ANNALS OF COMMUNICATIONS IN MATHEMATICS
Volume 5, Number 3 (2022), 161-180
ISSN: 2582-0818
© http://www.technoskypub.com

THE FORM OF THE SOLUTIONS OF FOURTH ORDER RATIONAL SYSTEMS OF DIFFERENCE EQUATIONS

E. M. ELASYED AND MAI T. ALHARTHI*

ABSTRACT. In this paper, we get the form of the solutions of the following difference equation systems of order four

$$z_{n+1} = \frac{z_n w_{n-2}}{w_{n-2} + w_{n-3}}, \quad w_{n+1} = \frac{w_n z_{n-2}}{\pm z_{n-2} \pm z_{n-3}},$$

where the initial conditions $z_{-3}, z_{-2}, z_{-1}, z_0, w_{-3}, w_{-2}, w_{-1}, w_0$ are arbitrary non-zero real numbers.

1. INTRODUCTION

Our aim in this paper to get the solutions of the system of following rational difference equations

$$z_{n+1} = \frac{z_n w_{n-2}}{w_{n-2} + w_{n-3}}, \quad w_{n+1} = \frac{w_n z_{n-2}}{\pm z_{n-2} \pm z_{n-3}},$$

where the initial conditions $z_{-3}, z_{-2}, z_{-1}, z_0, w_{-3}, w_{-2}, w_{-1}, w_0$ are arbitrary non-zero real numbers. Recently, there has been a great interest in studying nonlinear difference equations and systems. One reason for this is the need for methods that may be used to investigate equations that arise in mathematical models reflecting real life situations in population biology, economics, probability theory, genetics, psychology, and sociology. Furthermore, most nonlinear differential equations can be approximated into difference relations or systems. We highlight the following studies among the numerous that already exist.

Theoretical and numerical solutions to the systems of rational difference equations were obtained by Almatrafi and Elsayed [2].

$$z_{n+1} = \frac{w_{n-1} z_{n-3}}{w_{n-1}(1 + w_{n-1} z_{n-3})}, \quad w_{n+1} = \frac{z_{n-1} w_{n-3}}{z_{n-1}(1 + z_{n-1} w_{n-3})}.$$

Clark et al. [4] has examined the global stability characteristics and asymptotic behavior of solutions of the system

$$z_{n+1} = \frac{z_n}{a + c w_n}, \quad w_{n+1} = \frac{w_n}{b + d z_n}.$$

1991 *Mathematics Subject Classification.* 39A10.
Key words and phrases. difference equation; recursive sequences; system of difference equations.
Received: October 05, 2022. Accepted: November 15, 2022. Published: December 31, 2022.
*Corresponding author.

Cinar [5] has found the positive solution of the following systems of difference equations

$$z_{n+1}=\frac{1}{w_n},\quad w_{n+1}=\frac{w_n}{z_{n-1}w_{n-1}}.$$

In [26] Kurbanli presented the behavior of solutions of the difference equation system

$$z_{n+1}=\frac{z_{n-1}}{w_nz_{n-1}-1},\quad w_{n+1}=\frac{w_{n-1}}{z_nw_{n-1}-1}.$$

Touafek and Elsayed [33] studied the periodic nature and investigated specific solutions of the difference equation system presented below.

$$z_{n+1}=\frac{z_{n-3}}{\pm1\pm z_{n-3}w_{n-1}},\quad w_{n+1}=\frac{w_{n-3}}{\pm1\pm w_{n-3}z_{n-1}}.$$

Yalcinkaya [36] investigated the sufficient condition for the global asymptotic stability of the following system of difference equations

$$z_{n+1}=\frac{w_nz_{n-1}+a}{w_n+z_{n-1}},\quad w_{n+1}=\frac{z_nw_{n-1}+a}{z_n+w_{n-1}}.$$

Elsayed [15] discussed the periodic nature and found the solution of the difference equation

$$z_{n+1}=\frac{z_{n-3}w_{n-2}}{w_n(\pm1\pm z_{n-1}w_{n-2}z_{n-3})},\quad w_{n+1}=\frac{z_{n-2}w_{n-3}}{z_n(\pm1\pm w_{n-1}z_{n-2}w_{n-3})}.$$

For more researches about the difference equations systems see refs. [1]- [40]. Now, we will obtain the form of the solutions of some systems of difference equations.

2. On The System: $z_{n+1}=\frac{z_nw_{n-2}}{w_{n-2}+w_{n-3}},\quad w_{n+1}=\frac{w_nz_{n-2}}{z_{n-2}+z_{n-3}}$

In this section, we study the solutions of the following system of difference equations

$$z_{n+1}=\frac{z_nw_{n-2}}{w_{n-2}+w_{n-3}},\quad w_{n+1}=\frac{w_nz_{n-2}}{z_{n-2}+z_{n-3}},\tag{2.1}$$

where the initial conditions $z_{-3},\ z_{-2},\ z_{-1},\ z_0,\ w_{-3},\ w_{-2},\ w_{-1},\ w_0$ are arbitrary nonzero real numbers.

Theorem 1. *Suppose that $\{z_n,w_n\}_{n=-3}^{\infty}$ are solutions of the system (2.1). Then for n = 0,1,2,... , we have the following formula*

$$z_{6n-3}=\frac{d\ a^n\ h^n\ g^n\ f^n\ c^n\ b^n}{\prod_{i=0}^{n-1}[(2i+1)h+k][(2i+1)g+h][(2i+1)f+g][2ic+d][2ib+c][2ia+b]},$$

$$z_{6n-2}=\frac{a^n\ h^n\ g^n\ f^n\ c^{n+1}\ b^n}{\prod_{i=0}^{n-1}[(2i+1)h+k][(2i+1)g+h][(2i+1)f+g][2(i+1)c+d][2ib+c][2ia+b]},$$

$$z_{6n-1}=\frac{a^n\ h^n\ g^n\ f^n\ c^n\ b^{n+1}}{\prod_{i=0}^{n-1}[(2i+1)h+k][(2i+1)g+h][(2i+1)f+g][2(i+1)c+d][2(i+1)b+c][2ia+b]},$$

$$z_{6n}=\frac{a^{n+1}\ h^{n}\ g^{n}\ f^{n}\ c^{n}\ b^{n}}{\prod_{i=0}^{n-1}[(2i+1)h+k][(2i+1)g+h][(2i+1)f+g][2(i+1)c+d][2(i+1)b+c][2(i+1)a+b]},$$

$$z_{6n+1}=\frac{a^{n+1}\ h^{n+1}\ g^{n}\ f^{n}\ c^{n}\ b^{n}}{(h+k)\prod_{i=0}^{n-1}[(2i+3)h+k][(2i+1)g+h][(2i+1)f+g][2(i+1)c+d][2(i+1)b+c][2(i+1)a+b]},$$

$$\begin{aligned} z_{6n+2} &= \frac{1}{(h+k)(g+h)} \\ &\times\frac{a^{n+1}\ h^{n+1}\ g^{n+1}\ f^{n}\ c^{n}\ b^{n}}{\prod_{i=0}^{n-1}[(2i+3)h+k][(2i+3)g+h][(2i+1)f+g][2(i+1)c+d][2(i+1)b+c][2(i+1)a+b]},\end{aligned}$$

$$w_{6n-3}=\frac{k\ f^{n}\ c^{n}\ b^{n}\ a^{n}\ h^{n}\ g^{n}}{\prod_{i=0}^{n-1}[(2i+1)c+d][(2i+1)b+c][(2i+1)a+b][2ih+k][2ig+h][2if+g]},$$

$$w_{6n-2}=\frac{f^{n}\ c^{n}\ b^{n}\ a^{n}\ h^{n+1}\ g^{n}}{\prod_{i=0}^{n-1}[(2i+1)c+d][(2i+1)b+c][(2i+1)a+b][2(i+1)h+k][2ig+h][2if+g]},$$

$$w_{6n-1}=\frac{f^{n}\ c^{n}\ b^{n}\ a^{n}\ h^{n}\ g^{n+1}}{\prod_{i=0}^{n-1}[(2i+1)c+d][(2i+1)b+c][(2i+1)a+b][2(i+1)h+k][2(i+1)g+h][2if+g]},$$

$$w_{6n}=\frac{f^{n+1}\ c^{n}\ b^{n}\ a^{n}\ h^{n}\ g^{n}}{\prod_{i=0}^{n-1}[(2i+1)c+d][(2i+1)b+c][(2i+1)a+b][2(i+1)h+k][2(i+1)g+h][2(i+1)f+g]},$$

$$w_{6n+1}=\frac{f^{n+1}\ c^{n+1}\ b^{n}\ a^{n}\ h^{n}\ g^{n}}{(c+d)\prod_{i=0}^{n-1}[(2i+3)c+d][(2i+1)b+c][(2i+1)a+b][2(i+1)h+k][2(i+1)g+h][2(i+1)f+g]},$$

$$\begin{aligned} w_{6n+2} &= \frac{1}{(c+d)(b+c)} \\ &\times\frac{f^{n+1}\ c^{n+1}\ b^{n+1}\ a^{n}\ h^{n}\ g^{n}}{\prod_{i=0}^{n-1}[(2i+3)c+d][(2i+3)b+c][(2i+1)a+b][2(i+1)h+k][2(i+1)g+h][2(i+1)f+g]},\end{aligned}$$

where $z_{-3}=d$, $z_{-2}=c$, $z_{-1}=b$, $z_0=a$, $w_{-3}=k$, $w_{-2}=h$, $w_{-1}=g$, $w_0=f$.

Proof. By using mathematical induction. The result holds for $n=0$. Suppose that the result holds for $n-1$

$$z_{6n-9}=\frac{d\ a^{n-1}\ h^{n-1}\ g^{n-1}\ f^{n-1}\ c^{n-1}\ b^{n-1}}{\prod_{i=0}^{n-2}[(2i+1)h+k][(2i+1)g+h][(2i+1)f+g][2ic+d][2ib+c][2ia+b]},$$

$$z_{6n-8}=\frac{a^{n-1}\ h^{n-1}\ g^{n-1}\ f^{n-1}\ c^{n}\ b^{n-1}}{\prod_{i=0}^{n-2}[(2i+1)h+k][(2i+1)g+h][(2i+1)f+g][2(i+1)c+d][2ib+c][2ia+b]},$$

$$z_{6n-7}=\frac{a^{n-1}\ h^{n-1}\ g^{n-1}\ f^{n-1}\ c^{n-1}\ b^{n}}{\prod_{i=0}^{n-2}[(2i+1)h+k][(2i+1)g+h][(2i+1)f+g][2(i+1)c+d][2(i+1)b+c][2ia+b]},$$

$$z_{6n-6}=\frac{a^{n}\ h^{n-1}\ g^{n-1}\ f^{n-1}\ c^{n-1}\ b^{n-1}}{\prod_{i=0}^{n-2}[(2i+1)h+k][(2i+1)g+h][(2i+1)f+g][2(i+1)c+d][2(i+1)b+c][2(i+1)a+b]},$$

$$z_{6n-5}=\frac{a^{n}\ h^{n}\ g^{n-1}\ f^{n-1}\ c^{n-1}\ b^{n-1}}{(h+k)\prod_{i=0}^{n-2}[(2i+3)h+k][(2i+1)g+h][(2i+1)f+g][2(i+1)c+d][2(i+1)b+c][2(i+1)a+b]},$$

$$z_{6n-4}=\frac{a^{n}\ h^{n}\ g^{n}\ f^{n-1}\ c^{n-1}\ b^{n-1}}{(h+k)(g+h)\prod_{i=0}^{n-2}[(2i+3)h+k][(2i+3)g+h][(2i+1)f+g][2(i+1)c+d][2(i+1)b+c][2(i+1)a+b]},$$

$$w_{6n-9}=\frac{k\ f^{n-1}\ c^{n-1}\ b^{n-1}\ a^{n-1}\ h^{n-1}\ g^{n-1}}{\prod_{i=0}^{n-2}[(2i+1)c+d][(2i+1)b+c][(2i+1)a+b][2ih+k][2ig+h][2if+g]},$$

$$w_{6n-8}=\frac{f^{n-1}\ c^{n-1}\ b^{n-1}\ a^{n-1}\ h^{n}\ g^{n-1}}{\prod_{i=0}^{n-2}[(2i+1)c+d][(2i+1)b+c][(2i+1)a+b][2(i+1)h+k][2ig+h][2if+g]},$$

$$w_{6n-7}=\frac{f^{n-1}\ c^{n-1}\ b^{n-1}\ a^{n-1}\ h^{n-1}\ g^{n}}{\prod_{i=0}^{n-2}[(2i+1)c+d][(2i+1)b+c][(2i+1)a+b][2(i+1)h+k][2(i+1)g+h][2if+g]},$$

$$w_{6n-6}=\frac{f^{n}\ c^{n-1}\ b^{n-1}\ a^{n-1}\ h^{n-1}\ g^{n-1}}{\prod_{i=0}^{n-2}[(2i+1)c+d][(2i+1)b+c][(2i+1)a+b][2(i+1)h+k][2(i+1)g+h][2(i+1)f+g]},$$

$$w_{6n-5}=\frac{f^{n}\ c^{n}\ b^{n-1}\ a^{n-1}\ h^{n-1}\ g^{n-1}}{(c+d)\prod_{i=0}^{n-2}[(2i+3)c+d][(2i+1)b+c][(2i+1)a+b][2(i+1)h+k][2(i+1)g+h][2(i+1)f+g]},$$

$$w_{6n-4}=\frac{f^{n}\ c^{n}\ b^{n}\ a^{n-1}\ h^{n-1}\ g^{n-1}}{(c+d)(b+c)\prod_{i=0}^{n-2}[(2i+3)c+d][(2i+3)b+c][(2i+1)a+b][2(i+1)h+k][2(i+1)g+h][2(i+1)f+g]}.$$

From system (2.1) we can prove as follow but firstly, to prove z_{6n-3} we found the following relation

$$\begin{aligned} w_{6n-6} &= \frac{f\ w_{6n-7}}{(2n-2)f+g}, \\ &\Leftrightarrow\ w_{6n-7}=\frac{[(2n-2)f+g]w_{6n-6}}{f}. \end{aligned}$$

Therefore

$$\begin{aligned}
z_{6n-3} &= \frac{z_{6n-4}w_{6n-6}}{w_{6n-6}+w_{6n-7}} \\
&= \frac{z_{6n-4}w_{6n-6}}{w_{6n-6}+\frac{[(2n-2)f+g]w_{6n-6}}{f}} \\
&= \frac{f\ z_{6n-4}}{(2n-1)f+g} \\
&= \frac{f}{(2n-1)f+g}\cdot\frac{a^n\ \ h^n\ \ g^n\ \ f^{n-1}\ \ c^{n-1}\ b^{n-1}}{\begin{array}{c}(h+k)(g+h)\prod\limits_{i=0}^{n-2}[(2i+3)h+k][(2i+3)g+h][(2i+1)f+g]\\ [2(i+1)c+d][2(i+1)b+c][2(i+1)a+b]\end{array}} \\
&= \frac{f}{(2n-1)f+g}\cdot\frac{d\ \ c\ \ b\,a^n\ \ h^n\ \ g^n\ \ f^{n-1}\ \ c^{n-1}\ b^{n-1}}{\begin{array}{c}(h+k)(g+h)\prod\limits_{i=0}^{n-2}[(2i+3)h+k][(2i+3)g+h][(2i+1)f+g]d\\ [2(i+1)c+d]c[2(i+1)b+c]b[2(i+1)a+b]\end{array}} \\
&= \frac{d\,a^n\ \ h^n\ \ g^n\ \ f^n\ \ c^n\ b^n}{\prod\limits_{i=0}^{n-1}[(2i+1)h+k][(2i+1)g+h][(2i+1)f+g][2ic+d][2ib+c][2ia+b]},
\end{aligned}$$

to prove w_{6n-3} we found the following relation

$$\begin{aligned}
z_{6n-6} &= \frac{az_{6n-7}}{(2n-2)a+b}, \\
&\Leftrightarrow \quad z_{6n-7}=\frac{[(2n-2)a+b]z_{6n-6}}{a}.
\end{aligned}$$

Therefore

$$\begin{aligned}
w_{6n-3} &= \frac{w_{6n-4}z_{6n-6}}{z_{6n-6}+z_{6n-7}} \\
&= \frac{w_{6n-4}z_{6n-6}}{z_{6n-6}+\frac{[(2n-2)a+b]z_{6n-6}}{a}} \\
&= \frac{aw_{6n-4}}{(2n-1)a+b} \\
&= \frac{a}{(2n-1)a+b}\cdot\frac{f^n\ \ c^n\ \ b^n\ \ a^{n-1}\ \ h^{n-1}\ g^{n-1}}{\begin{array}{c}(c+d)(b+c)\prod\limits_{i=0}^{n-2}[(2i+3)c+d][(2i+3)b+c][(2i+1)a+b]\\ [2(i+1)h+k][2(i+1)g+h][2(i+1)f+g]\end{array}} \\
&= \frac{a}{(2n-1)a+b}\cdot\frac{k\ \ h\ \ g\ \ f^n\ \ c^n\ \ b^n\ \ a^{n-1}\ \ h^{n-1}\ g^{n-1}}{\begin{array}{c}(c+d)(b+c)\prod\limits_{i=0}^{n-2}[(2i+3)c+d][(2i+3)b+c][(2i+1)a+b]k\\ [2(i+1)h+k]h[2(i+1)g+h]g[2(i+1)f+g]\end{array}} \\
&= \frac{k\ \ f^n\ \ c^n\ \ b^n\ \ a^n\ \ h^n\ g^n}{\prod\limits_{i=0}^{n-1}[(2i+1)c+d][(2i+1)b+c][(2i+1)a+b][2ih+k][2ig+h][2if+g]}.
\end{aligned}$$

Similarly we can prove other relations and the proof is completed. □

3. On The System: $z_{n+1}=\frac{z_n w_{n-2}}{w_{n-2}+w_{n-3}}$, $w_{n+1}=\frac{w_n z_{n-2}}{z_{n-2}-z_{n-3}}$

We study in this section the solutions of the system of two difference equations

$$z_{n+1}=\frac{z_n w_{n-2}}{w_{n-2}+w_{n-3}},\quad w_{n+1}=\frac{w_n z_{n-2}}{z_{n-2}-z_{n-3}},\tag{3.1}$$

where the initial conditions $z_{-3},\ z_{-2},\ z_{-1},\ z_0,\ w_{-3},\ w_{-2},\ w_{-1},\ w_0$ are arbitrary non-zero real numbers.

Theorem 2. *Assume that $\{z_n,w_n\}_{n=-3}^{\infty}$ is a solution to system (3.1) and let $z_{-3}=d$, $z_{-2}=c$, $z_{-1}=b$, $z_0=a$, $w_{-3}=k$, $w_{-2}=h$, $w_{-1}=g$ and $w_0=f$ with $w_{-2}\neq\pm w_{-3}$, $w_{-1}\neq\pm w_{-2}$, $w_0\neq\pm w_{-1}$, $z_{-2}\neq\pm z_{-3}$, $z_{-1}\neq\pm z_{-2}$ and $z_0\neq\pm z_{-1}$. Then, for $n=0,1,...,$ we have*

$$z_{12n-3}=\frac{a^{2n}\,h^{2n}\,g^{2n}\,f^{2n}\,c^{n}\,b^{n}}{(h+k)^n(g+h)^n(f+g)^n(2c-d)^n(2b-c)^n(2a-b)^n(h-k)^n(g-h)^n(f-g)^n d^{n-1}},$$

$$z_{12n-2}=\frac{a^{2n}\,h^{2n}\,g^{2n}\,f^{2n}\,c^{n+1}\,b^{n}}{(h+k)^n(g+h)^n(f+g)^n(2c-d)^n(2b-c)^n(2a-b)^n(h-k)^n(g-h)^n(f-g)^n d^{n}},$$

$$z_{12n-1}=\frac{a^{2n}\,h^{2n}\,g^{2n}\,f^{2n}\,c^{n}\,b^{n+1}}{(h+k)^n(g+h)^n(f+g)^n(2c-d)^n(2b-c)^n(2a-b)^n(h-k)^n(g-h)^n(f-g)^n d^{n}},$$

$$z_{12n}=\frac{a^{2n+1}\,h^{2n}\,g^{2n}\,f^{2n}\,c^{n}\,b^{n}}{(h+k)^n(g+h)^n(f+g)^n(2c-d)^n(2b-c)^n(2a-b)^n(h-k)^n(g-h)^n(f-g)^n d^{n}},$$

$$z_{12n+1}=\frac{a^{2n+1}\,h^{2n+1}\,g^{2n}\,f^{2n}\,c^{n}\,b^{n}}{(h+k)^{n+1}(g+h)^n(f+g)^n(2c-d)^n(2b-c)^n(2a-b)^n(h-k)^n(g-h)^n(f-g)^n d^{n}},$$

$$z_{12n+2}=\frac{a^{2n+1}\,h^{2n+1}\,g^{2n+1}\,f^{2n}\,c^{n}\,b^{n}}{(h+k)^{n+1}(g+h)^{n+1}(f+g)^n(2c-d)^n(2b-c)^n(2a-b)^n(h-k)^n(g-h)^n(f-g)^n d^{n}},$$

$$z_{12n+3}=\frac{a^{2n+1}\,h^{2n+1}\,g^{2n+1}\,f^{2n+1}\,c^{n}\,b^{n}}{(h+k)^{n+1}(g+h)^{n+1}(f+g)^{n+1}(2c-d)^n(2b-c)^n(2a-b)^n(h-k)^n(g-h)^n(f-g)^n d^{n}},$$

$$z_{12n+4}=\frac{a^{2n+1}\,h^{2n+1}\,g^{2n+1}\,f^{2n+1}\,c^{n+1}\,b^{n}}{(h+k)^{n+1}(g+h)^{n+1}(f+g)^{n+1}(2c-d)^{n+1}(2b-c)^n(2a-b)^n(h-k)^n(g-h)^n(f-g)^n d^{n}},$$

$$z_{12n+5}=\frac{a^{2n+1}\,h^{2n+1}\,g^{2n+1}\,f^{2n+1}\,c^{n+1}\,b^{n+1}}{(h+k)^{n+1}(g+h)^{n+1}(f+g)^{n+1}(2c-d)^{n+1}(2b-c)^{n+1}(2a-b)^n(h-k)^n(g-h)^n(f-g)^n d^{n}},$$

$$z_{12n+6}=\frac{a^{2n+2}\,h^{2n+1}\,g^{2n+1}\,f^{2n+1}\,c^{n+1}\,b^{n+1}}{(h+k)^{n+1}(g+h)^{n+1}(f+g)^{n+1}(2c-d)^{n+1}(2b-c)^{n+1}(2a-b)^{n+1}(h-k)^n(g-h)^n(f-g)^n d^{n}},$$

$$z_{12n+7}=\frac{a^{2n+2}\,h^{2n+2}\,g^{2n+1}\,f^{2n+1}\,c^{n+1}\,b^{n+1}}{(h+k)^{n+1}(g+h)^{n+1}(f+g)^{n+1}(2c-d)^{n+1}(2b-c)^{n+1}(2a-b)^{n+1}(h-k)^{n+1}(g-h)^n(f-g)^n d^{n}},$$

$$z_{12n+8}=\frac{a^{2n+2}\,h^{2n+2}\,g^{2n+2}\,f^{2n+1}\,c^{n+1}\,b^{n+1}}{(h+k)^{n+1}(g+h)^{n+1}(f+g)^{n+1}(2c-d)^{n+1}(2b-c)^{n+1}(2a-b)^{n+1}(h-k)^{n+1}(g-h)^{n+1}(f-g)^n d^{n}},$$

$$w_{12n-3}=\frac{f^{2n}\ c^{2n}\ b^{2n}\ a^{2n}}{(c-d)^{2n}(b-c)^{2n}(a-b)^{2n}k^{2n-1}},\quad w_{12n-2}=\frac{f^{2n}\ c^{2n}\ b^{2n}\ a^{2n}\ h}{(c-d)^{2n}(b-c)^{2n}(a-b)^{2n}k^{2n}},$$

$$w_{12n-1}=\frac{f^{2n}\ c^{2n}\ b^{2n}\ a^{2n}\ g}{(c-d)^{2n}(b-c)^{2n}(a-b)^{2n}k^{2n}},\quad w_{12n}=\frac{f^{2n+1}\ c^{2n}\ b^{2n}\ a^{2n}}{(c-d)^{2n}(b-c)^{2n}(a-b)^{2n}k^{2n}},$$

$$w_{12n+1}=\frac{f^{2n+1}\ c^{2n+1}\ b^{2n}\ a^{2n}}{(c-d)^{2n+1}(b-c)^{2n}(a-b)^{2n}k^{2n}},\quad w_{12n+2}=\frac{f^{2n+1}\ c^{2n+1}\ b^{2n+1}\ a^{2n}}{(c-d)^{2n+1}(b-c)^{2n+1}(a-b)^{2n}k^{2n}},$$

$$w_{12n+3}=\frac{f^{2n+1}\ c^{2n+1}\ b^{2n+1}\ a^{2n+1}}{(c-d)^{2n+1}(b-c)^{2n+1}(a-b)^{2n+1}k^{2n}},\quad w_{12n+4}=\frac{-f^{2n+1}\ c^{2n+1}\ b^{2n+1}\ a^{2n+1}\ h}{(c-d)^{2n+1}(b-c)^{2n+1}(a-b)^{2n+1}k^{2n+1}},$$

$$w_{12n+5}=\frac{f^{2n+1}\ c^{2n+1}\ b^{2n+1}\ a^{2n+1}\ g}{(c-d)^{2n+1}(b-c)^{2n+1}(a-b)^{2n+1}k^{2n+1}},\quad w_{12n+6}=\frac{-f^{2n+2}\ c^{2n+1}\ b^{2n+1}\ a^{2n+1}}{(c-d)^{2n+1}(b-c)^{2n+1}(a-b)^{2n+1}k^{2n+1}},$$

$$w_{12n+7}=\frac{f^{2n+2}\ c^{2n+2}\ b^{2n+1}\ a^{2n+1}}{(c-d)^{2n+2}(b-c)^{2n+1}(a-b)^{2n+1}k^{2n+1}},\quad w_{12n+8}=\frac{-f^{2n+2}\ c^{2n+2}\ b^{2n+2}\ a^{2n+1}}{(c-d)^{2n+2}(b-c)^{2n+2}(a-b)^{2n+1}k^{2n+1}}.$$

Proof. For n = 0 the result holds. Now suppose that $n>0$ and that our assumption holds for $n-1$. That is,

$$z_{12n-15}=\frac{a^{2n-2}\ h^{2n-2}\ g^{2n-2}\ f^{2n-2}\ c^{n-1}\ b^{n-1}}{\begin{array}{c}(h+k)^{n-1}(g+h)^{n-1}(f+g)^{n-1}(2c-d)^{n-1}(2b-c)^{n-1}(2a-b)^{n-1}(h-k)^{n-1}\\(g-h)^{n-1}(f-g)^{n-1}d^{n-2}\end{array}},$$

$$z_{12n-14}=\frac{a^{2n-2}\ h^{2n-2}\ g^{2n-2}\ f^{2n-2}\ c^{n}\ b^{n-1}}{\begin{array}{c}(h+k)^{n-1}(g+h)^{n-1}(f+g)^{n-1}(2c-d)^{n-1}(2b-c)^{n-1}(2a-b)^{n-1}(h-k)^{n-1}\\(g-h)^{n-1}(f-g)^{n-1}d^{n-1}\end{array}},$$

$$z_{12n-13}=\frac{a^{2n-2}\ h^{2n-2}\ g^{2n-2}\ f^{2n-2}\ c^{n-1}\ b^{n}}{\begin{array}{c}(h+k)^{n-1}(g+h)^{n-1}(f+g)^{n-1}(2c-d)^{n-1}(2b-c)^{n-1}(2a-b)^{n-1}(h-k)^{n-1}\\(g-h)^{n-1}(f-g)^{n-1}d^{n-1}\end{array}},$$

$$z_{12n-12}=\frac{a^{2n-1}\ h^{2n-2}\ g^{2n-2}\ f^{2n-2}\ c^{n-1}\ b^{n-1}}{\begin{array}{c}(h+k)^{n-1}(g+h)^{n-1}(f+g)^{n-1}(2c-d)^{n-1}(2b-c)^{n-1}(2a-b)^{n-1}(h-k)^{n-1}\\(g-h)^{n-1}(f-g)^{n-1}d^{n-1}\end{array}},$$

$$z_{12n-11}=\frac{a^{2n-1}\ h^{2n-1}\ g^{2n-2}\ f^{2n-2}\ c^{n-1}\ b^{n-1}}{\begin{array}{c}(h+k)^{n}(g+h)^{n-1}(f+g)^{n-1}(2c-d)^{n-1}(2b-c)^{n-1}(2a-b)^{n-1}(h-k)^{n-1}\\(g-h)^{n-1}(f-g)^{n-1}d^{n-1}\end{array}},$$

$$z_{12n-10}=\frac{a^{2n-1}\ h^{2n-1}\ g^{2n-1}\ f^{2n-2}\ c^{n-1}\ b^{n-1}}{\begin{array}{c}(h+k)^{n}(g+h)^{n}(f+g)^{n-1}(2c-d)^{n-1}(2b-c)^{n-1}(2a-b)^{n-1}(h-k)^{n-1}\\(g-h)^{n-1}(f-g)^{n-1}d^{n-1}\end{array}},$$

$$z_{12n-9}=\frac{a^{2n-1}\ h^{2n-1}\ g^{2n-1}\ f^{2n-1}\ c^{n-1}\ b^{n-1}}{(h+k)^{n}(g+h)^{n}(f+g)^{n}(2c-d)^{n-1}(2b-c)^{n-1}(2a-b)^{n-1}(h-k)^{n-1}(g-h)^{n-1}(f-g)^{n-1}d^{n-1}},$$

$$z_{12n-8}=\frac{a^{2n-1}\ h^{2n-1}\ g^{2n-1}\ f^{2n-1}\ c^{n}\ b^{n-1}}{(h+k)^{n}(g+h)^{n}(f+g)^{n}(2c-d)^{n}(2b-c)^{n-1}(2a-b)^{n-1}(h-k)^{n-1}(g-h)^{n-1}(f-g)^{n-1}d^{n-1}},$$

$$z_{12n-7} = \frac{a^{2n-1}\ h^{2n-1}\ g^{2n-1}\ f^{2n-1}\ c^{n}\ b^{n}}{(h+k)^{n}(g+h)^{n}(f+g)^{n}(2c-d)^{n}(2b-c)^{n}(2a-b)^{n-1}(h-k)^{n-1}(g-h)^{n-1}(f-g)^{n-1}d^{n-1}},$$

$$z_{12n-6} = \frac{a^{2n}\ h^{2n-1}\ g^{2n-1}\ f^{2n-1}\ c^{n}\ b^{n}}{(h+k)^{n}(g+h)^{n}(f+g)^{n}(2c-d)^{n}(2b-c)^{n}(2a-b)^{n}(h-k)^{n-1}(g-h)^{n-1}(f-g)^{n-1}d^{n-1}},$$

$$z_{12n-5} = \frac{a^{2n}\ h^{2n}\ g^{2n-1}\ f^{2n-1}\ c^{n}\ b^{n}}{(h+k)^{n}(g+h)^{n}(f+g)^{n}(2c-d)^{n}(2b-c)^{n}(2a-b)^{n}(h-k)^{n}(g-h)^{n-1}(f-g)^{n-1}d^{n-1}},$$

$$z_{12n-4} = \frac{a^{2n}\ h^{2n}\ g^{2n}\ f^{2n-1}\ c^{n}\ b^{n}}{(h+k)^{n}(g+h)^{n}(f+g)^{n}(2c-d)^{n}(2b-c)^{n}(2a-b)^{n}(h-k)^{n}(g-h)^{n}(f-g)^{n-1}d^{n-1}},$$

$$w_{12n-15} = \frac{f^{2n-2}c^{2n-2}b^{2n-2}a^{2n-2}}{(c-d)^{2n-2}(b-c)^{2n-2}(a-b)^{2n-2}k^{2n-3}},$$

$$w_{12n-14} = \frac{f^{2n-2}c^{2n-2}b^{2n-2}a^{2n-2}h}{(c-d)^{2n-2}(b-c)^{2n-2}(a-b)^{2n-2}k^{2n-2}},$$

$$w_{12n-13}=\frac{f^{2n-2}\ c^{2n-2}\ b^{2n-2}\ a^{2n-2}\ g}{(c-d)^{2n-2}(b-c)^{2n-2}(a-b)^{2n-2}k^{2n-2}},$$

$$w_{12n-12}=\frac{f^{2n-1}\ c^{2n-2}\ b^{2n-2}\ a^{2n-2}}{(c-d)^{2n-2}(b-c)^{2n-2}(a-b)^{2n-2}k^{2n-2}},$$

$$w_{12n-11}=\frac{f^{2n-1}\ c^{2n-1}\ b^{2n-2}\ a^{2n-2}}{(c-d)^{2n-1}(b-c)^{2n-2}(a-b)^{2n-2}k^{2n-2}},$$

$$w_{12n-10}=\frac{f^{2n-1}\ c^{2n-1}\ b^{2n-1}\ a^{2n-2}}{(c-d)^{2n-1}(b-c)^{2n-1}(a-b)^{2n-2}k^{2n-2}},$$

$$w_{12n-9}=\frac{f^{2n-1}\ c^{2n-1}\ b^{2n-1}\ a^{2n-1}}{(c-d)^{2n-1}(b-c)^{2n-1}(a-b)^{2n-1}k^{2n-2}},$$

$$w_{12n-8}=\frac{-f^{2n-1}\ c^{2n-1}\ b^{2n-1}\ a^{2n-1}\ h}{(c-d)^{2n-1}(b-c)^{2n-1}(a-b)^{2n-1}k^{2n-1}},$$

$$w_{12n-7}=\frac{f^{2n-1}\ c^{2n-1}\ b^{2n-1}\ a^{2n-1}\ g}{(c-d)^{2n-1}(b-c)^{2n-1}(a-b)^{2n-1}k^{2n-1}},$$

$$w_{12n-6}=\frac{-f^{2n}\ c^{2n-1}\ b^{2n-1}\ a^{2n-1}}{(c-d)^{2n-1}(b-c)^{2n-1}(a-b)^{2n-1}k^{2n-1}},$$

$$w_{12n-5}=\frac{f^{2n}\ c^{2n}\ b^{2n-1}\ a^{2n-1}}{(c-d)^{2n}(b-c)^{2n-1}(a-b)^{2n-1}k^{2n-1}},$$

$$w_{12n-4}=\frac{-f^{2n}\ c^{2n}\ b^{2n}\ a^{2n-1}}{(c-d)^{2n}(b-c)^{2n}(a-b)^{2n-1}k^{2n-1}}.$$

From system (3.1) we can prove as follow

$$\begin{aligned}
z_{12n-3} &= \frac{z_{12n-4}w_{12n-6}}{w_{12n-6}+w_{12n-7}} \\
&= \frac{\left(\frac{a^{2n}\ h^{2n}\ g^{2n}\ f^{2n-1}\ c^{n}\ b^{n}}{(h+k)^{n}(g+h)^{n}(f+g)^{n}(2c-d)^{n}(2b-c)^{n}(2a-b)^{n}(h-k)^{n}(g-h)^{n}(f-g)^{n-1}d^{n-1}}\right) \times \left(\frac{-f^{2n}\ c^{2n-1}\ b^{2n-1}\ a^{2n-1}}{(c-d)^{2n-1}(b-c)^{2n-1}(a-b)^{2n-1}k^{2n-1}}\right)}{\left(\frac{-f^{2n}\ c^{2n-1}\ b^{2n-1}\ a^{2n-1}}{(c-d)^{2n-1}(b-c)^{2n-1}(a-b)^{2n-1}k^{2n-1}}\right)+\left(\frac{f^{2n-1}\ c^{2n-1}\ b^{2n-1}\ a^{2n-1}\ g}{(c-d)^{2n-1}(b-c)^{2n-1}(a-b)^{2n-1}k^{2n-1}}\right)} \\
&= \frac{\frac{-a^{2n}\ h^{2n}\ g^{2n}\ f^{2n-1}\ c^{n}\ b^{n}\ f}{(h+k)^{n}(g+h)^{n}(f+g)^{n}(2c-d)^{n}(2b-c)^{n}(2a-b)^{n}(h-k)^{n}(g-h)^{n}(f-g)^{n-1}d^{n-1}}}{-f+g} \\
&= \frac{a^{2n}\ h^{2n}\ g^{2n}\ f^{2n}\ c^{n}\ b^{n}}{(h+k)^{n}(g+h)^{n}(f+g)^{n}(2c-d)^{n}(2b-c)^{n}(2a-b)^{n}(h-k)^{n}(g-h)^{n}(f-g)^{n}d^{n-1}},
\end{aligned}$$

$$z_{12n-2} = \frac{z_{12n-3}w_{12n-5}}{w_{12n-5}+w_{12n-6}}$$

$$= \frac{\left(\frac{a^{2n}\,h^{2n}\,g^{2n}\,f^{2n}\,c^{n}\,b^{n}}{(h+k)^n(g+h)^n(f+g)^n(2c-d)^n(2b-c)^n(2a-b)^n(h-k)^n(g-h)^n(f-g)^n d^{n-1}}\right)\times\left(\frac{f^{2n}\,c^{2n}\,b^{2n-1}\,a^{2n-1}}{(c-d)^{2n}(b-c)^{2n-1}(a-b)^{2n-1}k^{2n-1}}\right)}{\left(\frac{f^{2n}\,c^{2n}\,b^{2n-1}\,a^{2n-1}}{(c-d)^{2n}(b-c)^{2n-1}(a-b)^{2n-1}k^{2n-1}}\right)+\left(\frac{-f^{2n}\,c^{2n-1}\,b^{2n-1}\,a^{2n-1}}{(c-d)^{2n-1}(b-c)^{2n-1}(a-b)^{2n-1}k^{2n-1}}\right)}$$

$$= \frac{\frac{a^{2n}\,h^{2n}\,g^{2n}\,f^{2n}\,c^{n+1}\,b^{n}}{(h+k)^n(g+h)^n(f+g)^n(2c-d)^n(2b-c)^n(2a-b)^n(h-k)^n(g-h)^n(f-g)^n d^{n-1}(c-d)}}{\frac{c}{c-d}-1}$$

$$= \frac{a^{2n}\,h^{2n}\,g^{2n}\,f^{2n}\,c^{n+1}\,b^{n}}{(h+k)^n(g+h)^n(f+g)^n(2c-d)^n(2b-c)^n(2a-b)^n(h-k)^n(g-h)^n(f-g)^n d^{n}},$$

$$w_{12n-3} = \frac{w_{12n-4}z_{12n-6}}{z_{12n-6}-z_{12n-7}}$$

$$= \frac{\left(\frac{-f^{2n}\,c^{2n}\,b^{2n}\,a^{2n-1}}{(c-d)^{2n}(b-c)^{2n}(a-b)^{2n-1}k^{2n-1}}\right)\times\left(\frac{a^{2n}\,h^{2n-1}\,g^{2n-1}\,f^{2n-1}\,c^{n}\,b^{n}}{(h+k)^n(g+h)^n(f+g)^n(2c-d)^n(2b-c)^n(2a-b)^n(h-k)^{n-1}(g-h)^{n-1}(f-g)^{n-1}d^{n-1}}\right)}{\frac{a^{2n}\,h^{2n-1}\,g^{2n-1}\,f^{2n-1}\,c^{n}\,b^{n}}{(h+k)^n(g+h)^n(f+g)^n(2c-d)^n(2b-c)^n(2a-b)^n(h-k)^{n-1}(g-h)^{n-1}(f-g)^{n-1}d^{n-1}}-\frac{a^{2n-1}\,h^{2n-1}\,g^{2n-1}\,f^{2n-1}\,c^{n}\,b^{n}}{(h+k)^n(g+h)^n(f+g)^n(2c-d)^n(2b-c)^n(2a-b)^{n-1}(h-k)^{n-1}(g-h)^{n-1}(f-g)^{n-1}d^{n-1}}}$$

$$= \frac{\frac{-f^{2n}\,c^{2n}\,b^{2n}\,a^{2n}}{(c-d)^{2n}(b-c)^{2n}(a-b)^{2n-1}k^{2n-1}(2a-b)}}{\frac{a}{2a-b}-1}$$

$$= \frac{f^{2n}\,c^{2n}\,b^{2n}\,a^{2n}}{(c-d)^{2n}(b-c)^{2n}(a-b)^{2n}k^{2n-1}},$$

$$w_{12n-2} = \frac{w_{12n-3}z_{12n-5}}{z_{12n-5}-z_{12n-6}}$$

$$= \frac{\left(\frac{f^{2n}\,c^{2n}\,b^{2n}\,a^{2n}}{(c-d)^{2n}(b-c)^{2n}(a-b)^{2n}k^{2n-1}}\right)\times\left(\frac{a^{2n}\,h^{2n}\,g^{2n-1}\,f^{2n-1}\,c^{n}\,b^{n}}{(h+k)^n(g+h)^n(f+g)^n(2c-d)^n(2b-c)^n(2a-b)^n(h-k)^n(g-h)^{n-1}(f-g)^{n-1}d^{n-1}}\right)}{\frac{a^{2n}\,h^{2n}\,g^{2n-1}\,f^{2n-1}\,c^{n}\,b^{n}}{(h+k)^n(g+h)^n(f+g)^n(2c-d)^n(2b-c)^n(2a-b)^n(h-k)^n(g-h)^{n-1}(f-g)^{n-1}d^{n-1}}-\frac{a^{2n}\,h^{2n-1}\,g^{2n-1}\,f^{2n-1}\,c^{n}\,b^{n}}{(h+k)^n(g+h)^n(f+g)^n(2c-d)^n(2b-c)^n(2a-b)^n(h-k)^{n-1}(g-h)^{n-1}(f-g)^{n-1}d^{n-1}}}$$

$$= \frac{\frac{f^{2n}\,c^{2n}\,b^{2n}\,a^{2n}\,h}{(c-d)^{2n}(b-c)^{2n}(a-b)^{2n}k^{2n-1}(h-k)}}{\frac{h}{h-k}-1}$$

$$= \frac{f^{2n}\,c^{2n}\,b^{2n}\,a^{2n}\,h}{(c-d)^{2n}(b-c)^{2n}(a-b)^{2n}k^{2n}}.$$

Similarly we can prove other relations and the proof is completed. □

4. On The System: $z_{n+1}=\frac{z_n w_{n-2}}{w_{n-2}+w_{n-3}}, \quad w_{n+1}=\frac{w_n z_{n-2}}{-z_{n-2}+z_{n-3}}$

In this section, we deal with the solutions of the system of the difference equations

$$z_{n+1}=\frac{z_n w_{n-2}}{w_{n-2}+w_{n-3}}, \quad w_{n+1}=\frac{w_n z_{n-2}}{-z_{n-2}+z_{n-3}}, \tag{4.1}$$

where the initial conditions $z_{-3},\ z_{-2},\ z_{-1},\ z_0,\ w_{-3},\ w_{-2},\ w_{-1},\ w_0$ are arbitrary non-zero real numbers.

Theorem 3. *Suppose that* $\{z_n, w_n\}_{n=-3}^{\infty}$ *are solutions of system (4.1). Then for* $n = 0, 1, 2, ...,$

$$z_{6n-3} = \frac{a^n h^n g^n f^n}{(h+k)^n (g+h)^n (f+g)^n d^{n-1}}, \quad z_{6n-2} = \frac{c\, a^n h^n g^n f^n}{(h+k)^n (g+h)^n (f+g)^n d^n},$$

$$z_{6n-1} = \frac{b\, a^n h^n g^n f^n}{(h+k)^n (g+h)^n (f+g)^n d^n}, \quad z_{6n} = \frac{a^{n+1} h^n g^n f^n}{(h+k)^n (g+h)^n (f+g)^n d^n},$$

$$z_{6n+1} = \frac{a^{n+1} h^{n+1} g^n f^n}{(h+k)^{n+1} (g+h)^n (f+g)^n d^n}, \quad z_{6n+2} = \frac{a^{n+1} h^{n+1} g^{n+1} f^n}{(h+k)^{n+1} (g+h)^{n+1} (f+g)^n d^n},$$

$$w_{6n-3} = \frac{f^n c^n b^n a^n}{(-c+d)^n (-b+c)^n (-a+b)^n k^{n-1}}, \quad w_{6n-2} = \frac{h\, f^n c^n b^n a^n}{(-c+d)^n (-b+c)^n (-a+b)^n k^n},$$

$$w_{6n-1} = \frac{g\, f^n c^n b^n a^n}{(-c+d)^n (-b+c)^n (-a+b)^n k^n}, \quad w_{6n} = \frac{f^{n+1} c^n b^n a^n}{(-c+d)^n (-b+c)^n (-a+b)^n k^n},$$

$$w_{6n+1} = \frac{f^{n+1} c^{n+1} b^n a^n}{(-c+d)^{n+1} (-b+c)^n (-a+b)^n k^n}, \quad w_{6n+2} = \frac{f^{n+1} c^{n+1} b^{n+1} a^n}{(-c+d)^{n+1} (-b+c)^{n+1} (-a+b)^n k^n},$$

where $z_{-3} = d,\ z_{-2} = c,\ z_{-1} = b,\ z_0 = a,\ w_{-3} = k,\ w_{-2} = h,\ w_{-1} = g$ and $w_0 = f$ with $w_{-2} \neq \pm w_{-3}$, $w_{-1} \neq \pm w_{-2}$, $w_0 \neq \pm w_{-1}$, $z_{-2} \neq \pm z_{-3}$, $z_{-1} \neq \pm z_{-2}$ and $z_0 \neq \pm z_{-1}$.

Proof. By using mathematical induction. The result holds for $n = 0$. Suppose that the result holds for $n - 1$

$$z_{6n-9} = \frac{a^{n-1} h^{n-1} g^{n-1} f^{n-1}}{(h+k)^{n-1} (g+h)^{n-1} (f+g)^{n-1} d^{n-2}}, \quad z_{6n-8} = \frac{c\, a^{n-1} h^{n-1} g^{n-1} f^{n-1}}{(h+k)^{n-1} (g+h)^{n-1} (f+g)^{n-1} d^{n-1}},$$

$$z_{6n-7} = \frac{b\, a^{n-1} h^{n-1} g^{n-1} f^{n-1}}{(h+k)^{n-1} (g+h)^{n-1} (f+g)^{n-1} d^{n-1}}, \quad z_{6n-6} = \frac{a^n h^{n-1} g^{n-1} f^{n-1}}{(h+k)^{n-1} (g+h)^{n-1} (f+g)^{n-1} d^{n-1}},$$

$$z_{6n-5} = \frac{a^n h^n g^{n-1} f^{n-1}}{(h+k)^n (g+h)^{n-1} (f+g)^{n-1} d^{n-1}}, \quad z_{6n-4} = \frac{a^n h^n g^n f^{n-1}}{(h+k)^n (g+h)^n (f+g)^{n-1} d^{n-1}},$$

$$w_{6n-9} = \frac{f^{n-1} c^{n-1} b^{n-1} a^{n-1}}{(-c+d)^{n-1} (-b+c)^{n-1} (-a+b)^{n-1} k^{n-2}}, \quad w_{6n-8} = \frac{h\, f^{n-1} c^{n-1} b^{n-1} a^{n-1}}{(-c+d)^{n-1} (-b+c)^{n-1} (-a+b)^{n-1} k^{n-1}},$$

$$w_{6n-7} = \frac{g\, f^{n-1} c^{n-1} b^{n-1} a^{n-1}}{(-c+d)^{n-1} (-b+c)^{n-1} (-a+b)^{n-1} k^{n-1}}, \quad w_{6n-6} = \frac{f^n c^{n-1} b^{n-1} a^{n-1}}{(-c+d)^{n-1} (-b+c)^{n-1} (-a+b)^{n-1} k^{n-1}},$$

$$w_{6n-5} = \frac{f^n c^n b^{n-1} a^{n-1}}{(-c+d)^n (-b+c)^{n-1} (-a+b)^{n-1} k^{n-1}}, \quad w_{6n-4} = \frac{f^n c^n b^n a^{n-1}}{(-c+d)^n (-b+c)^n (-a+b)^{n-1} k^{n-1}}.$$

From system (4.1) we can prove as follow

$$\begin{aligned}
z_{6n-3} &= \frac{z_{6n-4}w_{6n-6}}{w_{6n-6}+w_{6n-7}} \\
&= \frac{\frac{a^n h^n g^n f^{n-1}}{(h+k)^n(g+h)^n(f+g)^{n-1}d^{n-1}} \times \frac{f^n c^{n-1} b^{n-1} a^{n-1}}{(-c+d)^{n-1}(-b+c)^{n-1}(-a+b)^{n-1}k^{n-1}}}{\frac{f^n c^{n-1} b^{n-1} a^{n-1}}{(-c+d)^{n-1}(-b+c)^{n-1}(-a+b)^{n-1}k^{n-1}} + \frac{g\, f^{n-1} c^{n-1} b^{n-1} a^{n-1}}{(-c+d)^{n-1}(-b+c)^{n-1}(-a+b)^{n-1}k^{n-1}}} \\
&= \frac{\frac{a^n h^n g^n f^n}{(h+k)^n(g+h)^n(f+g)^{n-1}d^{n-1}}}{f+g} \\
&= \frac{a^n h^n g^n f^n}{(h+k)^n(g+h)^n(f+g)^n d^{n-1}}.
\end{aligned}$$

$$\begin{aligned}
z_{6n-2} &= \frac{z_{6n-3}w_{6n-5}}{w_{6n-5}+w_{6n-6}} \\
&= \frac{\frac{a^n h^n g^n f^n}{(h+k)^n(g+h)^n(f+g)^n d^{n-1}} \times \frac{f^n c^n b^{n-1} a^{n-1}}{(-c+d)^n(-b+c)^{n-1}(-a+b)^{n-1}k^{n-1}}}{\frac{f^n c^n b^{n-1} a^{n-1}}{(-c+d)^n(-b+c)^{n-1}(-a+b)^{n-1}k^{n-1}} + \frac{f^n c^{n-1} b^{n-1} a^{n-1}}{(-c+d)^{n-1}(-b+c)^{n-1}(-a+b)^{n-1}k^{n-1}}} \\
&= \frac{\frac{c\, a^n h^n g^n f^n}{(h+k)^n(g+h)^n(f+g)^n d^{n-1}(-c+d)}}{\frac{c}{(-c+d)}+1} \\
&= \frac{c\, a^n h^n g^n f^n}{(h+k)^n(g+h)^n(f+g)^n d^n}.
\end{aligned}$$

$$\begin{aligned}
w_{6n-3} &= \frac{w_{6n-4}z_{6n-6}}{-z_{6n-6}+z_{6n-7}} \\
&= \frac{\frac{f^n c^n b^n a^{n-1}}{(-c+d)^n(-b+c)^n(-a+b)^{n-1}k^{n-1}} \times \frac{a^n h^{n-1} g^{n-1} f^{n-1}}{(h+k)^{n-1}(g+h)^{n-1}(f+g)^{n-1}d^{n-1}}}{\frac{a^n h^{n-1} g^{n-1} f^{n-1}}{(h+k)^{n-1}(g+h)^{n-1}(f+g)^{n-1}d^{n-1}} + \frac{b\, a^{n-1} h^{n-1} g^{n-1} f^{n-1}}{(h+k)^{n-1}(g+h)^{n-1}(f+g)^{n-1}d^{n-1}}} \\
&= \frac{\frac{f^n c^n b^n a^{n-1} a}{(-c+d)^n(-b+c)^n(-a+b)^{n-1}k^{n-1}}}{(-a+b)} \\
&= \frac{f^n c^n b^n a^n}{(-c+d)^n(-b+c)^n(-a+b)^n k^{n-1}}.
\end{aligned}$$

$$\begin{aligned}
w_{6n-2} &= \frac{w_{6n-3}z_{6n-5}}{-z_{6n-5}+z_{6n-6}} \\
&= \frac{\frac{f^n c^n b^n a^n}{(-c+d)^n(-b+c)^n(-a+b)^n k^{n-1}} \times \frac{a^n h^n g^{n-1} f^{n-1}}{(h+k)^n(g+h)^{n-1}(f+g)^{n-1}d^{n-1}}}{\frac{a^n h^n g^{n-1} f^{n-1}}{(h+k)^n(g+h)^{n-1}(f+g)^{n-1}d^{n-1}} + \frac{a^n h^{n-1} g^{n-1} f^{n-1}}{(h+k)^{n-1}(g+h)^{n-1}(f+g)^{n-1}d^{n-1}}} \\
&= \frac{\frac{f^n c^n b^n a^n\, h}{(-c+d)^n(-b+c)^n(-a+b)^n k^{n-1}(h+k)}}{\frac{-h}{h+k}+1} \\
&= \frac{f^n c^n b^n a^n\, h}{(-c+d)^n(-b+c)^n(-a+b)^n k^n}.
\end{aligned}$$

So, we can prove the other relations and the proof is completed. □

5. On The System: $z_{n+1}=\frac{z_nw_{n-2}}{w_{n-2}+w_{n-3}},\quad w_{n+1}=\frac{w_nz_{n-2}}{-z_{n-2}-z_{n-3}}$

We study in this section the solutions of the system of two difference equations

$$z_{n+1}=\frac{z_nw_{n-2}}{w_{n-2}+w_{n-3}},\quad w_{n+1}=\frac{w_nz_{n-2}}{-z_{n-2}-z_{n-3}} \tag{5.1}$$

where the initial conditions $z_{-3},\ z_{-2},\ z_{-1},\ z_0,\ w_{-3},\ w_{-2},\ w_{-1},\ w_0$ are arbitrary non-zero real numbers.

Theorem 4. *Assume that $\{z_n,w_n\}_{n=-3}^{\infty}$ is a solution to system (5.1) and let $z_{-3}=d$, $z_{-2}=c, z_{-1}=b, z_0=a, w_{-3}=k, w_{-2}=h, w_{-1}=g$ and $w_0=f$, with $w_{-2}\neq\pm w_{-3}$, $w_{-1}\neq\pm w_{-2},\ w_0\neq\pm w_{-1},\ z_{-2}\neq\pm z_{-3},\ z_{-1}\neq\pm z_{-2}$ and $z_0\neq\pm z_{-1}$. Then, for $n=0,1,\ldots$, we have*

$$z_{12n-3}=\frac{a^{2n}\ h^{2n}\ g^{2n}\ f^{2n}}{(h+k)^{2n}(g+h)^{2n}(f+g)^{2n}d^{2n-1}},\quad z_{12n-2}=\frac{c\ a^{2n}\ h^{2n}\ g^{2n}\ f^{2n}}{(h+k)^{2n}(g+h)^{2n}(f+g)^{2n}d^{2n}},$$

$$z_{12n-1}=\frac{b\ a^{2n}\ h^{2n}\ g^{2n}\ f^{2n}}{(h+k)^{2n}(g+h)^{2n}(f+g)^{2n}d^{2n}},\quad z_{12n}=\frac{a^{2n+1}\ h^{2n}\ g^{2n}\ f^{2n}}{(h+k)^{2n}(g+h)^{2n}(f+g)^{2n}d^{2n}},$$

$$z_{12n+1}=\frac{a^{2n+1}\ h^{2n+1}\ g^{2n}\ f^{2n}}{(h+k)^{2n+1}(g+h)^{2n}(f+g)^{2n}d^{2n}},\quad z_{12n+2}=\frac{a^{2n+1}\ h^{2n+1}\ g^{2n+1}\ f^{2n}}{(h+k)^{2n+1}(g+h)^{2n+1}(f+g)^{2n}d^{2n}},$$

$$z_{12n+3}=\frac{a^{2n+1}\ h^{2n+1}\ g^{2n+1}\ f^{2n+1}}{(h+k)^{2n+1}(g+h)^{2n+1}(f+g)^{2n+1}d^{2n}},\quad z_{12n+4}=\frac{-c\ a^{2n+1}\ h^{2n+1}\ g^{2n+1}\ f^{2n+1}}{(h+k)^{2n+1}(g+h)^{2n+1}(f+g)^{2n+1}d^{2n+1}},$$

$$z_{12n+5}=\frac{b\ a^{2n+1}\ h^{2n+1}\ g^{2n+1}\ f^{2n+1}}{(h+k)^{2n+1}(g+h)^{2n+1}(f+g)^{2n+1}d^{2n+1}},\quad z_{12n+6}=\frac{-\ a^{2n+2}\ h^{2n+1}\ g^{2n+1}\ f^{2n+1}}{(h+k)^{2n+1}(g+h)^{2n+1}(f+g)^{2n+1}d^{2n+1}},$$

$$z_{12n+7}=\frac{a^{2n+2}\ h^{2n+2}\ g^{2n+1}\ f^{2n+1}}{(h+k)^{2n+2}(g+h)^{2n+1}(f+g)^{2n+1}d^{2n+1}},\quad z_{12n+8}=\frac{-\ a^{2n+2}\ h^{2n+2}\ g^{2n+2}\ f^{2n+1}}{(h+k)^{2n+2}(g+h)^{2n+2}(f+g)^{2n+1}d^{2n+1}},$$

$$w_{12n-3}=\frac{(-1)^nf^{2n}c^{2n}b^{2n}a^{2n}h^ng^n}{(c+d)^n(b+c)^n(a+b)^n(2h+k)^n(2g+h)^n(2f+g)^n(c-d)^n(b-c)^n(a-b)^nk^{n-1}},$$

$$w_{12n-2}=\frac{(-1)^nf^{2n}c^{2n}b^{2n}a^{2n}h^{n+1}g^n}{(c+d)^n(b+c)^n(a+b)^n(2h+k)^n(2g+h)^n(2f+g)^n(c-d)^n(b-c)^n(a-b)^nk^{n}},$$

$$w_{12n-1}=\frac{(-1)^nf^{2n}c^{2n}b^{2n}a^{2n}h^ng^{n+1}}{(c+d)^n(b+c)^n(a+b)^n(2h+k)^n(2g+h)^n(2f+g)^n(c-d)^n(b-c)^n(a-b)^nk^{n}},$$

$$w_{12n}=\frac{(-1)^nf^{2n+1}c^{2n}b^{2n}a^{2n}h^ng^n}{(c+d)^n(b+c)^n(a+b)^n(2h+k)^n(2g+h)^n(2f+g)^n(c-d)^n(b-c)^n(a-b)^nk^{n}},$$

$$w_{12n+1}=\frac{(-1)^{n+1}f^{2n+1}c^{2n+1}b^{2n}a^{2n}h^ng^n}{(c+d)^{n+1}(b+c)^n(a+b)^n(2h+k)^n(2g+h)^n(2f+g)^n(c-d)^n(b-c)^n(a-b)^nk^{n}},$$

$$w_{12n+2}=\frac{(-1)^nf^{2n+1}c^{2n+1}b^{2n+1}a^{2n}h^ng^n}{(c+d)^{n+1}(b+c)^{n+1}(a+b)^n(2h+k)^n(2g+h)^n(2f+g)^n(c-d)^n(b-c)^n(a-b)^nk^{n}},$$

$$w_{12n+3}=\frac{(-1)^{n+1}f^{2n+1}c^{2n+1}b^{2n+1}a^{2n+1}h^{n}g^{n}}{(c+d)^{n+1}(b+c)^{n+1}(a+b)^{n+1}(2h+k)^{n}(2g+h)^{n}(2f+g)^{n}(c-d)^{n}(b-c)^{n}(a-b)^{n}k^{n}},$$

$$w_{12n+4}=\frac{(-1)^{n}f^{2n+1}c^{2n+1}b^{2n+1}a^{2n+1}h^{n+1}g^{n}}{(c+d)^{n+1}(b+c)^{n+1}(a+b)^{n+1}(2h+k)^{n+1}(2g+h)^{n}(2f+g)^{n}(c-d)^{n}(b-c)^{n}(a-b)^{n}k^{n}},$$

$$w_{12n+5}=\frac{(-1)^{n+1}f^{2n+1}c^{2n+1}b^{2n+1}a^{2n+1}h^{n+1}g^{n+1}}{(c+d)^{n+1}(b+c)^{n+1}(a+b)^{n+1}(2h+k)^{n+1}(2g+h)^{n+1}(2f+g)^{n}(c-d)^{n}(b-c)^{n}(a-b)^{n}k^{n}},$$

$$w_{12n+6}=\frac{(-1)^{n}f^{2n+2}c^{2n+1}b^{2n+1}a^{2n+1}h^{n+1}g^{n+1}}{(c+d)^{n+1}(b+c)^{n+1}(a+b)^{n+1}(2h+k)^{n+1}(2g+h)^{n+1}(2f+g)^{n+1}(c-d)^{n}(b-c)^{n}(a-b)^{n}k^{n}},$$

$$w_{12n+7}=\frac{(-1)^{n+1}f^{2n+2}c^{2n+2}b^{2n+1}a^{2n+1}h^{n+1}g^{n+1}}{(c+d)^{n+1}(b+c)^{n+1}(a+b)^{n+1}(2h+k)^{n+1}(2g+h)^{n+1}(2f+g)^{n+1}(c-d)^{n+1}(b-c)^{n}(a-b)^{n}k^{n}},$$

$$w_{12n+8}=\frac{(-1)^{n}f^{2n+2}c^{2n+2}b^{2n+2}a^{2n+1}h^{n+1}g^{n+1}}{\begin{array}{c}(c+d)^{n+1}(b+c)^{n+1}(a+b)^{n+1}(2h+k)^{n+1}(2g+h)^{n+1}(2f+g)^{n+1}\\(c-d)^{n+1}(b-c)^{n+1}(a-b)^{n}k^{n}\end{array}}.$$

Proof. For n = 0 the result holds. Now suppose that n ¿ 0 and that our assumption holds for n - 1. That is,

$$z_{12n-15}=\frac{a^{2n-2}\,h^{2n-2}\,g^{2n-2}\,f^{2n-2}}{(h+k)^{2n-2}(g+h)^{2n-2}(f+g)^{2n-2}d^{2n-3}},$$

$$z_{12n-14}=\frac{c\,a^{2n-2}\,h^{2n-2}\,g^{2n-2}\,f^{2n-2}}{(h+k)^{2n-2}(g+h)^{2n-2}(f+g)^{2n-2}d^{2n-2}},$$

$$z_{12n-13}=\frac{b\,a^{2n-2}\,h^{2n-2}\,g^{2n-2}\,f^{2n-2}}{(h+k)^{2n-2}(g+h)^{2n-2}(f+g)^{2n-2}d^{2n-2}},$$

$$z_{12n-12}=\frac{a^{2n-1}\,h^{2n-2}\,g^{2n-2}\,f^{2n-2}}{(h+k)^{2n-2}(g+h)^{2n-2}(f+g)^{2n-2}d^{2n-2}},$$

$$z_{12n-11}=\frac{a^{2n-1}\,h^{2n-1}\,g^{2n-2}\,f^{2n-2}}{(h+k)^{2n-1}(g+h)^{2n-2}(f+g)^{2n-2}d^{2n-2}},$$

$$z_{12n-10}=\frac{a^{2n-1}\,h^{2n-1}\,g^{2n-1}\,f^{2n-2}}{(h+k)^{2n-1}(g+h)^{2n-1}(f+g)^{2n-2}d^{2n-2}},$$

$$z_{12n-9}=\frac{a^{2n-1}\,h^{2n-1}\,g^{2n-1}\,f^{2n-1}}{(h+k)^{2n-1}(g+h)^{2n-1}(f+g)^{2n-1}d^{2n-2}},$$

$$z_{12n-8}=\frac{-c\,a^{2n-1}\,h^{2n-1}\,g^{2n-1}\,f^{2n-1}}{(h+k)^{2n-1}(g+h)^{2n-1}(f+g)^{2n-1}d^{2n-1}},$$

$$z_{12n-7}=\frac{b\,a^{2n-1}\,h^{2n-1}\,g^{2n-1}\,f^{2n-1}}{(h+k)^{2n-1}(g+h)^{2n-1}(f+g)^{2n-1}d^{2n-1}},$$

$$z_{12n-6}=\frac{-\,a^{2n}\,h^{2n-1}\,g^{2n-1}\,f^{2n-1}}{(h+k)^{2n-1}(g+h)^{2n-1}(f+g)^{2n-1}d^{2n-1}},$$

$$z_{12n-5}=\frac{a^{2n}\,h^{2n}\,g^{2n-1}\,f^{2n-1}}{(h+k)^{2n}(g+h)^{2n-1}(f+g)^{2n-1}d^{2n-1}},$$

$$z_{12n-4}=\frac{-\,a^{2n}\,h^{2n}\,g^{2n}\,f^{2n-1}}{(h+k)^{2n}(g+h)^{2n}(f+g)^{2n-1}d^{2n-1}},$$

$$w_{12n-15} = \frac{(-1)^{n-1} f^{2n-2} c^{2n-2} b^{2n-2} a^{2n-2} h^{n-1} g^{n-1}}{(c+d)^{n-1}(b+c)^{n-1}(a+b)^{n-1}(2h+k)^{n-1}(2g+h)^{n-1}(2f+g)^{n-1}(c-d)^{n-1} (b-c)^{n-1}(a-b)^{n-1}k^{n-2}},$$

$$w_{12n-14} = \frac{(-1)^{n-1} f^{2n-2} c^{2n-2} b^{2n-2} a^{2n-2} h^{n} g^{n-1}}{(c+d)^{n-1}(b+c)^{n-1}(a+b)^{n-1}(2h+k)^{n-1}(2g+h)^{n-1}(2f+g)^{n-1}(c-d)^{n-1} (b-c)^{n-1}(a-b)^{n-1}k^{n-1}},$$

$$w_{12n-13} = \frac{(-1)^{n-1} f^{2n-2} c^{2n-2} b^{2n-2} a^{2n-2} h^{n-1} g^{n}}{(c+d)^{n-1}(b+c)^{n-1}(a+b)^{n-1}(2h+k)^{n-1}(2g+h)^{n-1}(2f+g)^{n-1}(c-d)^{n-1} (b-c)^{n-1}(a-b)^{n-1}k^{n-1}},$$

$$w_{12n-12} = \frac{(-1)^{n-1} f^{2n-1} c^{2n-2} b^{2n-2} a^{2n-2} h^{n-1} g^{n-1}}{(c+d)^{n-1}(b+c)^{n-1}(a+b)^{n-1}(2h+k)^{n-1}(2g+h)^{n-1}(2f+g)^{n-1}(c-d)^{n-1} (b-c)^{n-1}(a-b)^{n-1}k^{n-1}},$$

$$w_{12n-11} = \frac{(-1)^{n} f^{2n-1} c^{2n-1} b^{2n-2} a^{2n-2} h^{n-1} g^{n-1}}{(c+d)^{n}(b+c)^{n-1}(a+b)^{n-1}(2h+k)^{n-1}(2g+h)^{n-1}(2f+g)^{n-1}(c-d)^{n-1} (b-c)^{n-1}(a-b)^{n-1}k^{n-1}},$$

$$w_{12n-10} = \frac{(-1)^{n-1} f^{2n-1} c^{2n-1} b^{2n-1} a^{2n-2} h^{n-1} g^{n-1}}{(c+d)^{n}(b+c)^{n}(a+b)^{n-1}(2h+k)^{n-1}(2g+h)^{n-1}(2f+g)^{n-1}(c-d)^{n-1} (b-c)^{n-1}(a-b)^{n-1}k^{n-1}},$$

$$w_{12n-9} = \frac{(-1)^{n} f^{2n-1} c^{2n-1} b^{2n-1} a^{2n-1} h^{n-1} g^{n-1}}{(c+d)^{n}(b+c)^{n}(a+b)^{n}(2h+k)^{n-1}(2g+h)^{n-1}(2f+g)^{n-1}(c-d)^{n-1} (b-c)^{n-1}(a-b)^{n-1}k^{n-1}},$$

$$w_{12n-8} = \frac{(-1)^{n-1} f^{2n-1} c^{2n-1} b^{2n-1} a^{2n-1} h^{n} g^{n-1}}{(c+d)^{n}(b+c)^{n}(a+b)^{n}(2h+k)^{n}(2g+h)^{n-1}(2f+g)^{n-1}(c-d)^{n-1} (b-c)^{n-1}(a-b)^{n-1}k^{n-1}},$$

$$w_{12n-7} = \frac{(-1)^{n} f^{2n-1} c^{2n-1} b^{2n-1} a^{2n-1} h^{n} g^{n}}{(c+d)^{n}(b+c)^{n}(a+b)^{n}(2h+k)^{n}(2g+h)^{n}(2f+g)^{n-1}(c-d)^{n-1}(b-c)^{n-1}(a-b)^{n-1}k^{n-1}},$$

$$w_{12n-6} = \frac{(-1)^{n-1} f^{2n} c^{2n-1} b^{2n-1} a^{2n-1} h^{n} g^{n}}{(c+d)^{n}(b+c)^{n}(a+b)^{n}(2h+k)^{n}(2g+h)^{n}(2f+g)^{n}(c-d)^{n-1}(b-c)^{n-1}(a-b)^{n-1}k^{n-1}},$$

$$w_{12n-5} = \frac{(-1)^{n} f^{2n} c^{2n} b^{2n-1} a^{2n-1} h^{n} g^{n}}{(c+d)^{n}(b+c)^{n}(a+b)^{n}(2h+k)^{n}(2g+h)^{n}(2f+g)^{n}(c-d)^{n}(b-c)^{n-1}(a-b)^{n-1}k^{n-1}},$$

$$w_{12n-4} = \frac{(-1)^{n-1} f^{2n} c^{2n} b^{2n} a^{2n-1} h^{n} g^{n}}{(c+d)^{n}(b+c)^{n}(a+b)^{n}(2h+k)^{n}(2g+h)^{n}(2f+g)^{n}(c-d)^{n}(b-c)^{n}(a-b)^{n-1}k^{n-1}}.$$

From system (5.1) we have

$$
\begin{aligned}
z_{12n-3} &= \frac{z_{12n-4}w_{12n-6}}{w_{12n-6}+w_{12n-7}} \\
&= \frac{\frac{-a^{2n}\,h^{2n}\,g^{2n}\,f^{2n-1}}{(h+k)^{2n}(g+h)^{2n}(f+g)^{2n-1}d^{2n-1}} \times \frac{(-1)^{n-1}f^{2n}c^{2n-1}b^{2n-1}a^{2n-1}h^{n}g^{n}}{(c+d)^{n}(b+c)^{n}(a+b)^{n}(2h+k)^{n}(2g+h)^{n}(2f+g)^{n}(c-d)^{n-1}(b-c)^{n-1}(a-b)^{n-1}k^{n-1}}}{\frac{(-1)^{n-1}f^{2n}c^{2n-1}b^{2n-1}a^{2n-1}h^{n}g^{n}}{(c+d)^{n}(b+c)^{n}(a+b)^{n}(2h+k)^{n}(2g+h)^{n}(2f+g)^{n}(c-d)^{n-1}(b-c)^{n-1}(a-b)^{n-1}k^{n-1}} + \frac{(-1)^{n}f^{2n-1}c^{2n-1}b^{2n-1}a^{2n-1}h^{n}g^{n}}{(c+d)^{n}(b+c)^{n}(a+b)^{n}(2h+k)^{n}(2g+h)^{n}(2f+g)^{n-1}(c-d)^{n-1}(b-c)^{n-1}(a-b)^{n-1}k^{n-1}}} \\
&= \frac{\frac{-a^{2n}\,h^{2n}\,g^{2n}\,f^{2n}}{(h+k)^{2n}(g+h)^{2n}(f+g)^{2n-1}d^{2n-1}(2f+g)}}{\frac{f}{(2f+g)}-1} \\
&= \frac{a^{2n}\,h^{2n}\,g^{2n}\,f^{2n}}{(h+k)^{2n}(g+h)^{2n}(f+g)^{2n}d^{2n-1}}
\end{aligned}
$$

$$
\begin{aligned}
z_{12n-2} &= \frac{z_{12n-3}w_{12n-5}}{w_{12n-5}+w_{12n-6}} \\
&= \frac{\frac{a^{2n}\,h^{2n}\,g^{2n}\,f^{2n}}{(h+k)^{2n}(g+h)^{2n}(f+g)^{2n}d^{2n-1}} \times \frac{(-1)^{n}f^{2n}c^{2n}b^{2n-1}a^{2n-1}h^{n}g^{n}}{(c+d)^{n}(b+c)^{n}(a+b)^{n}(2h+k)^{n}(2g+h)^{n}(2f+g)^{n}(c-d)^{n}(b-c)^{n-1}(a-b)^{n-1}k^{n-1}}}{\frac{(-1)^{n}f^{2n}c^{2n}b^{2n-1}a^{2n-1}h^{n}g^{n}}{(c+d)^{n}(b+c)^{n}(a+b)^{n}(2h+k)^{n}(2g+h)^{n}(2f+g)^{n}(c-d)^{n}(b-c)^{n-1}(a-b)^{n-1}k^{n-1}} + \frac{(-1)^{n-1}f^{2n}c^{2n-1}b^{2n-1}a^{2n-1}h^{n}g^{n}}{(c+d)^{n}(b+c)^{n}(a+b)^{n}(2h+k)^{n}(2g+h)^{n}(2f+g)^{n}(c-d)^{n-1}(b-c)^{n-1}(a-b)^{n-1}k^{n-1}}} \\
&= \frac{\frac{a^{2n}\,h^{2n}\,g^{2n}\,f^{2n}(-c)}{(h+k)^{2n}(g+h)^{2n}(f+g)^{2n}d^{2n-1}(c-d)}}{\frac{-c}{c-d}+1} \\
&= \frac{a^{2n}\,h^{2n}\,g^{2n}\,f^{2n}c}{(h+k)^{2n}(g+h)^{2n}(f+g)^{2n}d^{2n}}
\end{aligned}
$$

$$
\begin{aligned}
w_{12n-3} &= \frac{w_{12n-4}z_{12n-6}}{-z_{12n-6}-z_{12n-7}} \\
&= \frac{\frac{(-1)^{n-1}f^{2n}c^{2n}b^{2n}a^{2n-1}h^{n}g^{n}}{(c+d)^{n}(b+c)^{n}(a+b)^{n}(2h+k)^{n}(2g+h)^{n}(2f+g)^{n}(c-d)^{n}(b-c)^{n}(a-b)^{n-1}k^{n-1}} \times \frac{-a^{2n}\,h^{2n-1}\,g^{2n-1}\,f^{2n-1}}{(h+k)^{2n-1}(g+h)^{2n-1}(f+g)^{2n-1}d^{2n-1}}}{\frac{a^{2n}\,h^{2n-1}\,g^{2n-1}\,f^{2n-1}}{(h+k)^{2n-1}(g+h)^{2n-1}(f+g)^{2n-1}d^{2n-1}} - \frac{b\,a^{2n-1}\,h^{2n-1}\,g^{2n-1}\,f^{2n-1}}{(h+k)^{2n-1}(g+h)^{2n-1}(f+g)^{2n-1}d^{2n-1}}} \\
&= \frac{\frac{(-1)^{n-1}f^{2n}c^{2n}b^{2n}a^{2n-1}h^{n}g^{n}a}{(c+d)^{n}(b+c)^{n}(a+b)^{n}(2h+k)^{n}(2g+h)^{n}(2f+g)^{n}(c-d)^{n}(b-c)^{n}(a-b)^{n-1}k^{n-1}}}{(a-b)} \\
&= \frac{(-1)^{n}f^{2n}c^{2n}b^{2n}a^{2n}h^{n}g^{n}}{(c+d)^{n}(b+c)^{n}(a+b)^{n}(2h+k)^{n}(2g+h)^{n}(2f+g)^{n}(c-d)^{n}(b-c)^{n}(a-b)^{n}k^{n-1}}
\end{aligned}
$$

$$
\begin{aligned}
w_{12n-2} &= \frac{w_{12n-3}z_{12n-5}}{-z_{12n-5}-z_{12n-6}} \\
&= \frac{\frac{(-1)^n f^{2n}c^{2n}b^{2n}a^{2n}h^n g^n}{(c+d)^n(b+c)^n(a+b)^n(2h+k)^n(2g+h)^n(2f+g)^n(c-d)^n(b-c)^n(a-b)^n k^{n-1}} \times \frac{a^{2n}\ h^{2n}\ g^{2n-1}\ f^{2n-1}}{(h+k)^{2n}(g+h)^{2n-1}(f+g)^{2n-1}d^{2n-1}}}{\frac{-a^{2n}\ h^{2n}\ g^{2n-1}\ f^{2n-1}}{(h+k)^{2n}(g+h)^{2n-1}(f+g)^{2n-1}d^{2n-1}} + \frac{a^{2n}\ h^{2n-1}\ g^{2n-1}\ f^{2n-1}}{(h+k)^{2n-1}(g+h)^{2n-1}(f+g)^{2n-1}d^{2n-1}}} \\
&= \frac{\frac{(-1)^n f^{2n}c^{2n}b^{2n}a^{2n}h^n g^n h}{(c+d)^n(b+c)^n(a+b)^n(2h+k)^n(2g+h)^n(2f+g)^n(c-d)^n(b-c)^n(a-b)^n k^{n-1}(h+k)}}{\frac{-h}{h+k}+1} \\
&= \frac{(-1)^n f^{2n}c^{2n}b^{2n}a^{2n}h^{n+1}g^n}{(c+d)^n(b+c)^n(a+b)^n(2h+k)^n(2g+h)^n(2f+g)^n(c-d)^n(b-c)^n(a-b)^n k^n}
\end{aligned}
$$

So, we can prove the other relations and the proof is completed.

□

6. NUMERICAL EXAMPLES

In order to illustrate the results of the previous sections and to support our theoretical discussions, we present several interesting numerical examples in this section.

Example 1. *We take the initial conditions, for the system (2.1), as follows* $z_{-3} = 7.5$, $z_{-2} = -12$,$z_{-1} = 9$, $z_0 = 6$, $w_{-3} = 4.75$, $w_{-2} = 24$, $w_{-1} = 13$ *and* $w_0 = -2$.*See Fig. 1.*

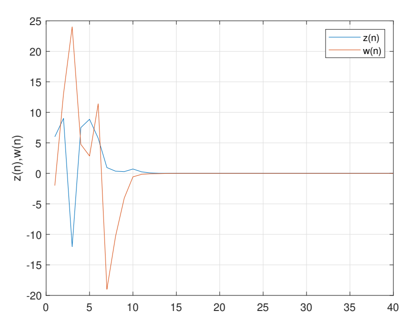

FIGURE 1

Example 2. *We consider numerical example for the difference system (3.1) with the initial conditions* $z_{-3} = 5.3$, $z_{-2} = -.45$,$z_{-1} = 3.5$, $z_0 = 11$, $w_{-3} = 8$, $w_{-2} = 26$, $w_{-1} = 14$ *and* $w_0 = 2$. *See Fig. 2.*

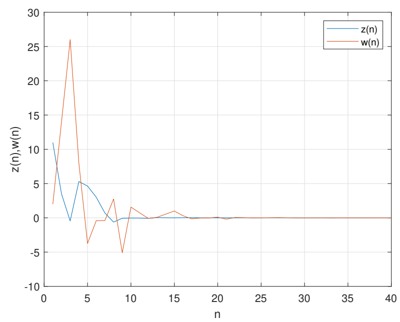

FIGURE 2

Example 3. *See Figure 3. We put the initial condition* $z_{-3} = 20$, $z_{-2} = -3$,$z_{-1} = 9$, $z_0 = 6$, $w_{-3} = 4.75$, $w_{-2} = 24$, $w_{-1} = 13$ *and* $w_0 = 18$ *on the difference system (4.1)*

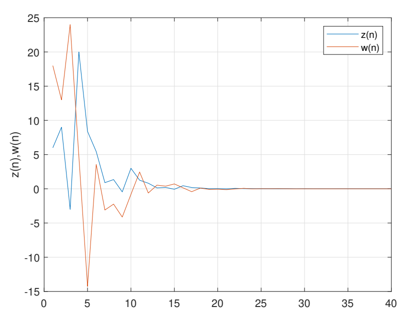

FIGURE 3

Example 4. *Fig. 4 shows the behavior of the solution of the difference system (5.1) with the initial conditions* $z_{-3} = 24$, $z_{-2} = -30$,$z_{-1} = 11$, $z_0 = -5.5$, $w_{-3} = 7$, $w_{-2} = 13$, $w_{-1} = 8.75$ *and* $w_0 = -14$.

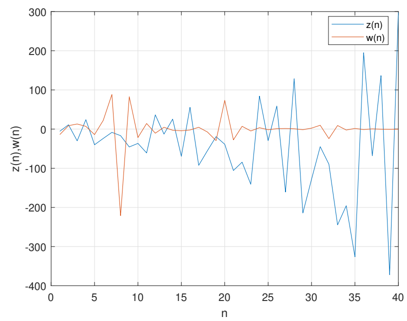

FIGURE 4

CONCLUSION

This paper discussed the expression's form of systems of rational Fourth order difference equations. In Section 2, we obtained the form of the solutions of system $z_{n+1} = \frac{z_n w_{n-2}}{w_{n-2}+w_{n-3}}, w_{n+1} = \frac{w_n z_{n-2}}{z_{n-2}+z_{n-3}}$. Also In section 3, we found the solution's form of the system $z_{n+1} = \frac{z_n w_{n-2}}{w_{n-2}+w_{n-3}}, w_{n+1} = \frac{w_n z_{n-2}}{z_{n-2}-z_{n-3}}$. In section 4,5 , we got the solution of the following systems respectively, $z_{n+1} = \frac{z_n w_{n-2}}{w_{n-2}+w_{n-3}}, w_{n+1} = \frac{w_n z_{n-2}}{-z_{n-2}+z_{n-3}}$ and $z_{n+1} = \frac{z_n w_{n-2}}{w_{n-2}+w_{n-3}}, w_{n+1} = \frac{w_n z_{n-2}}{-z_{n-2}-z_{n-3}}$. Finally, we gave a numerical example in each case to illustrates the results.

ACKNOWLEDGEMENT

The authors are thankful to the Editor-in-Chief, Prof. G. Muhiuddin for the technical comments and to the anonymous referee(s) for a careful checking of the details and for suggestions for improvement of this paper.

REFERENCES

[1] M. Alghamdi, E. M. Elsayed and M. M. El-Dessoky, On the Solutions of Some Systems of Second Order Rational Difference Equations, Life Sci J., 10 (3) (2013), 344-351.

[2] M. B. Almatrafi and E. M. Elsayed, Solutions and Formulae For Some Systems Of Difference Equations, MathLAB Journal, 1 (3) (2018), 356-369.

[3] F. Belhannache, Asymptotic stability of a higher order rational difference equation, Electronic Journal of Mathematical Analysis and Applications, 2019, 7(2),1–8.

[4] D. Clark, and M. R. S. Kulenovic, A coupled system of rational difference equations, Comput. Math. Appl., 43 (2002), 849-867.

[5] C. Cinar, On the positive solutions of the difference equation system $z_{n+1} = \frac{1}{w_n}, w_{n+1} = \frac{w_n}{z_{n-1}w_{n-1}}$, Applied Mathematics and Computation, 158 (2) (2004), 303–305.

[6] Q. Din and E. M. Elsayed, Stability analysis of a discrete ecological model, Computational Ecology and Software, 4 (2) (2014), 89–103.

[7] Q. Din, M. N. Qureshi and A. Q. Khan, Dynamics of a fourth-order system of rational difference equations, Advances in Difference Equations, (2012), 1—15.

[8] E. M. Elabbasy, H. El-Metwally and E. M. Elsayed, On the solutions of a class of difference equations systems, Demonstratio Mathematica, 41 (1) (2008), 109–122.

[9] E. M. Elabbasy, H. El-Metwally and E. M. Elsayed, Global behavior of the solutions of difference equation, Advances in Difference Equations, (2011).

[10] E. M. Elabbasy and E. M. Elsayed, Global attractivity and periodic nature of a difference equation, World Applied Sciences Journal, 12 (1) (2011), 39–47.

[11] S. N. Elaydi, An Introduction to Difference Equations, Springer, New York, NY, USA, 2nd edition, 1999.
[12] E. M. Elsayed, M. M. El-Dessoky and Abdullah Alotaibi, On the Solutions of a General System of Difference Equations,Discrete Dynamics in Nature and Society, (2012), 12.
[13] M. M. El-Dessoky and M. El-Moneam, On The Higher Order Difference Equation $x_{n+1} = Ax_n + Bx_{n-1} + Cx_{n-k} + (\gamma x_{n-k})/(Dx_{n-s} + Ex_{n-t})$, Journal of Computational Analysis and Applications, 2018, 25(2), 342–354.
[14] E. M. Elsayed, B. S. Alofi, A. Q. Khan, Qualitative behavior of solutions of tenth-order recursive sequence equation, Math. Probl. Eng., 2022 (2022).
[15] E. M. Elsayed, On the solutions and periodicity of some rational systems of difference equations, Bull. Math. Soc. Sci. Math. Roumanie Tome 60 (108) (2) (2017), 159–171.
[16] N. Haddad, N. Touafek and J. F. T. Rabago, Well-defined solutions of a system of difference equations, Journal of Applied Mathematics and Computing, 56 (2018), 439-458.
[17] J. Hale and H. Kocak, Dynamics and Bifwcations, Springer-Verlag, New York, (1991) .
[18] Y. Halim, N. Touafek and E. M. Elsayed, Closed form solutions of some systems of rational difference equations in terms of Fibonacci numbers, Dynamics of Continuous, Discrete and Impulsive Systems, Series A: Mathematical Analysis, 21 (2014), 473-486.
[19] A. Gelisken and M. Kara, Some general systems of rational difference equations, J. Differ. Equ., 2015 (2015).
[20] M. Gumus, Global Dynamics of Solutions of A New Class of Rational Difference Equations, Konuralp Journal of Mathematics, 2019, 7(2), 380–387.
[21] T. F. Ibrahim and N. Touafek, Max-type system of difference equations with positive two-periodic sequences, Math. Methods Appl. Sci., 37 (2014), 2541–2553.
[22] A. Khaliq and E. M. Elsayed, Qualitative properties of difference equation of order six, Math., 4 (2016), 14.
[23] A. Q. Khan and M. N. Qureshi, Global dynamics of a competitive system of rational difference equations, Math. Methods Appl. Sci., 38 (2015), 4786–4796.
[24] A. Khelifa, Y. Halim, M. Berkal, Solutions of a system of two higher-order difference equations in terms of Lucas sequence, Univ. J. Math. Appl., 2 (2019), 202–211.
[25] M. R. S. Kulenovic and M. Pilling, Global Dynamics of a Certain Two-dimensional Competitive System of Rational Difference Equations with Quadratic Terms, Journal of Computational Analysis and Applications, 19 (1) (2015), 156-166.
[26] A. S. Kurbanli, On the behavior of solutions of the system of rational difference equations $z_{n+1} = \frac{z_{n-1}}{w_n z_{n-1} - 1}$, $w_{n+1} = \frac{w_{n-1}}{z_n w_{n-1} - 1}$,World Appl. Sci. J., 10(11) (2010), 1344-1350.
[27] A. S. Kurbanli, C. Cinar and I. Yalcınkaya, On the behavior of positive solutions of the system of rational difference equations, Math. Comput. Modelling, 53 (2011), 1261–1267.
[28] H. Ma and H. Feng, On positive solutions for the rational difference equation systems, Int. Scho. Res. Not., 2014 (2014), 4.
[29] M. Mansour, M. M. El-Dessoky and E. M. Elsayed, The form of the solutions and periodicity of some systems of difference equations, Discrete Dynamics in Nature and Society,(2012), 17.
[30] M. Mansour, M. M. El-Dessoky and E. M. Elsayed, On the solution of rational systems of difference equations, J. Comp. Anal. Appl., 15 (5) (2013), 967-976.
[31] H.L. Smith, Planar competitive and cooperative difference equations, J. Dif. EqzLa., 3 (1998), 335-357.
[32] D. Tollu, I. Yalcınkaya, H. Ahmad and S. Yao, A detailed study on a solvable system related to the linear fractional difference equation, Math. Biosci. Eng., 18 (2021), 5392–5408.
[33] N. Touafek and E. M. Elsayed, On the solutions of systems of rational difference equations, Math. Comput. Mod., 55 (2012), 1987—1997.
[34] N. Touafek and E. M. Elsayed, On the periodicity of some systems of nonlinear difference equations, Bull. Math. Soc. Sci. Math. Roumanie, 55 (2012), 217–224.
[35] N. Touafek and E. M. Elsayed, On a second order rational systems of difference equation, Hokkaido Math. J., 44 (2015), 29–45.
[36] I. Yalcinkaya, On the global asymptotic stability of a second-order system of difference equations, Disc. Dyn. Nat. Soc., (2008), 12.
[37] I. Yalcınkaya, On the global asymptotic behavior of a system of two nonlinear difference equations, Ars Combin., 95 (2010), 151–159.
[38] Y. Yazlik, D. T. Tollu, and N .Taskara, On the Behaviour of Solutions for Some Systems of Difference Equations, Journal of Computational Analysis and Applications, 18 (1) (2015), 166-178.
[39] Q. Zhang, J. Liu, and Z. Luo, Dynamical Behavior of a System of Third-Order Rational Difference Equation, Discrete Dynamics in Nature and Society, (2015), 6.

[40] Q. Zhang and W. Zhang, On a system of two high-order nonlinear difference equations, Advances in Mathematical Physics, 2014 (2014), 8.

MAI T. ALHARTHI
DEPARTMENT OF MATHEMATICS, FACULTY OF SCIENCE, KING ABDULAZIZ UNIVERSITY, JEDDAH, KINGDOM OF SAUDI ARABIA.
DEPARTMENT OF MATHEMATICS, FACULTY OF SCIENCE, JEDDAH UNIVERSITY, JEDDAH, KINGDOM OF SAUDI ARABIA.
Email address: miami-992@hotmail.com

E. M. ELASYED
DEPARTMENT OF MATHEMATICS, FACULTY OF SCIENCE, KING ABDULAZIZ UNIVERSITY, JEDDAH, KINGDOM OF SAUDI ARABIA.
DEPARTMENT OF MATHEMATICS, FACULTY OF SCIENCE, MANSOURA UNIVERSITY, MANSOURA, 35516, EGYPT.
Email address: emmelsayed@yahoo.com.