) $A \oplus A \neq A$,
(iv) $A \ominus A \neq A$ *(*$A \ominus A = \emptyset$*).*

Proof. Straightforward. □

Proposition 3.3. *Let $A, B \in FMS(X)$. Then, the following properties hold:*

(i) $A \cap B = B \cap A$,
(i) $A \cup B = B \cup A$,
(iii) $A \oplus B = B \oplus A$,
(iv) $A \ominus B \neq B \oplus A$.

Proof. Straightforward. □

Proposition 3.4. *Let $A, B, C \in FMS(X)$. Then, the following properties hold:*

(i) $A \cap (B \cap C) = (A \cap B) \cap C$,
(ii) $A \cup (B \cup C) = (A \cup B) \cup C$,
(iii) $A \oplus (B \oplus C) = (A \oplus B) \oplus C$,
(iv) $A \ominus (B \ominus C) \neq (A \ominus B) \ominus C$.

Proof. (i) Let $x \in X$. Then, we have

$$\begin{aligned} CM_{A\cap(B\cap C)}(x) &= CM_A(x) \wedge CM_{B\cap C}(x) \\ &= CM_A(x) \wedge [CM_B(x) \wedge CM_C(x)] \\ &= [CM_A(x) \wedge CM_B(x)] \wedge CM_C(x) \\ &= CM_{A\cap B}(x) \wedge CM_C(x) \\ &= CM_{(A\cap B)\cap C}(x). \end{aligned}$$

Hence, the result. The proofs of (ii)–(iv) follow from (i). □

Proposition 3.5. *Let $A, B, C \in FMS(X)$. Then, the following properties hold:*

(i) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$,
(ii) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$,
(iii) $A \oplus (B \cup C) = (A \oplus B) \cup (A \oplus C)$,
(iv) $A \oplus (B \cap C) = (A \oplus B) \cap (A \oplus C)$,
(v) $A \ominus (B \cup C) = (A \ominus B) \cup (A \ominus C)$,
(vi) $A \ominus (B \cap C) = (A \ominus B) \cap (A \ominus C)$.

Proof. (i) Let $x \in X$. Then, we have

$$\begin{aligned} CM_{A\cup(B\cap C)}(x) &= CM_A(x) \vee CM_{B\cap C}(x) \\ &= CM_A(x) \vee [CM_B(x) \wedge CM_C(x)] \\ &= [CM_A(x) \vee CM_B(x)] \wedge [CM_A(x) \vee CM_C(x)] \\ &= CM_{A\cup B}(x) \wedge CM_{A\cup C}(x) \\ &= CM_{(A\cup B)\cap(A\cup C)}(x). \end{aligned}$$

Hence, the result. The proofs of (ii)–(vi) are similar to (i). □

Proposition 3.6. *Let $A, B \in FMS(X)$ such that $B \subseteq A$. Then $A \ominus B = A \ominus (A \cap B)$.*

Proof. For $x \in X$, we have

$$\begin{aligned} CM_{A\ominus(A\cap B)}(x) &= [CM_A(x) - CM_{A\cap B}(x)] \vee 0 \\ &= (CM_A(x) - [CM_A(x) \wedge CM_B(x)]) \vee 0 \\ &= [CM_A(x) - CM_B(x)] \vee 0 \\ &= CM_{A\ominus B}(x). \end{aligned}$$

This completes the proof. □

Proposition 3.7. *Let $A, B, C \in FMS(X)$. Then, the following properties hold:*

(i) $A \cap (A \oplus B) = A$,
(ii) $A \cup (A \oplus B) = A \oplus B$,
(iii) $A \oplus B = (A \cup B) \oplus (A \cap B)$.

Proof. For all $x \in X$, we have

(i)

$$\begin{aligned} CM_{A\cap(A\oplus B)}(x) &= CM_A(x) \wedge CM_{A\oplus B}(x) \\ &= CM_A(x) \wedge [CM_A(x) + CM_B(x)] \\ &= CM_A(x). \end{aligned}$$

Hence, $A \cap (A \oplus B) = A$.

(ii)

$$\begin{aligned} CM_{A\cup(A\oplus B)}(x) &= CM_A(x) \vee CM_{A\oplus B}(x) \\ &= CM_A(x) \vee [CM_A(x) + CM_B(x)] \\ &= CM_A(x) + CM_B(x). \end{aligned}$$

Hence, $A \cup (A \oplus B) = A \oplus B$.

(iii)

$$\begin{aligned} CM_{A\oplus B}(x) &= CM_A(x) + CM_B(x) \\ &= [CM_A(x) \vee CM_B(x)] + [CM_A(x) \wedge CM_B(x)] \\ &= CM_{A\cup B}(x) + CM_{A\cap B}(x). \end{aligned}$$

Hence, $A \oplus B = (A \cup B) \oplus (A \cap B)$. □

Proposition 3.8. *Let $A, B \in FMS(X)$. Then $(A')' = A$.*

Proof. Given that $A, B \in FMS(X)$. Then, for all $x \in X$, it follows that

$$CM_{A'}(x) = 1 - CM_A(x).$$

Certainly,

$$CM_{(A')'}(x) = 1 - [1 - CM_A(x)] = CM_A(x).$$

Thus $(A')' = A$. □

Proposition 3.9. *Let $A, B \in FMS(X)$. Then*

(i) $(A \cap B)' = A' \cup B'$.
(ii) $(A \cup B)' = A' \cap B'$.

Proof.

(i) Given that $A, B \in FMS(X)$. For all $x \in X$, we have

$$\begin{aligned} CM_{(A\cap B)'}(x) &= 1 - CM_{A\cap B}(x) \\ &= 1 - [CM_A(x) \wedge CM_B(x)] \\ &= [1 - CM_A(x)] \vee [1 - CM_B(x)] \\ &= CM_{(A'\cup B')}(x). \end{aligned}$$

Hence, $(A \cap B)' = A' \cup B'$.

(ii) Straightforward from (i). □

Proposition 3.10. *Let $A, B \in FMS(X)$. Then $A \cap (A \cup B) = A \cup (A \cap B)$.*

Proof. For all $x \in X$, we get

$$\begin{aligned} CM_{A\cap(A\cup B)}(x) &= CM_A(x) \wedge CM_{A\cup B}(x) \\ &= CM_A(x) \wedge [CM_A(x) \vee CM_B(x)] \\ &= [CM_A(x) \wedge CM_A(x)] \vee [CM_A(x) \wedge CM_B(x)) \\ &= CM_A(x) \vee CM_{A\cap B}(x) \\ &= CM_{A\cup(A\cap B)}(x) \end{aligned}$$

$\Rightarrow A \cap (A \cup B) \subseteq A \cup (A \cap B)$.

Again,

$$\begin{aligned} CM_{A\cup(A\cap B)}(x) &= CM_A(x) \vee CM_{A\cap B}(x) \\ &= CM_A(x) \vee [CM_A(x) \wedge CM_B(x)] \\ &= [CM_A(x) \vee CM_A(x)] \wedge [CM_A(x) \vee CM_B(x)] \\ &= CM_A(x) \wedge CM_{A\cup B}(x) \\ &= CM_{A\cap(A\cup B)}(x) \end{aligned}$$

$\Rightarrow\ A \cup (A \cap B) \subseteq A \cap (A \cup B)$. These complete the result. □

Theorem 3.11. *Suppose A and B are fuzzy submultisets of $C \in FMS(X)$ such that $A = B'$ and $B = A'$. Then*

(i) $(A' \cup B) \cap (A \cup B') = (A' \cap B') \cup (A \cap B)$.
(ii) $(A' \cap B) \cup (A \cap B') = (A' \cup B') \cap (A \cup B)$.

Proof. The proof is comparable to Theorem 2.10. □

Theorem 3.12. *Let $A_1, A_2, A_3 \in FMS(X)$. Then*

$$|A_1 \cup A_2 \cup A_3| = |A_1| + |A_2| + |A_3| - |A_1 \cap A_2| - |A_2 \cap A_3| - |A_1 \cap A_3| + |A_1 \cap A_2 \cap A_3|.$$

Proof. Firstly, we show that

$$|A_1 \cup A_2| = |A_1| + |A_2| - |A_1 \cap A_2|.$$

Thus,

$$\begin{aligned} |A_1 \cup A_2| + |A_1 \cap A_2| &= \Sigma_{x\in X} CM_{A_1\cup A_2}(x) + \Sigma_{x\in X} CM_{A_1\cap A_2}(x)\ \forall x \in X \\ &= \Sigma_{x\in X} CM_{A_1}(x) \vee CM_{A_2}(x) + \Sigma_{x\in X} CM_{A_1}(x) \wedge CM_{A_2}(x) \\ &= \Sigma_{x\in X}[CM_{A_1}(x) \vee CM_{A_2}(x) + CM_{A_1}(x) \wedge CM_{A_2}(x)] \\ &= \Sigma_{x\in X}[CM_{A_1}(x) + CM_{A_2}(x)] \\ &= \Sigma_{x\in X} CM_{A_1}(x) + \Sigma_{x\in X} CM_{A_2}(x) \\ &= |A_1| + |A_2|. \end{aligned}$$

Hence, $|A_1 \cup A_2| = |A_1| + |A_2| - |A_1 \cap A_2|$.

Now,

$$\begin{aligned} |A_1 \cup A_2 \cup A_3| &= |A_1| + |\tilde{A}_2 \cup \tilde{A}_3| - |\tilde{A}_1 \cap (\tilde{A}_2 \cup \tilde{A}_3)| \\ &= |\tilde{A}_1| + |\tilde{A}_2| + |\tilde{A}_3| \\ &- |\tilde{A}_2 \cap \tilde{A}_3| - |(\tilde{A}_1 \cap \tilde{A}_2) \cup (\tilde{A}_1 \cap \tilde{A}_3)| \\ &= |\tilde{A}_1| + |\tilde{A}_2| + |\tilde{A}_3| - |\tilde{A}_2 \cap \tilde{A}_3| \\ &- (|\tilde{A}_1 \cap \tilde{A}_2| + |\tilde{A}_1 \cap \tilde{A}_3| - |(\tilde{A}_1 \cap \tilde{A}_2) \cap (\tilde{A}_1 \cap \tilde{A}_3)|) \\ &= |\tilde{A}_1| + |\tilde{A}_2| + |\tilde{A}_3| - |\tilde{A}_2 \cap \tilde{A}_3| \\ &- |\tilde{A}_1 \cap \tilde{A}_2| - |\tilde{A}_1 \cap \tilde{A}_3| + |\tilde{A}_1 \cap (\tilde{A}_2 \cap \tilde{A}_3)| \\ &= |\tilde{A}_1| + |\tilde{A}_2| + |\tilde{A}_3| - |\tilde{A}_1 \cap \tilde{A}_2| \\ &- |\tilde{A}_2 \cap \tilde{A}_3| - |\tilde{A}_1 \cap \tilde{A}_3| + |\tilde{A}_1 \cap \tilde{A}_2 \cap \tilde{A}_3|. \end{aligned}$$

□

Theorem 3.13. *Let $A_1, A_2, A_3, A_4 \in FMS(X)$. Then*

$$\begin{aligned}|A_1 \cup A_2 \cup A_3 \cup A_4| &= |A_1| + |A_2| + |A_3| + |A_4| - |A_1 \cap A_2| - |A_1 \cap A_3| \\ &- |A_1 \cap A_4| - |A_2 \cap A_3| - |A_2 \cap A_4| - |A_3 \cap A_4| \\ &+ |A_1 \cap A_3 \cap A_4| + |A_2 \cap A_3 \cap A_4| \\ &+ |A_1 \cap A_2 \cap (A_3 \cup A_4)|.\end{aligned}$$

Proof. The result is analogous to Theorem 2.11. □

4. CONCLUSIONS

We have vividly covered an abridge account on the theories of multisets and fuzzy multisets, which conspicuously juxtaposed the concepts. By describing the operations between multisets and fuzzy multiset, we established some relevant results. This paper shall serves as a readily needed material for computer scientists and experts in control, to mention but a few.

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