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RANDOM COUPLED CAPUTO–HADAMARD FRACTIONAL DIFFERENTIAL SYSTEMS WITH FOUR-POINT BOUNDARY CONDITIONS IN GENERALIZED BANACH SPACES

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ABSTRACT. This paper deals with some existence and uniqueness of random solutions for a coupled system of Caputo–Hadamard fractional differential equations with four-point boundary conditions and random effects in generalized Banach spaces. Some applications are made of generalizations of classical random fixed point theorems on generalized Banach spaces. An illustrative example is presented in the last section.

1. INTRODUCTION

Fractional differential equations have recently been applied in various areas of engineering, mathematics, physics and bio-engineering, and other applied sciences [23]. For some fundamental results in the theory of fractional calculus and fractional differential equations we refer the reader to the monographs of Abbas *et al.* [1, 3, 4], Samko *et al.* [21], Kilbas *et al.* [17] and Zhou [27], and the references therein.

In [2], the authors studied a class of fractional differential equations.involving the Caputo–Hadamard fractional derivative, and in [11], the authors established a four-point boundary value problem for fractional integro-differential equations. In this article we discuss the existence and uniqueness of solutions for the following coupled system of Caputo–Hadamard fractional differential equations

$$\begin{cases} ({}^{Hc}D_1^{\alpha_1}u)(t,w) = f_1(t,u(t,w),v(t,w),w) \\ ({}^{Hc}D_1^{\alpha_2}v)(t,w) = f_2(t,u(t,w),v(t,w),w) \end{cases} ;\ t \in I := [1,T],\ w \in \Omega, \qquad (1.1)$$

with the four-point boundary conditions

$$\begin{cases} a_1u(1,w) - b_1u'(1,w) = d_1u(\xi_1,w) \\ a_2u(T,w) + b_2u'(T,w) = d_2u(\xi_2,w) \\ a_3v(1,w) - b_3v'(1,w) = d_3v(\xi_3,w) \\ a_4v(T,w) + b_4v'(T,w) = d_4v(\xi_4,w) \end{cases} ;\ w \in \Omega, \qquad (1.2)$$

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where $T > 1,\ a_i, b_i, d_i \in \mathbb{R},\ \xi_i \in (1, T);\ i = 1, 2, 3, 4,\ (\Omega, \mathcal{A})$ is a measurable space, $f_1, f_2 : I \times \mathbb{R}^m \times \mathbb{R}^m \times \Omega \to \mathbb{R}^m$; are given functions, $\mathbb{R}^m$; $m \in \mathbb{N}^*$ is the Euclidian Banach space with a suitable norm $\|\cdot\|$, ${}^{Hc}D_1^{\alpha_i}$ is the Caputo–Hadamard fractional derivative of order $\alpha_i \in (1, 2], ;\ i = 1, 2$.

2. Preliminaries

Let C be the Banach space of all continuous functions v from I into $\mathbb{R}^m$ with the supremum (uniform) norm

$$\|v\|_\infty := \sup_{t \in I} \|v(t)\|.$$

By $L^\infty(\Omega, \mathbb{R}_+)$ we denote the Banach space of measurable functions from Ω into $\mathbb{R}_+$ which are essentially bounded.

As usual, $AC(I)$ denotes the space of absolutely continuous functions from I into $\mathbb{R}^m$, and $L^1(I)$ denotes the space of Lebesgue-integrable functions $v : I \to \mathbb{R}^m$ with the norm

$$\|v\|_1 = \int_I \|v(t)\| dt.$$

For any $n \in \mathbb{N}^*$, we denote by $AC^n(I)$ the space defined by

$$AC^n(I) := \{w : I \to E : \frac{d^n}{dt^n} w(\cdot) \in AC(I)\}.$$

Let

$$\delta = t\frac{d}{dt},\ q > 0,\ n = [q] + 1,$$

where $[q]$ is the integer part of q. Define the space

$$AC^n_\delta := \{u : I \to E : \delta^{n-1}[u(\cdot)] \in AC(I)\}.$$

Also, by $\mathcal{C} := C \times C$ we denote the Banach space with the norm

$$\|(u, v)\|_{\mathcal{C}} = \|u\|_\infty + \|v\|_\infty.$$

Let $\beta_{\mathbb{R}^m}$ be the σ-algebra of Borel subsets of $\mathbb{R}^m$. A mapping $v : \Omega \to \mathbb{R}^m$ is said to be measurable if for any $B \in \beta_{\mathbb{R}^m}$, one has

$$v^{-1}(B) = \{w \in \Omega : v(w) \in B\} \subset \mathcal{A}.$$

To define integrals of sample paths of random process, it is necessary to define a jointly measurable map.

Definition 2.1. A mapping $T : \Omega \times \mathbb{R}^m \to \mathbb{R}^m$ is called jointly measurable if for any $B \in \beta_{\mathbb{R}^m}$, one has

$$T^{-1}(B) = \{(w, v) \in \Omega \times \mathbb{R}^m : T(w, v) \in B\} \subset \mathcal{A} \times \beta_{\mathbb{R}^m},$$

where $\mathcal{A} \times \beta_{\mathbb{R}^m}$ is the direct product of the σ-algebras $\mathcal{A}$ and $\beta_{\mathbb{R}^m}$ those defined in Ω and $\mathbb{R}^m$ respectively.

Definition 2.2. A function $T : \Omega \times \mathbb{R}^m \to \mathbb{R}^m$ is called jointly measurable if $T(\cdot, u)$ is measurable for all $u \in \mathbb{R}^m$ and $T(w, \cdot)$ is continuous for all $w \in \Omega$.

Let $T : \Omega \times \mathbb{R}^m \to \mathbb{R}^m$ be a mapping. Then T is called a random operator if $T(w, u)$ is measurable in w for all $u \in \mathbb{R}^m$ and it expressed as $T(w)u = T(w, u)$. In this case we also say that $T(w)$ is a random operator on $\mathbb{R}^m$. A random operator $T(w)$ on E is called continuous (resp. compact, totally bounded and completely continuous) if $T(w, u)$

is continuous (resp. compact, totally bounded and completely continuous) in u for all $w \in \Omega$. The details of completely continuous random operators in Banach spaces and their properties appear in Itoh [14].

Definition 2.3. [8] Let $\mathcal{P}(Y)$ be the family of all nonempty subsets of Y and C be a mapping from Ω into $\mathcal{P}(Y)$. A mapping $T : \{(w,y) : w \in \Omega,\ y \in C(w)\} \to Y$ is called random operator with stochastic domain C if C is measurable (i.e., for all closed $A \subset Y,\ \{w \in \Omega, C(w) \cap A \neq \emptyset\}$ is measurable) and for all open $D \subset Y$ and all $y \in Y,\ \{w \in \Omega : y \in C(w), T(w,y) \in D\}$ is measurable. T will be called continuous if every $T(w)$ is continuous. For a random operator T, a mapping $y : \Omega \to Y$ is called random (stochastic) fixed point of T if for $P-$almost all $w \in \Omega,\ y(w) \in C(w)$ and $T(w)y(w) = y(w)$ and for all open $D \subset Y,\ \{w \in \Omega : y(w) \in D\}$ is measurable.

Definition 2.4. A function $f : I \times \mathbb{R}^m \times \Omega \to \mathbb{R}^m$ is called random Carathéodory if the following conditions are satisfied:

(i) The map $(t,w) \to f(x,y,u,w)$ is jointly measurable for all $u \in \mathbb{R}^m$, and
(ii) The map $u \to f(t,u,w)$ is continuous for all $t \in I$ and $w \in \Omega$.

Let $x,,y \in \mathbb{R}^m$ with $x = (x_1, x_2, \dots, x_m),\ y = (y_1, y_2, \dots, y_m)$.
By $x \le y$ we mean $x_i \le y_i;\ i = 1, \dots, m$. Also $|x| = (|x_1|, |x_2|, \dots, |x_m|)$, $\max(x,y) = (\max(x_1,y_1), \max(x_2,y_2), \dots, \max(x_m,y_m))$, and $\mathbb{R}^m_+ = \{x \in \mathbb{R}^m : x_i \in \mathbb{R}_+,\ i = 1, \dots, m\}$. If $c \in \mathbb{R}$, then $x \le c$ means $x_i \le c;\ i = 1, \dots, m$.

Definition 2.5. Let X be a nonempty set. By a vector-valued metric on X we mean a map $d : X \times X \to \mathbb{R}^m$ with the following properties:

(i) $d(x,y) \ge 0$ for all $x,y \in X$, and if $d(x,y) = 0$, then $x = y$;
(ii) $d(x,y) = d(y,x)$ for all $x,y \in X$;
(iii) $d(x,z) \le d(x,y) + d(y,z)$ for all $x,y,z \in X$.

We call the pair (X,d) a generalized metric space with $d(x,y) := \begin{pmatrix} d_1(x,y) \\ d_2(x,y) \\ \cdot \\ \cdot \\ \cdot \\ d_m(x,y) \end{pmatrix}$.

Notice that d is a generalized metric space on X if and only if $d_i;\ i = 1, \dots, m$ are metrics on X. Similarly, we can define a generalized normed space $(X, \|\cdot\|)$ with $\|x-y\| := \begin{pmatrix} \|x-y\|_1 \\ \|x-y\|_2 \\ \cdot \\ \cdot \\ \cdot \\ \|x-y\|_m \end{pmatrix}$.

For $r = (r_1, \dots, r_m) \in \mathbb{R}^m$ and $x_0 \in X$, we will denote by

$$B_r(x_0) := \{x \in X : d(x_0,x) < r\} = \{x \in X : d_i(x_0,x) < r_i;\ i = 1, \dots, m\}$$

the open ball centered in x_0 with radius r and

$$\overline{B}_r(x_0) := \{x \in X : d(x_0,x) \le r\} = \{x \in X : d_i(x_0,x) \le r_i;\ i = 1, \dots, m\}$$

the closed ball centered in x_0 with radius r.

We mention that for generalized metric spaces, the notations of open, closed, compact, convex sets, convergence, and Cauchy sequence are similar to those in usual metric spaces.

Definition 2.6. [5, 25] A square matrix M of real numbers is said to be convergent to zero if and only if its spectral radius $\rho(M)$ is strictly less than 1. In other words, this means that all the eigenvalues of M are in the open unit disc i.e. $|\lambda| < 1$; for every $\lambda \in \mathbb{C}$ with $det(M - \lambda I) = 0$; where I denotes the unit matrix of $M_{m\times m}(\mathbb{R})$.

Example 2.7. The matrix $A \in M_{2\times 2}(\mathbb{R})$ defined by

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

converges to zero in the following cases:

(1) $b = c = 0,\ a, d > 0$ and $\max\{a, d\} < 1$.
(2) $c = 0,\ a, d > 0,\ a + d < 1$ and $-1 < b < 0$.
(3) $a + b = c + d = 0,\ a > 1,\ c > 0$ and $|a - c| < 1$.

Let us recall some definitions and properties of Hadamard fractional integration and differentiation. We refer to [17] for a more detailed analysis.

Definition 2.8. [17] (Hadamard fractional integral) The Hadamard fractional integral of order $q > 0$ for a function $u \in L^1(I)$, is defined as

$$({}^H I_1^q u)(x) = \frac{1}{\Gamma(q)} \int_1^x \left(\ln \frac{x}{s}\right)^{q-1} \frac{u(s)}{s} ds,$$

provided the integral exists.

Example 2.9. Let $0 < q < 1$. Then

$${}^H I_1^q \ln t = \frac{1}{\Gamma(2+q)} (\ln t)^{1+q},\ for\ a.e.\ t \in [1, e].$$

Definition 2.10. [17] (Hadamard fractional derivative) The Hadamard fractional derivative of order $q > 0$ applied to the function $u \in AC_\delta^n(I)$ is defined as

$$({}^H D_1^q u)(x) = \delta^n ({}^H I_1^{n-q} u)(x).$$

In particular, if $q \in (0, 1]$, then

$$({}^H D_1^q u)(x) = \delta ({}^H I_1^{1-q} u)(x).$$

Example 2.11. Let $0 < q < 1$. Then

$${}^H D_1^q \ln t = \frac{1}{\Gamma(2-q)} (\ln t)^{1-q},\ for\ a.e.\ t \in [1, e].$$

It has been proved (see e.g. Kilbas [[16], Theorem 4.8]) that in the space $L^1(I)$, the Hadamard fractional derivative is the left-inverse operator to the Hadamard fractional integral, i.e.

$$({}^H D_1^q)({}^H I_1^q w)(x) = w(x).$$

From Theorem 2.3 of [17], we have

$$({}^H I_1^q)({}^H D_1^q w)(x) = w(x) - \frac{({}^H I_1^{1-q} w)(1)}{\Gamma(q)} (\ln x)^{q-1}.$$

Analogous to the Hadamard fractional calculus, the Caputo–Hadamard fractional derivative is defined in the following way:

Definition 2.12. (Caputo–Hadamard fractional derivative) The Caputo–Hadamard fractional derivative of order $q > 0$ applied to the function $u \in AC^n_\delta$ is defined as

$$({}^{HC}D^q_1 u)(x) = ({}^H I^{n-q}_1 \delta^n u)(x).$$

In particular, if $q \in (0, 1]$, then

$$({}^{HC}D^q_1 u)(x) = ({}^H I^{1-q}_1 \delta u)(x).$$

Now, we prove the following lemma.

Lemma 2.1. *Let $h \in C(I)$, and $\alpha \in (1, 2]$. Then the unique solution of problem*

$$\begin{cases} ({}^{Hc}D^\alpha_1 u)(t) = h(t);\ t \in I, \\ a_1 u(1) - b_1 u'(1) = d_1 u(\xi_1), \\ a_2 u(T) + b_2 u'(T) = d_2 u(\xi_2), \end{cases}$$

is given by

$$u(t) = \int_1^T G(t, s) h(s) ds,$$

where the Green function G is given by: $G(t,s) =$

$$\begin{cases}
\frac{1}{s\Gamma(\alpha)}(ln\frac{t}{s})^{\alpha-1} + \frac{d_1(ln\xi_1)^{\alpha-2}(lnt)^{\alpha-1}}{s\Delta}[\frac{a_2}{\Gamma(\alpha)}(ln\frac{T}{s})^{\alpha-1} + \frac{b_2}{\Gamma(\alpha-1)}(ln\frac{T}{s})^{\alpha-2} - \frac{d_2}{\Gamma(\alpha)}(ln\frac{\xi_2}{s})^{\alpha-1}] \\
-\frac{d_1(lnt)^{\alpha-1}}{s\Delta\Gamma(\alpha)}(ln\frac{\xi_1}{s})^{\alpha-1}[a_2(lnT)^{\alpha-2} + \frac{b_2}{T}(\alpha-2)(lnT)^{\alpha-3} - d_2(ln\xi_2)^{\alpha-2}] \\
+\frac{d_1(lnt)^{\alpha-2}}{s\Delta\Gamma(\alpha)}(ln\frac{\xi_1}{s})^{\alpha-1}[a_2(lnT)^{\alpha-1} + \frac{b_2}{T}(\alpha-1)(lnT)^{\alpha-2} - d_2(ln\xi_2)^{\alpha-1}] \\
-\frac{d_1(ln\xi_1)^{\alpha-1}(lnt)^{\alpha-2}}{s\Delta}[\frac{a_2}{\Gamma(\alpha)}(ln\frac{T}{s})^{\alpha-1} + \frac{b_2}{\Gamma(\alpha-1)}(ln\frac{T}{s})^{\alpha-2} - \frac{d_2}{\Gamma(\alpha)}(ln\frac{\xi_2}{s})^{\alpha-1}];\ s \le \xi_1,\ s \le t \\
\\
\frac{d_1(ln\xi_1)^{\alpha-2}(lnt)^{\alpha-1}}{s\Delta}[\frac{a_2}{\Gamma(\alpha)}(ln\frac{T}{s})^{\alpha-1} + \frac{b_2}{\Gamma(\alpha-1)}(ln\frac{T}{s})^{\alpha-2} - \frac{d_2}{\Gamma(\alpha)}(ln\frac{\xi_2}{s})^{\alpha-1}] \\
-\frac{d_1(lnt)^{\alpha-1}}{s\Delta\Gamma(\alpha)}(ln\frac{\xi_1}{s})^{\alpha-1}[a_2(lnT)^{\alpha-2} + \frac{b_2}{T}(\alpha-2)(lnT)^{\alpha-3} - d_2(ln\xi_2)^{\alpha-2}] \\
+\frac{d_1(lnt)^{\alpha-2}}{s\Delta\Gamma(\alpha)}(ln\frac{\xi_1}{s})^{\alpha-1}[a_2(lnT)^{\alpha-1} + \frac{b_2}{T}(\alpha-1)(lnT)^{\alpha-2} - d_2(ln\xi_2)^{\alpha-1}] \\
-\frac{d_1(ln\xi_1)^{\alpha-1}(lnt)^{\alpha-2}}{s\Delta}[\frac{a_2}{\Gamma(\alpha)}(ln\frac{T}{s})^{\alpha-1} + \frac{b_2}{\Gamma(\alpha-1)}(ln\frac{T}{s})^{\alpha-2} - \frac{d_2}{\Gamma(\alpha)}(ln\frac{\xi_2}{s})^{\alpha-1}];\ s \le \xi_1,\ t \le s \\
\\
\frac{1}{s\Gamma(\alpha)}(ln\frac{t}{s})^{\alpha-1} \\
+\frac{d_1(ln\xi_1)^{\alpha-2}(lnt)^{\alpha-1}}{s\Delta}[\frac{a_2}{\Gamma(\alpha)}(ln\frac{T}{s})^{\alpha-1} + \frac{b_2}{\Gamma(\alpha-1)}(ln\frac{T}{s})^{\alpha-2} - \frac{d_2}{\Gamma(\alpha)}(ln\frac{\xi_2}{s})^{\alpha-1}] \\
-\frac{d_1(ln\xi_1)^{\alpha-1}(lnt)^{\alpha-2}}{s\Delta}[\frac{a_2}{\Gamma(\alpha)}(ln\frac{T}{s})^{\alpha-1} + \frac{b_2}{\Gamma(\alpha-1)}(ln\frac{T}{s})^{\alpha-2} - \frac{d_2}{\Gamma(\alpha)}(ln\frac{\xi_2}{s})^{\alpha-1}];\ \xi_1 \le s \le \xi_2,\ s \le t \\
\\
\frac{d_1(ln\xi_1)^{\alpha-2}(lnt)^{\alpha-1}}{s\Delta}[\frac{a_2}{\Gamma(\alpha)}(ln\frac{T}{s})^{\alpha-1} + \frac{b_2}{\Gamma(\alpha-1)}(ln\frac{T}{s})^{\alpha-2} - \frac{d_2}{\Gamma(\alpha)}(ln\frac{\xi_2}{s})^{\alpha-1}] \\
-\frac{d_1(ln\xi_1)^{\alpha-1}(lnt)^{\alpha-2}}{s\Delta}[\frac{a_2}{\Gamma(\alpha)}(ln\frac{T}{s})^{\alpha-1} + \frac{b_2}{\Gamma(\alpha-1)}(ln\frac{T}{s})^{\alpha-2} - \frac{d_2}{\Gamma(\alpha)}(ln\frac{\xi_2}{s})^{\alpha-1}];\ \xi_1 \le s \le \xi_2,\ t \le s \\
\\
\frac{1}{s\Gamma(\alpha)}(ln\frac{t}{s})^{\alpha-1} \\
+\frac{d_1(ln\xi_1)^{\alpha-2}(lnt)^{\alpha-1}}{s\Delta}[\frac{a_2}{\Gamma(\alpha)}(ln\frac{T}{s})^{\alpha-1} + \frac{b_2}{\Gamma(\alpha-1)}(ln\frac{T}{s})^{\alpha-2}] \\
-\frac{d_1(ln\xi_1)^{\alpha-1}(lnt)^{\alpha-2}}{s\Delta}[\frac{a_2}{\Gamma(\alpha)}(ln\frac{T}{s})^{\alpha-1} + \frac{b_2}{\Gamma(\alpha-1)}(ln\frac{T}{s})^{\alpha-2}];\ \xi_2 \le s,\ s \le t \\
\\
\frac{d_1(ln\xi_1)^{\alpha-2}(lnt)^{\alpha-1}}{s\Delta}[\frac{a_2}{\Gamma(\alpha)}(ln\frac{T}{s})^{\alpha-1} + \frac{b_2}{\Gamma(\alpha-1)}(ln\frac{T}{s})^{\alpha-2}] \\
-\frac{d_1(ln\xi_1)^{\alpha-1}(lnt)^{\alpha-2}}{s\Delta}[\frac{a_2}{\Gamma(\alpha)}(ln\frac{T}{s})^{\alpha-1} + \frac{b_2}{\Gamma(\alpha-1)}(ln\frac{T}{s})^{\alpha-2}];\ \xi_2 \le s,\ t \le s
\end{cases}$$

with

$$\begin{aligned}
\Delta \quad = \quad & d_1(ln\xi_1)^{\alpha-1}[a_2(lnT)^{\alpha-2} + \frac{b_2}{T}(\alpha-2)(lnT)^{\alpha-3} - d_2(ln\xi_2)^{\alpha-2}] \\
- \quad & d_1(ln\xi_1)^{\alpha-2}[a_2(lnT)^{\alpha-1} + \frac{b_2}{T}(\alpha-1)(lnT)^{\alpha-2} - d_2(ln\xi_2)^{\alpha-1}] \neq 0.
\end{aligned}$$

Proof. Solving the linear equation

$$({}^{Hc}D_1^{\alpha}u)(t) = h(t),$$

we get

$$u(t) = {}^H I_1^{\alpha}h(t) + c_1(lnt)^{\alpha-1} + c_2(lnt)^{\alpha-2}. \tag{2.1}$$

On the other hand, by the relation $D_1^{\beta}I_1^{\alpha}u(t) = I_1^{\alpha-\beta}u(t)$, we get

$$\begin{aligned}
u'(t) \quad = \quad & \frac{1}{\Gamma(\alpha-1)}\int_1^t(ln\frac{t}{s})^{\alpha-2}h(s)\frac{ds}{s} \\
+ \quad & \frac{\alpha-1}{t}c_1(lnt)^{\alpha-2} + \frac{\alpha-2}{t}c_2(lnt)^{\alpha-3}.
\end{aligned}$$

From the boundary conditions, we have

$$\begin{cases} [d_1(ln\xi_1)^{\alpha-1}]c_1+[d_1(ln\xi_1)^{\alpha-2}]c_2 = a_1^H I_1^\alpha h(1) - b_1^H I_1^{\alpha-1}h(1) - d_1^H I_1^\alpha h(\xi_1) \\ \\ [a_2(lnT)^{\alpha-1}+\frac{b_2}{T}(\alpha-1)(lnT)^{\alpha-2}-d_2(ln\xi_2)^{\alpha-1}]c_1 \\ +[a_2(lnT)^{\alpha-2}+\frac{b_2}{T}(\alpha-2)(lnT)^{\alpha-3}-d_2(ln\xi_2)^{\alpha-2}]c_2 \\ = d_2^H I_1^\alpha h(\xi_2) - a_2^H I_1^\alpha h(T) - b_2^H I_1^{\alpha-1}h(T). \end{cases}$$

Thus we get

$$\begin{aligned} c_1 &= \tfrac{d_1(ln\xi_1)^{\alpha-2}}{\Delta}(a_2\int_1^T(ln\tfrac{T}{s})^{\alpha-1}\tfrac{h(s)}{s\Gamma(\alpha)}ds \\ &+ b_2\int_1^T(ln\tfrac{T}{s})^{\alpha-2}\tfrac{h(s)}{s\Gamma(\alpha-1)}ds - d_2\int_1^{\xi_2}(ln\tfrac{\xi_2}{s})^{\alpha-1}\tfrac{h(s)}{s\Gamma(\alpha)}ds \\ &- \tfrac{d_1}{\Delta\Gamma(\alpha)}(a_2(lnT)^{\alpha-2}+\tfrac{b_2}{T}(\alpha-2)(lnT)^{\alpha-3} \\ &- d_2(ln\xi_2)^{\alpha-2})\int_1^{\xi_1}(ln\tfrac{\xi_1}{s})^{\alpha-1}\tfrac{h(s)}{s}ds, \end{aligned}$$

and

$$\begin{aligned} c_2 &= \tfrac{d_1}{\Delta\Gamma(\alpha)}(a_2(lnT)^{\alpha-1}+\tfrac{b_2}{T}(\alpha-1)(lnT)^{\alpha-2} \\ &- d_2(ln\xi_2)^{\alpha-1})\int_1^{\xi_1}(ln\tfrac{\xi_1}{s})^{\alpha-1}\tfrac{h(s)}{s}ds \\ &- \tfrac{d_1(ln\xi_1)^{\alpha-1}}{\Delta}(a_2\int_1^T(ln\tfrac{T}{s})^{\alpha-1}\tfrac{h(s)}{s\Gamma(\alpha)}ds \\ &+ b_2\int_1^T(ln\tfrac{T}{s})^{\alpha-2}\tfrac{h(s)}{s\Gamma(\alpha-1)}ds - d_2\int_1^{\xi_2}(ln\tfrac{\xi_2}{s})^{\alpha-1}\tfrac{h(s)}{s\Gamma(\alpha)}ds. \end{aligned}$$

Substituting the values of c_1 and c_2 in (2.1), we get

$$\begin{aligned} u(t) &= \tfrac{1}{\Gamma(\alpha)}\int_1^t(ln\tfrac{t}{s})^{\alpha-1}\tfrac{h(s)}{s}ds \\ \\ &+ \tfrac{d_1(ln\xi_1)^{\alpha-2}(lnt)^{\alpha-1}}{\Delta}[a_2\int_1^T(ln\tfrac{T}{s})^{\alpha-1}\tfrac{h(s)}{s\Gamma(\alpha)}ds \\ &+ b_2\int_1^T(ln\tfrac{T}{s})^{\alpha-2}\tfrac{h(s)}{s\Gamma(\alpha-1)}ds - d_2\int_1^{\xi_2}(ln\tfrac{\xi_2}{s})^{\alpha-1}\tfrac{h(s)}{s\Gamma(\alpha)}ds] \\ &- \tfrac{d_1(lnt)^{\alpha-1}}{\Delta\Gamma(\alpha)}[a_2(lnT)^{\alpha-2}+\tfrac{b_2}{T}(\alpha-2)(lnT)^{\alpha-3} \\ &- d_2(ln\xi_2)^{\alpha-2}]\int_1^{\xi_1}(ln\tfrac{\xi_1}{s})^{\alpha-1}\tfrac{h(s)}{s}ds \\ \\ &+ \tfrac{d_1(lnt)^{\alpha-2}}{\Delta\Gamma(\alpha)}[a_2(lnT)^{\alpha-1}+\tfrac{b_2}{T}(\alpha-1)(lnT)^{\alpha-2} \\ &- d_2(ln\xi_2)^{\alpha-1}]\int_1^{\xi_1}(ln\tfrac{\xi_1}{s})^{\alpha-1}\tfrac{h(s)}{s}ds \\ \\ &- \tfrac{d_1(ln\xi_1)^{\alpha-1}(lnt)^{\alpha-2}}{\Delta}[a_2\int_1^T(ln\tfrac{T}{s})^{\alpha-1}\tfrac{h(s)}{s\Gamma(\alpha)}ds \\ &+ b_2\int_1^T(ln\tfrac{T}{s})^{\alpha-2}\tfrac{h(s)}{s\Gamma(\alpha-1)}ds - d_2\int_1^{\xi_2}(ln\tfrac{\xi_2}{s})^{\alpha-1}\tfrac{h(s)}{s\Gamma(\alpha)}ds] \\ \\ &= \int_1^T G(t,s)h(s)ds. \end{aligned}$$

Remark. *Notice that the function $G(\cdot,\cdot)$ is not continuous over $[1,T]\times[1,T]$, however the function $t\mapsto\int_1^t G(t,s)ds$ is continuous on $[1,T]$.*

In the sequel we will make use of the following random fixed point theorems:

Theorem 2.2. [12, 20] *Let $(\Omega,\mathcal{F})$ be a measurable space, X be a real separable generalized Banach space and $F:\Omega\times X\to X$ be a continuous random operator, and let $M(w)\in\mathcal{M}_{n\times n}(\mathbb{R}_+)$ be a random variable matric such that for every $w\in\Omega$, the matrix $M(w)$ converges to 0 and*

$$d(F(w,x_1),F(w,x_2))\le M(w)d(x_1,x_2);\ for\ each\ x_1,x_2\in X\ and\ w\in\Omega,$$

then there exists a random variable $x : \Omega \to X$ which is the unique random fixed point of F.

Theorem 2.3. [12, 20] *Let $(\Omega, \mathcal{F})$ be a measurable space, X be a real separable generalized Banach space and $F : \Omega \times X \to X$ be a completely continuous random operator, Then, either*

(i) *the random equation $F(w, x) = x$ has a random solution, i.e., there is a measurable function $x : \Omega \to X$ such that $F(w, x(w)) = x(w)$ for all $w \in \Omega$, or*
(ii) *the set $M = \{x : \Omega \to X \; is \; measurable : \; \lambda(w)F(w, x) = x\}$ is unbounded for some measurable function $\lambda : \Omega \to X$ with $0 < \lambda(w) < 1$ on Ω.*

3. Main Results

In this section, we are concerned with the existence and uniqueness results of the coupled system (1.1)-(1.2).

Definition 3.1. By a random solution of the problem (1.1)-(1.2) we mean a coupled measurable functions $(u, v) \in C(I) \times C(I)$ satisfying the boundary conditions (1.2), and the equations (1.1) on I.

The following hypotheses will be used in the sequel.

(H_1) The functions $f_i; \; i = 1, 2$ are Carathéodory.
(H_2) There exist continuous functions $p_i, q_i : I \to L^\infty(\Omega, \mathbb{R}_+); \; i = 1, 2$ such that
$$\|f_i(t, u_1, v_1) - f_i(t, u_2, v_2)\| \leq p_i(t, w)\|u_1 - u_2\| + q_i(t, w)\|v_1 - v_2\|;$$
$$for \; a.e. \; t \in I, \; and \; each \; u_i, v_i \in \mathbb{R}^m, \; i = 1, 2.$$
(H_3) There exist continuous functions $h_i, k_i, l_i : I \to L^\infty(\Omega, \mathbb{R}_+); \; i = 1, 2$ such that
$$\|f_i(t, u, v)\| \leq h_i(t, w) + k_i(t, w)\|u\| + l_i(t, w)\|v\|; \; for \; a.e. \; t \in I, \; and \; each \; u, v \in \mathbb{R}^m.$$

First, we prove an existence and uniqueness result for the coupled system (1.1)- (1.2) by using Banach's random fixed point theorem type in generalized Banach spaces. As a consequence of Lemma 2.1, we define the operators $N_1, N_2 : \mathcal{C} \times \Omega \to C(I)$ by
$$(N_1(u, v))(t, w) = \int_1^T G_1(t, s) f_1(s, u(s, w), w) ds, \tag{3.1}$$
and
$$(N_2(u, v))(t, w) = \int_1^T G_2(t, s) f_2(s, u(s, w), w) ds, \tag{3.2}$$

where $G_1(t,s) =$

$$
\begin{cases}
\frac{1}{s\Gamma(\alpha_1)}(ln\frac{t}{s})^{\alpha_1-1}+\frac{d_1(ln\xi_1)^{\alpha_1-2}(lnt)^{\alpha_1-1}}{s\Delta_1}[\frac{a_2}{\Gamma(\alpha_1)}(ln\frac{T}{s})^{\alpha_1-1}+\frac{b_2}{\Gamma(\alpha_1-1)}(ln\frac{T}{s})^{\alpha_1-2}-\frac{d_2}{\Gamma(\alpha_1)}(ln\frac{\xi_2}{s})^{\alpha_1-1}] \\
-\frac{d_1(lnt)^{\alpha_1-1}}{s\Delta_1\Gamma(\alpha_1)}(ln\frac{\xi_1}{s})^{\alpha_1-1}[a_2(lnT)^{\alpha_1-2}+\frac{b_2}{T}(\alpha_1-2)(lnT)^{\alpha_1-3}-d_2(ln\xi_2)^{\alpha_1-2}] \\
+\frac{d_1(lnt)^{\alpha_1-2}}{s\Delta_1\Gamma(\alpha_1)}(ln\frac{\xi_1}{s})^{\alpha_1-1}[a_2(lnT)^{\alpha_1-1}+\frac{b_2}{T}(\alpha_1-1)(lnT)^{\alpha_1-2}-d_2(ln\xi_2)^{\alpha_1-1}] \\
-\frac{d_1(ln\xi_1)^{\alpha_1-1}(lnt)^{\alpha_1-2}}{s\Delta_1}[\frac{a_2}{\Gamma(\alpha_1)}(ln\frac{T}{s})^{\alpha_1-1}+\frac{b_2}{\Gamma(\alpha_1-1)}(ln\frac{T}{s})^{\alpha_1-2}-\frac{d_2}{\Gamma(\alpha_1)}(ln\frac{\xi_2}{s})^{\alpha_1-1}];\ s\le\xi_1,\ s\le t \\
\\
\frac{d_1(ln\xi_1)^{\alpha_1-2}(lnt)^{\alpha_1-1}}{s\Delta_1}[\frac{a_2}{\Gamma(\alpha_1)}(ln\frac{T}{s})^{\alpha_1-1}+\frac{b_2}{\Gamma(\alpha_1-1)}(ln\frac{T}{s})^{\alpha_1-2}-\frac{d_2}{\Gamma(\alpha_1)}(ln\frac{\xi_2}{s})^{\alpha_1-1}] \\
-\frac{d_1(lnt)^{\alpha_1-1}}{s\Delta_1\Gamma(\alpha_1)}(ln\frac{\xi_1}{s})^{\alpha_1-1}[a_2(lnT)^{\alpha_1-2}+\frac{b_2}{T}(\alpha_1-2)(lnT)^{\alpha_1-3}-d_2(ln\xi_2)^{\alpha_1-2}] \\
+\frac{d_1(lnt)^{\alpha_1-2}}{s\Delta_1\Gamma(\alpha_1)}(ln\frac{\xi_1}{s})^{\alpha_1-1}[a_2(lnT)^{\alpha_1-1}+\frac{b_2}{T}(\alpha_1-1)(lnT)^{\alpha_1-2}-d_2(ln\xi_2)^{\alpha_1-1}] \\
-\frac{d_1(ln\xi_1)^{\alpha_1-1}(lnt)^{\alpha_1-2}}{s\Delta_1}[\frac{a_2}{\Gamma(\alpha_1)}(ln\frac{T}{s})^{\alpha_1-1}+\frac{b_2}{\Gamma(\alpha_1-1)}(ln\frac{T}{s})^{\alpha_1-2}-\frac{d_2}{\Gamma(\alpha_1)}(ln\frac{\xi_2}{s})^{\alpha_1-1}];\ s\le\xi_1,\ t\le s \\
\\
\frac{1}{s\Gamma(\alpha_1)}(ln\frac{t}{s})^{\alpha_1-1} \\
+\frac{d_1(ln\xi_1)^{\alpha_1-2}(lnt)^{\alpha_1-1}}{s\Delta_1}[\frac{a_2}{\Gamma(\alpha_1)}(ln\frac{T}{s})^{\alpha_1-1}+\frac{b_2}{\Gamma(\alpha_1-1)}(ln\frac{T}{s})^{\alpha_1-2}-\frac{d_2}{\Gamma(\alpha_1)}(ln\frac{\xi_2}{s})^{\alpha_1-1}] \\
-\frac{d_1(ln\xi_1)^{\alpha_1-1}(lnt)^{\alpha_1-2}}{s\Delta_1}[\frac{a_2}{\Gamma(\alpha_1)}(ln\frac{T}{s})^{\alpha_1-1}+\frac{b_2}{\Gamma(\alpha_1-1)}(ln\frac{T}{s})^{\alpha_1-2}-\frac{d_2}{\Gamma(\alpha_1)}(ln\frac{\xi_2}{s})^{\alpha_1-1}];\ \xi_1\le s\le\xi_2,\ s\le t \\
\\
\frac{d_1(ln\xi_1)^{\alpha_1-2}(lnt)^{\alpha_1-1}}{s\Delta_1}[\frac{a_2}{\Gamma(\alpha_1)}(ln\frac{T}{s})^{\alpha_1-1}+\frac{b_2}{\Gamma(\alpha_1-1)}(ln\frac{T}{s})^{\alpha_1-2}-\frac{d_2}{\Gamma(\alpha_1)}(ln\frac{\xi_2}{s})^{\alpha_1-1}] \\
-\frac{d_1(ln\xi_1)^{\alpha_1-1}(lnt)^{\alpha_1-2}}{s\Delta_1}[\frac{a_2}{\Gamma(\alpha_1)}(ln\frac{T}{s})^{\alpha_1-1}+\frac{b_2}{\Gamma(\alpha_1-1)}(ln\frac{T}{s})^{\alpha_1-2}-\frac{d_2}{\Gamma(\alpha_1)}(ln\frac{\xi_2}{s})^{\alpha_1-1}];\ \xi_1\le s\le\xi_2,\ t\le s \\
\\
\frac{1}{s\Gamma(\alpha_1)}(ln\frac{t}{s})^{\alpha_1-1} \\
+\frac{d_1(ln\xi_1)^{\alpha_1-2}(lnt)^{\alpha_1-1}}{s\Delta_1}[\frac{a_2}{\Gamma(\alpha_1)}(ln\frac{T}{s})^{\alpha_1-1}+\frac{b_2}{\Gamma(\alpha_1-1)}(ln\frac{T}{s})^{\alpha_1-2}] \\
-\frac{d_1(ln\xi_1)^{\alpha_1-1}(lnt)^{\alpha_1-2}}{s\Delta_1}[\frac{a_2}{\Gamma(\alpha_1)}(ln\frac{T}{s})^{\alpha_1-1}+\frac{b_2}{\Gamma(\alpha_1-1)}(ln\frac{T}{s})^{\alpha_1-2}];\ \xi_2\le s,\ s\le t \\
\\
\frac{d_1(ln\xi_1)^{\alpha_1-2}(lnt)^{\alpha_1-1}}{s\Delta_1}[\frac{a_2}{\Gamma(\alpha_1)}(ln\frac{T}{s})^{\alpha_1-1}+\frac{b_2}{\Gamma(\alpha_1-1)}(ln\frac{T}{s})^{\alpha_1-2}] \\
-\frac{d_1(ln\xi_1)^{\alpha_1-1}(lnt)^{\alpha_1-2}}{s\Delta_1}[\frac{a_2}{\Gamma(\alpha_1)}(ln\frac{T}{s})^{\alpha_1-1}+\frac{b_2}{\Gamma(\alpha_1-1)}(ln\frac{T}{s})^{\alpha_1-2}];\ \xi_2\le s,\ t\le s
\end{cases}
$$

with

$$
\begin{aligned}
\Delta_1 &= d_1(ln\xi_1)^{\alpha_1-1}[a_2(lnT)^{\alpha_1-2}+\frac{b_2}{T}(\alpha_1-2)(lnT)^{\alpha_1-3}-d_2(ln\xi_2)^{\alpha_1-2}] \\
&- d_1(ln\xi_1)^{\alpha_1-2}[a_2(lnT)^{\alpha_1-1}+\frac{b_2}{T}(\alpha_1-1)(lnT)^{\alpha_1-2}-d_2(ln\xi_2)^{\alpha_1-1}] \neq 0.
\end{aligned}
$$

and $G_2(t,s) =$

$$\begin{cases} \frac{1}{s\Gamma(\alpha_2)}(ln\frac{t}{s})^{\alpha_2-1} + \frac{d_3(ln\xi_3)^{\alpha_2-2}(lnt)^{\alpha_2-1}}{s\Delta_2}[\frac{a_4}{\Gamma(\alpha_2)}(ln\frac{T}{s})^{\alpha_2-1} + \frac{b_4}{\Gamma(\alpha_2-1)}(ln\frac{T}{s})^{\alpha_2-2} - \frac{d_4}{\Gamma(\alpha_2)}(ln\frac{\xi_4}{s})^{\alpha_2-1}] \\ -\frac{d_3(lnt)^{\alpha_2-1}}{s\Delta_2\Gamma(\alpha_2)}(ln\frac{\xi_3}{s})^{\alpha_2-1}[a_4(lnT)^{\alpha_2-2} + \frac{b_4}{T}(\alpha_2-2)(lnT)^{\alpha_2-3} - d_4(ln\xi_4)^{\alpha_2-2}] \\ +\frac{d_3(lnt)^{\alpha_2-2}}{s\Delta_2\Gamma(\alpha_2)}(ln\frac{\xi_3}{s})^{\alpha_2-1}[a_4(lnT)^{\alpha_2-1} + \frac{b_4}{T}(\alpha_2-1)(lnT)^{\alpha_2-2} - d_4(ln\xi_4)^{\alpha_2-1}] \\ -\frac{d_3(ln\xi_3)^{\alpha_2-1}(lnt)^{\alpha_2-2}}{s\Delta_2}[\frac{a_4}{\Gamma(\alpha_2)}(ln\frac{T}{s})^{\alpha_2-1} + \frac{b_4}{\Gamma(\alpha_2-1)}(ln\frac{T}{s})^{\alpha_2-2} - \frac{d_4}{\Gamma(\alpha_2)}(ln\frac{\xi_4}{s})^{\alpha_2-1}];\ s \le \xi_3,\ s \le t \\ \\ \frac{d_3(ln\xi_3)^{\alpha_2-2}(lnt)^{\alpha_2-1}}{s\Delta_2}[\frac{a_4}{\Gamma(\alpha_2)}(ln\frac{T}{s})^{\alpha_2-1} + \frac{b_4}{\Gamma(\alpha_2-1)}(ln\frac{T}{s})^{\alpha_2-2} - \frac{d_4}{\Gamma(\alpha_2)}(ln\frac{\xi_4}{s})^{\alpha_2-1}] \\ -\frac{d_3(lnt)^{\alpha_2-1}}{s\Delta_2\Gamma(\alpha_2)}(ln\frac{\xi_3}{s})^{\alpha_2-1}[a_4(lnT)^{\alpha_2-2} + \frac{b_4}{T}(\alpha_2-2)(lnT)^{\alpha_2-3} - d_4(ln\xi_4)^{\alpha_2-2}] \\ +\frac{d_3(lnt)^{\alpha_2-2}}{s\Delta_2\Gamma(\alpha_2)}(ln\frac{\xi_3}{s})^{\alpha_2-1}[a_4(lnT)^{\alpha_2-1} + \frac{b_4}{T}(\alpha_2-1)(lnT)^{\alpha_2-2} - d_4(ln\xi_4)^{\alpha_2-1}] \\ -\frac{d_3(ln\xi_3)^{\alpha_2-1}(lnt)^{\alpha_2-2}}{s\Delta_2}[\frac{a_4}{\Gamma(\alpha_2)}(ln\frac{T}{s})^{\alpha_2-1} + \frac{b_4}{\Gamma(\alpha_2-1)}(ln\frac{T}{s})^{\alpha_2-2} - \frac{d_4}{\Gamma(\alpha_2)}(ln\frac{\xi_4}{s})^{\alpha_2-1}];\ s \le \xi_3,\ t \le s \\ \\ \frac{1}{s\Gamma(\alpha_2)}(ln\frac{t}{s})^{\alpha_2-1} \\ +\frac{d_3(ln\xi_3)^{\alpha_2-2}(lnt)^{\alpha_2-1}}{s\Delta_2}[\frac{a_4}{\Gamma(\alpha_2)}(ln\frac{T}{s})^{\alpha_2-1} + \frac{b_4}{\Gamma(\alpha_2-1)}(ln\frac{T}{s})^{\alpha_2-2} - \frac{d_4}{\Gamma(\alpha_2)}(ln\frac{\xi_4}{s})^{\alpha_2-1}] \\ -\frac{d_3(ln\xi_3)^{\alpha_2-1}(lnt)^{\alpha_2-2}}{s\Delta_2}[\frac{a_4}{\Gamma(\alpha_2)}(ln\frac{T}{s})^{\alpha_2-1} + \frac{b_4}{\Gamma(\alpha_2-1)}(ln\frac{T}{s})^{\alpha_2-2} - \frac{d_4}{\Gamma(\alpha_2)}(ln\frac{\xi_4}{s})^{\alpha_2-1}];\ \xi_3 \le s \le \xi_4,\ s \le t \\ \\ \frac{d_3(ln\xi_3)^{\alpha_2-2}(lnt)^{\alpha_2-1}}{s\Delta_2}[\frac{a_4}{\Gamma(\alpha_2)}(ln\frac{T}{s})^{\alpha_2-1} + \frac{b_4}{\Gamma(\alpha_2-1)}(ln\frac{T}{s})^{\alpha_2-2} - \frac{d_4}{\Gamma(\alpha_2)}(ln\frac{\xi_4}{s})^{\alpha_2-1}] \\ -\frac{d_3(ln\xi_3)^{\alpha_2-1}(lnt)^{\alpha_2-2}}{s\Delta_2}[\frac{a_4}{\Gamma(\alpha_2)}(ln\frac{T}{s})^{\alpha_2-1} + \frac{b_4}{\Gamma(\alpha_2-1)}(ln\frac{T}{s})^{\alpha_2-2} - \frac{d_4}{\Gamma(\alpha_2)}(ln\frac{\xi_4}{s})^{\alpha_2-1}];\ \xi_3 \le s \le \xi_4,\ t \le s \\ \\ \frac{1}{s\Gamma(\alpha_2)}(ln\frac{t}{s})^{\alpha_2-1} \\ +\frac{d_3(ln\xi_3)^{\alpha_2-2}(lnt)^{\alpha_2-1}}{s\Delta_2}[\frac{a_4}{\Gamma(\alpha_2)}(ln\frac{T}{s})^{\alpha_2-1} + \frac{b_4}{\Gamma(\alpha_2-1)}(ln\frac{T}{s})^{\alpha_2-2}] \\ -\frac{d_3(ln\xi_3)^{\alpha_2-1}(lnt)^{\alpha_2-2}}{s\Delta_2}[\frac{a_4}{\Gamma(\alpha_2)}(ln\frac{T}{s})^{\alpha_2-1} + \frac{b_4}{\Gamma(\alpha_2-1)}(ln\frac{T}{s})^{\alpha_2-2}];\ \xi_4 \le s,\ s \le t \\ \\ \frac{d_3(ln\xi_3)^{\alpha_2-2}(lnt)^{\alpha_2-1}}{s\Delta_2}[\frac{a_4}{\Gamma(\alpha_2)}(ln\frac{T}{s})^{\alpha_2-1} + \frac{b_4}{\Gamma(\alpha_2-1)}(ln\frac{T}{s})^{\alpha_2-2}] \\ -\frac{d_3(ln\xi_3)^{\alpha_2-1}(lnt)^{\alpha_2-2}}{s\Delta_2}[\frac{a_4}{\Gamma(\alpha_2)}(ln\frac{T}{s})^{\alpha_2-1} + \frac{b_4}{\Gamma(\alpha_2-1)}(ln\frac{T}{s})^{\alpha_2-2}];\ \xi_4 \le s,\ t \le s \end{cases}$$

with

$$\begin{aligned} \Delta_2 &= d_3(ln\xi_3)^{\alpha_2-1}[a_4(lnT)^{\alpha_2-2} + \frac{b_4}{T}(\alpha_2-2)(lnT)^{\alpha_2-3} - d_4(ln\xi_4)^{\alpha_2-2}] \\ &- d_3(ln\xi_3)^{\alpha_2-2}[a_4(lnT)^{\alpha_2-1} + \frac{b_4}{T}(\alpha_2-1)(lnT)^{\alpha_2-2} - d_4(ln\xi_4)^{\alpha_2-1}] \neq 0. \end{aligned}$$

Set

$$G_i^* = \sup_{t\in[1,T]} \int_1^t |G_i(t,s)|ds;\ i = 1,2.$$

Theorem 3.1. *Assume that the hypotheses* (H_1) *and* (H_2) *hold. If for every* $w \in \Omega$, *the matrix*

$$M(w) := \begin{pmatrix} G_1^*\|p_1(\cdot,w)\|_\infty & G_1^*\|q_1(\cdot,w)\|_\infty \\ G_2^*\|p_2(\cdot,w)\|_\infty & G_2^*\|q_2(\cdot,w)\|_\infty \end{pmatrix}$$

converges to 0, *then the coupled system (1.1)-(1.2) has a unique random solution.*

Proof. Consider the operator $N : \mathcal{C} \times \Omega \to \mathcal{C}$ defined by

$$(N(u,v))(t,w) = ((N_1(u,v))(t,w), (N_2(u,v))(t,w)). \tag{3.3}$$

Clearly, the fixed points of the operator N are random solutions of the coupled system (1.1)-(1.2). Let us show that N is a random operator on $\mathcal{C}$. Since $f_i;\ i = 1,2$ are Carathéodory functions, then $w \to f_i(t,u,v,w)$ are measurable maps in view of Proposition 4.1.2 we conclude that the maps

$$w \to (N_1(u,v))(t,w) \ and \ w \to (N_2(u,v))(t,w),$$

are measurable. As a result, N is a random operator on $\mathcal{C} \times \Omega$ into $\mathcal{C}$. We show that N satisfies all conditions of Theorem 2.2.
For any $w \in \Omega$ and each $(u_1,v_1),(u_2,v_2) \in \mathcal{C}$ and $t \in I$, we have

$$\begin{aligned} & \|(N_1(u_1,v_1))(t,w) - (N_1(u_2,v_2))(t,w)\| \\ \leq & \int_1^T |G_1(t,s)| \|f_1(s,u_1(s,w),v_1(s,w),w) - f_1(s,u_2(s,w),v_2(s,w),w)\| ds \\ \leq & \int_1^T |G_1(t,s)| (p_1(s,w)\|u_1(s,w) - u_2(s,w)\| + q_1(s,w)\|v_1(s,w) - v_2(s,w)\|) ds \\ \leq & G_1^*(\|p_1(\cdot,w)\|_\infty \|u_1(\cdot,w) - u_2(\cdot,w)\|_\infty + \|q_1(\cdot,w)\|_\infty \|v_1(\cdot,w) - v_2(\cdot,w)\|_\infty). \end{aligned}$$

Then,

$$\begin{aligned} & \|(N_1(u_1,v_1))(\cdot,w) - (N_1(u_2,v_2))(\cdot,w)\|_\infty \\ \leq & G_1^*(\|p_1(\cdot,w)\|_\infty \|u_1(\cdot,w) - u_2(\cdot,w)\|_\infty + \|q_1(\cdot,w)\|_\infty \|v_1(\cdot,w) - v_2(\cdot,w)\|_\infty).. \end{aligned}$$

Also, for any $w \in \Omega$ and each $(u_1,v_1),(u_2,v_2) \in \mathcal{C}$ and $t \in I$, we get

$$\begin{aligned} & \|(N_2(u_1,v_1))(\cdot,w) - (N_2(u_2,v_2))(\cdot,w)\|_\infty \\ \leq & G_2^*(\|p_2(\cdot,w)\|_\infty \|u_1(\cdot,w) - u_2(\cdot,w)\|_\infty + \|q_2(\cdot,w)\|_\infty \|v_1(\cdot,w) - v_2(\cdot,w)\|_\infty). \end{aligned}$$

Thus,

$$d((N(u_1,v_1))(\cdot,w),(N(u_2,v_2))(\cdot,w)) \leq M(w)d((u_1(\cdot,w),v_1(\cdot,w)),(u_2(\cdot,w),v_2(\cdot,w))),$$

where

$$d((u_1(\cdot,w),v_1(\cdot,w)),(u_2(\cdot,w),v_2(\cdot,w))) = \begin{pmatrix} \|u_1(\cdot,w) - u_2(\cdot,w)\|_\infty \\ \|v_1(\cdot,w) - v_2(\cdot,w)\|_\infty \end{pmatrix}.$$

Since for every $w \in \Omega$, the matrix $M(w)$ converges to zero, then Theorem 2.2 implies that coupled system (1.1)- (1.2) has a unique random solution.

Set

$$A(w) := (G_1^*\|h_1(\cdot,w)\|_\infty + G_2^*\|h_2(\cdot,w)\|_\infty),$$

and

$$C(w) := \max\{G_1^*\|k_1(\cdot,w)\|_\infty + G_2^*\|k_2(\cdot,w)\|_\infty, G_1^*\|l_1(\cdot,w)\|_\infty + G_2^*\|l_2(\cdot,w)\|_\infty\}.$$

Now, we prove an existence result for the coupled system (1.1)- (1.2) by using Leray–Schauder random fixed point theorem type in generalized Banach space.

Theorem 3.2. *Assume that the hypotheses (H_1) and (H_3) hold. If $C(w) < 1$, then the coupled system (1.1)-(1.2) has at least a random solution.*

Proof. Let $N : \mathcal{C} \times \Omega \to \mathcal{C}$ be the operator defined in (3.3). We show that N satisfies all conditions of Theorem 2.3. The proof will be given in several steps.

Step 1. $N(\cdot,\cdot,w)$ *is continuous.*
Let $(u_n,v_n)_n$ be a sequence such that $(u_n,v_n) \to (u,v) \in \mathcal{C}$ as $n \to \infty$. For any $w \in \Omega$

and each $t \in I$, we have

$$\begin{aligned} & \|(N_1(u_n,v_n))(t,w)-(N_1(u,v))(t,w)\| \\ \leq & \int_1^T |G_1(t,s)|\|f_1(s,u_n(s,w),v_n(s,w),w)-f_1(s,u(s,w),v(s,w),w)\|ds \\ \leq & G_1^*\|f_1(\cdot,u_n(\cdot,w),v_n(\cdot,w),w)-f_1(\cdot,u(\cdot,w),v(\cdot,w),w)\|_\infty. \end{aligned}$$

Since f_1 is Carathéodory, we have

$$\|(N_1(u_n,v_n))(\cdot,w)-(N_1(u,v))(\cdot,w)\|_\infty \to 0 \; as \; n\to\infty.$$

On the other hand, for any $w \in \Omega$ and each $t \in I$, we obtain

$$\begin{aligned} & \|(N_2(u_n,v_n))(t,w)-(N_2(u,v))(t,w)\| \\ \leq & \int_1^T |G_2(t,s)|\|f_2(\cdot,u_n(\cdot,w),v_n(\cdot,w),w)-f_2(\cdot,u(\cdot,w),v(\cdot,w),w)\|_\infty \\ \leq & G_2^*\|f_2(\cdot,u_n(\cdot,w),v_n(\cdot,w),w)-f_2(\cdot,u(\cdot,w),v(\cdot,w),w)\|_\infty. \end{aligned}$$

Also the fact that f_2 is Carathéodory, we get

$$\|(N_2(u_n,v_n))(\cdot,w)-(N_2(u,v))(\cdot,w)\|_\infty \to 0 \; as \; n\to\infty.$$

Hence $N(\cdot,\cdot,w)$ is continuous.

Step 2. $N(\cdot,\cdot,w)$ *maps bounded sets into bounded sets in* $\mathcal{C}$.
Let $R>0$ *and set*

$$B_R := \{(\mu,\nu)\in\mathcal{C} : \|\mu\|_\infty \leq R, \|\nu\|_\infty \leq R\}.$$

For any $w\in\Omega$ *and each* $(u,v)\in B_R$ *and* $t\in I$, *we have*

$$\begin{aligned} \|(N_1(u,v))(t,w)\| &\leq \int_1^T |G_1(t,s)|\|f_1(s,u(s,w),v(s,w),w)\|ds \\ &\leq \int_1^T |G_1(t,s)|(h_1(s,w)+k_1(w)\|u(s,w\|+l_1(s,w)\|v(s,w\|)ds \\ &\leq \int_1^T |G_1(t,s)|(\|h_1(\cdot,w)\|_\infty+R\|k_1(\cdot,w)\|_\infty+R\|l_1(\cdot,w)\|_\infty)ds \\ &\leq G_1^*(\|h_1(\cdot,w)\|_\infty+R\|k_1(\cdot,w)\|_\infty+R\|l_1(\cdot,w)\|_\infty) \\ &:= \ell_1(w). \end{aligned}$$

Thus,

$$\|(N_1(u,v))(\cdot,w)\|_\infty \leq \ell_1(w).$$

Also, for any $w\in\Omega$ *and each* $(u,v)\in B_R$ *and* $t\in I$, *we get*

$$\begin{aligned} \|(N_2(u,v))(\cdot,w)\|_\infty &\leq G_2^*(\|h_2(\cdot,w)\|_\infty+R\|k_2(\cdot,w)\|_\infty+R\|l_2(\cdot,w)\|_\infty) \\ &:= \ell_2(w). \end{aligned}$$

Hence,

$$\|(N(u,v))(\cdot,w)\|_{\mathcal{C}} \leq (\ell_1(w),\ell_2(w)) := \ell(w).$$

Step 3. $N(\cdot,\cdot,w)$ maps bounded sets into equicontinuous sets in $\mathcal{C}$.
Let B_R be the ball defined in Step 2. For each $t_1,t_2\in I$ with $t_1\leq t_2$ and any $(u,v)\in B_R$

and $w \in \Omega$, we have

$$\begin{aligned}
& \|(N_1(u,v))(t_1,w) - (N_1(u,v))(t_2,w)\| \\
\leq & \int_1^T |G_1(t_1,s) - G_1(t_2,s)| \|f_1(s,u(s,w),v(s,w),w)\| ds \\
\leq & \int_1^T |G_1(t_1,s) - G_1(t_2,s)| (h_1(s,w) + k_1(s,w)\|u(\cdot,w)\|_\infty + l_1(s,w)\|v(\cdot,w)\|_\infty) ds \\
\leq & (\|h_1(\cdot,w)\|_\infty + R\|k_1(\cdot,w)\|_\infty + R\|l_1(\cdot,w)\|_\infty) \int_1^T |G_1(t_1,s) - G_1(t_2,s)| ds \\
& \to 0 \; as \; t_1 \to t_2.
\end{aligned}$$

Also, we get

$$\begin{aligned}
& \|(N_2(u,v))(t_1,w) - (N_2(u,v))(t_2,w)\| \\
\leq & (\|h_2(\cdot,w)\|_\infty + R\|k_2(\cdot,w)\|_\infty + R\|l_2(\cdot,w)\|_\infty) \int_1^T |G_2(t_1,s) - G_2(t_2,s)| ds \\
& \to 0 \; as \; t_1 \to t_2.
\end{aligned}$$

As a consequence of Steps 1 to 3, with the Arzela–Ascoli theorem, we conclude that $N(\cdot,\cdot,w)$ maps B_R into a precompact set in $\mathcal{C}$.

Step 4. The set $E(w)$ consisting of $(u(\cdot,w), v(\cdot,w)) \in \mathcal{C}$ such that $(u(\cdot,w), v(\cdot,w)) = \lambda(w)(N((u,v))(\cdot,w)$ for some measurable function $\lambda : \Omega \to (0,1)$ is bounded in $\mathcal{C}$.
Let $(u(\cdot,w), v(\cdot,w)) \in \mathcal{C}$ such that $(u(\cdot,w), v(\cdot,w)) = \lambda(w)(N((u,v))(\cdot,w)$. Then $u(\cdot,w) = \lambda(w)(N_1((u,v))(\cdot,w)$ and $v(\cdot,w) = \lambda(w)(N_2((u,v))(\cdot,w)$. Thus, for any $w \in \Omega$ and each $t \in I$, we have

$$\begin{aligned}
\|u(t,w)\| & \leq \int_1^T |G_1(t,s)| \|f_1(s,u(s,w),v(s,w),w)\| ds \\
& \leq \int_1^T |G_1(t,s)| (h_1(s,w) + k_1(s,w)\|u(s,w\| + l_1(s,w)\|v(s,w\|) ds \\
& \leq G_1^*(\|h_1(\cdot,w)\|_\infty + \|k_1(\cdot,w)\|_\infty \|u(\cdot,w\|_\infty + \|l_1(\cdot,w)\|_\infty \|v(\cdot,w\|_\infty).
\end{aligned}$$

So, we get

$$\|u(\cdot,w)\|_\infty \leq G_1^*(\|h_1(\cdot,w)\|_\infty + \|k_1(\cdot,w)\|_\infty \|u(\cdot,w\|_\infty + \|l_1(\cdot,w)\|_\infty \|v(\cdot,w\|_\infty).$$

Also, we get

$$\|v(\cdot,w)\|_\infty \leq G_2^*(\|h_2(\cdot,w)\|_\infty + \|k_2(\cdot,w)\|_\infty \|u(\cdot,w\|_\infty + \|l_2(\cdot,w)\|_\infty \|v(\cdot,w\|)_\infty.$$

We obtain

$$\|u(\cdot,w)\|_\infty + \|v(\cdot,w)\|_\infty \leq A(w) + C(w)(\|u(\cdot,w)\|_\infty + \|v(\cdot,w)\|_\infty).$$

It follows that

$$\begin{aligned}
\|u(\cdot,w)\|_\infty + \|v(\cdot,w)\|_\infty & \leq \frac{A(w)}{1 - C(w)} \\
& := L(w).
\end{aligned}$$

Hence we get

$$\|(u(\cdot,w), v(\cdot,w))\|_{\mathcal{C}} \leq L(w).$$

This shows that the set $E(w)$ is bounded.

As a consequence of steps 1 to 4 together with Theorem 2.3, we can conclude that N has at least one random fixed point in B_R which is a random solution of the coupled system (1.1)- (1.2).

4. An Example

Let $\Omega = (-\infty, 0)$ be equipped with the usual σ-algebra consisting of Lebesgue measurable subsets of $(-\infty, 0)$. Consider the following random coupled system of Caputo–Hadamard fractional differential equations

$$\begin{cases} ({}^{Hc}D_1^{\frac{3}{2}}u)(t,w) = f(t,u(t,w),v(t,w),w); \\ ({}^{Hc}D_1^{\frac{3}{2}}v)(t,w) = g(t,u(t,w),v(t,w),w); \end{cases} ;w \in \Omega,\ t \in [1,e], \tag{4.1}$$

with the four-point boundary conditions

$$\begin{cases} u(1,w) - u'(1,w) = u(2,w), \\ 2u(T,w) + u'(T,w) = 2u(\frac{3}{2},w), \\ 3v(1,w) - v'(1,w) = 3v(\frac{5}{4},w), \\ v(T,w) + 2v'(T,w) = v(2,w), \end{cases} ;\ w \in \Omega, \tag{4.2}$$

where

$$f(t,u,v,w) = \frac{ct^{\frac{-1}{4}}w^2u(t)\sin t}{64(1+w^2+\sqrt{t})(1+|u|+|v|)};\ t \in [1,e],$$

$$g(t,u,v,w) = \frac{cw^2v(t)\cos t}{64(1+w^2+|u|+|v|)};\ w \in \Omega,\ t \in [1,e],$$

and $c < \frac{1}{\max\{G_1^*,G_2^*\}}$. The hypothesis (H_2) is satisfied with

$$p_2(t,w) = q_1(t,w) = 0,\ p_1(t,w) = \frac{cw^2\sin t}{64(1+w^2)},\ q_2(t,w) = \frac{cw^2\cos t}{64(1+w^2)}.$$

Also, if for every $w \in \Omega$, the matrix

$$\frac{cw^2}{64(1+w^2)}\begin{pmatrix} G_1^* & 0 \\ 0 & G_2^* \end{pmatrix},$$

converges to 0. Hence, Theorem 3.1 implies that the system (4.1)-(4.2) has a unique random solution defined on $[1,e]$.

5. Conclusion

In this paper, we provided some sufficient conditions ensuring the existence and uniqueness of random solutions for a new class of boundary value problems of coupled Caputo–Hadamard fractional differential systems with random effects in generalized Banach spaces.

Authors' contributions

All authors contributed equally to the manuscript. All authors read and approved the final manuscript.

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