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A NEW VIEW ON (2,2)-REGULAR AG–GROUPOID VIA DFS SETS WITH APPLICATIONS IN DECISION MAKING

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ABSTRACT. In this paper, we study DFS left (right, two-sided) ideals, DFS (generalized) bi-ideals, DFS interior ideals and DFS (1, 2)-ideals of (2,2)-regular AG–Groupoid over an initial universe set U. We have shown that these DFS ideals are coincides in a (2,2)-regular unitary AG–Groupoid. Further we investigate some useful conditions for an AG–Groupoid to become a (2,2)-regular AG–Groupoid and characterize a (2,2)-regular AG–Groupoid in terms of DFS ideals. Finally we apply DFS expert sets to develop a decision making scheme for everyday problems.

1. Introduction

The (crisp) set theory is a main mathematical approach to deal with a class of problems that are characterized by precision, exactness, specificity, perfection, and certainty. However, many problems in the real-life inherently involve inconsistency, imprecision, ambiguity, and uncertainties. In particular, such classes of problems arise in engineering, economics, medical sciences, environmental sciences, social sciences, and many different scopes. The crisp (classical) mathematical tools fail to model or solve these types of problems.

In the course of time, mathematicians, engineers, and scientists, particularly those who focus on artificial intelligence, are seeking for alternative mathematical approaches to solve the problems that contain uncertainty or vagueness. They initiated several set theories such as probability theory, fuzzy set, intuitionistic fuzzy set, and rough set.

In 1999, Molodtsov [29] proposed the concept of soft sets, which has wide range applications in artificial intelligence, computer engineering, control engineering, robotices, medical diagnosis, forcasting, operation research, management science and many more. The theory of soft sets is a novel mathematical approach as concerns with the uncertainties. Now a days, the concept of soft sets obtain a lot of reputation for its parameteric nature. Due to its dynamical behaviour, the soft sets victoriously made its place and now comprehensively used in many applied areas. For example, soft sets are applied in decision making problems [7, 11, 25], soft integrals, soft derivatives and soft numbers along with

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their applications in [35]. In international business, soft sets are applied for forecasting the import and export volumes [35]. Maji et al. [27]. Maji et al. [26] gives many operations of algebraic structures in the form of soft sets that is further elaborated by Ali et al. [1, 2]. The major areas of application of soft computing are: robotics and machine control (path planning, control, coordination, and decision making [6]), natural language processing (representation and understanding), speech and character recognition (understanding, image processing, and biometrics [43]), biomedical systems and bioinformatics (Santos-Buitrago et al. [36] define a real-life application for decision making under incomplete information in the field of symbolic computational biology [39]), and big data and data mining (extract rules, features, analysis, and trends from large databases, e.g. social networks or financial series).

Currently, Jun et al. [16] further extend the notion of soft set into double-framed soft sets and apply double-framed soft set to BCK/BCI algebra and studied its related properties. Jun et al. [16] also introduced the concept of a double-framed soft ideal (briefly, DFS ideal) of a BCK/BCI-algebra and produce much valuable results. In [18], Khan et al. have applied the idea of double-framed soft set to LA-semigroup and defined double-framed soft LA-semigroup (briefly. DFS-LA-semigroup). Khan et al. have also characterized different classes of LA-semigroups by using different DFS ideals. Iftikhar and Mahmood [14] produced several results on lattice ordered double-framed soft semirings, Bordar et al. [5] applied the said concept to hyper BCK-algebra. In addition, Jayaraman et al. [15] introduced double-framed soft lattices, distributive double-framed soft lattice and double-framed soft chain. Khan and Mahmood [21] developed the concept of double-framed T-soft fuzzy set and applied the concept into BCK/BCI-algebra. Park [33] developed, double-framed soft deductive system in subtraction algebra and Hussain [12] produced the applications of double-framed soft ideal in gamma near-rings. Also, Hussain at al. [13] introduced double-framed fuzzy quotient lattices. For further study on double-famed soft sets, the readers refer to [4, 20, 22, 23, 32].

In this paper, we investigate the notions of DFS left (right, two-sided) ideals, DFS (generalized) bi-ideals, DFS interior ideals and DFS (1, 2)-ideals over an initial universe set U. We study the relationship between these DFS ideals in a (2,2)-regular class of an AG–Groupoid in detail. An application of our results we get characterizations of a (2,2)-regular AG–Groupoids in terms of DFS left (right, two-sided) ideals, DFS (generalized) bi-ideals, DFS interior ideals and DFS (1, 2)-ideals over U. Moreover, we apply DFS expert sets to develop a decision making scheme for everyday problems.

2. Preliminaries

2.1. **AG–Groupoids.** An AG–Groupoid is a non-associative and non-commutative algebraic structure lying in between a groupoid and a commutative semigroup. Commutative law is given by $abc = cba$ in ternary operations. By putting brackets on the left of this equation, i.e. $(ab)c = (cb)a$, in 1972, M. A. Kazim and M. Naseeruddin introduced a new algebraic structure called a left almost semigroup abbreviated as an LA-semigroup [17]. This identity is called the left invertive law. P. V. Protic and N. Stevanovic called the same structure an Abel-Grassmann's groupoid abbreviated as an AG–Groupoid [34].

This structure is closely related to a commutative semigroup because a commutative AG–Groupoid is a semigroup [30]. It was proved in [17] that an AG–Groupoid S is medial, that is, $(ab)(cd) = (ac)(bd)$ holds for all a,b,c,$d \in S$. An AG–Groupoid may or may not contain a left identity. The left identity of an AG–Groupoid permits the inverses of elements in the structure. If an AG–Groupoid contains a left identity, then this left identity is unique [30]. In an AG–Groupoid S with left identity, the paramedial law $(ab)(cd) = (dc)(ba)$ holds for all $a,b,c,d \in S$. By using medial law with left identity, we get $a(bc) = b(ac)$ for all $a,b,c \in S$. We should genuinely acknowledge that much of the ground work has been done by M. A. Kazim, M. Naseeruddin, Q. Mushtaq, M. S. Kamran, P. V. Protic, N. Stevanovic, M. Khan, W. A. Dudek and R. S. Gigon. One can be referred to [8, 9, 19, 30, 31, 34, 37] in this regard.

• A non-empty subset A of an AG–Groupoid S is sub-AG–Groupoid of S if $A^2 \subseteq A$;

•A non-empty subset A of an AG–Groupoid S is called a left (right) ideal of S if $SA \subseteq A$ $(AS \subseteq A)$;

• By a two-sided ideal or simply ideal, we mean a non-empty subset of an AG–Groupoid S which is both left and right ideal of S.

• By an interior ideal of S, we means a non-empty subset A of S such that $(SA)S \subseteq A$.

• By a bi-ideal of S, we means an sub-AG–Groupoid A of S such that $(AS)A \subseteq A$.

2.2. **Double framed soft sets.** In [38], Sezgin and Atagun introduce few new operations on soft set and defined soft sets in the following way:

Suppose a universal set is U, set of parameters is E , power set of U is $P(U)$ and $A \subseteq E$. Then a *soft set* K_A over U is a mapping described by:

$$K_A : E \to P(U) \text{ such that } K_A(u) = \emptyset, \text{ if } u \notin A.$$

Here K_A is known as *approximate mapping*. A soft set over U is denoted by the set of ordered pairs as:

$$K_A = \{(u,\ K_A(u)) :\ u \in E,\ K_A(u) \in P(U)\}.$$

Note that a soft set is a parameterized family of subsets of U. By $S(U)$ means set of all soft sets.

• Suppose K_A, $K_B \in S(U)$. Then K_A is a soft subset of K_B, represented by $K_A \mathrel{\widetilde{\subseteq}} K_B$ if $K_A(u) \subseteq K_B(u)\ \forall\ u \in S$. Two soft sets K_A, K_B are called equal, if $K_A \mathrel{\widetilde{\subseteq}} K_B$ and $K_B \mathrel{\widetilde{\subseteq}} K_A$ and is represented by $K_A \mathrel{\widetilde{=}} K_B$. The union of K_A and K_B is defined by $K_A \mathbin{\widetilde{\cup}} K_B = K_{A\cup B}$, where $K_{A\cup B}(u) = K_A(u) \cup K_B(u)$, $\forall\ u \in E$. The intersection of K_A and K_B is defined in similar way.

• Suppose S is an AG–Groupoid, and let K_A, $L_B \in S(U)$. The *soft product* [38] of K_A and L_B, represented by $K_A \mathbin{\widetilde{\circ}} L_B$, is defined as:

$$(K_A \mathbin{\widetilde{\circ}} L_B)(u) = \begin{cases} \bigcup\limits_{u=vw} \{K_A^-(v) \cap L_B(w)\} & \text{if } u = vw \text{ for } u, v \in S \\ \emptyset & \text{otherwise} \end{cases},$$

• A double-framed soft pair $\left\langle (K_A^+,\ K_A^-; A\right\rangle$ is called a *double-framed soft set* (briefly, $DFSS$) [16] of A over U denoted by K_A, where K_A^+ and K_A^- are mappings from A to $P(U)$. The set of all $DFSS$s of A over U is denoted by $DFS(U)$.

• Suppose $K_A = \left\langle (K_A^+,\ K_A^-); A\right\rangle$ and $L_A = \left\langle (L_A^+,\ L_A^-); A\right\rangle$ be two $DFSS$s of an AG–Groupoid S over U. The *soft uni-int product* [42], represented by $K_A \diamond L_A = \left\langle (K_A^+ \mathbin{\widetilde{\circ}} L_A^+,\ K_A^- \mathbin{\widetilde{\star}} L_A^-); A\right\rangle$ is defined as a $DFSS$ of S over U, in which $K_A^+ \mathbin{\widetilde{\circ}} L_A^+$ and $K_A^- \mathbin{\widetilde{\star}} L_A^-$ are mapping from S to $P(U)$, as:

$$K_A^+ \widetilde{\circ} L_A^+ : S \longrightarrow P(U), u \longmapsto \begin{cases} \bigcup\limits_{u=vw} \{K_A^+(v) \cap L_A^+(w)\} & \text{if } u = vw \text{ for } v, w \in S \\ \emptyset & \text{otherwise} \end{cases},$$

$$K_A^- \widetilde{\star} L_A^- : S \longrightarrow P(U), u \longmapsto \begin{cases} \bigcap_{u=vw} \{K_A^-(v) \cup L_A^-(w)\} & \text{if } u = vw \text{ for } v, w \in S \\ U & \text{otherwise} \end{cases}.$$

• Suppose $K_A = \langle (K_A^+, K_A^-); A \rangle$ and $L_A = \langle (L_A^+, L_A^-); A \rangle$ be two $DFSS$s over U. Then $\langle (K_A^+, K_A^-); A \rangle$ is known as a DFS *subset* [42] of $\langle (L_A^+, L_A^-); A \rangle$, denote by $\langle (K_A^+, K_A^-); A \rangle \sqsubseteq \langle (L_A^+, L_A^-); A \rangle$ if

(i) $A \subseteq B$,

(ii) $(\forall e \in A) \begin{pmatrix} K_A^+ \text{ and } L_A^+ \text{ are same approximations } (K_A^+(e) \subseteq L_A^+(e)) \\ K_A^- \text{ and } L_A^- \text{ are same approximations } (K_A^-(e) \supseteq L_A^-(e)) \end{pmatrix}$.

• For two $DFSS$s $K_A = \langle (K_A^+, K_A^-); A \rangle$ and $L_A = \langle (L_A^+, L_A^-); A \rangle$ over U are known as *equal*, represented by $\langle (K_A^+, K_A^-); A \rangle = \langle (L_A^+, L_A^-); A \rangle$, if $\langle (K_A^+, K_A^-); A \rangle \sqsubseteq \langle (L_A^+, L_A^-); A \rangle$ and $\langle (L_A^+, L_A^-); A \rangle \sqsubseteq \langle (K_A^+, K_A^-); A \rangle$.

• For two $DFSS$s $K_A = \langle (K_A^+, K_A^-); A \rangle$ and $L_A = \langle (L_A^+, L_A^-); A \rangle$ over U, the DFS int-uni set [42] of $\langle (K_A^+, K_A^-); A \rangle$ and $\langle (L_A^+, L_A^-); A \rangle$, is defined as a $DFSS$ $\langle (K_A^+ \cap L_A^+, K_A^- \cup L_A^-); A \rangle$, where $K_A^+ \cap L_A^+$ and $K_A^- \cup L_A^-$ are mappings as fallow:

$$\begin{aligned} K_A^+ \cap L_A^+ &: A \longrightarrow P(U),\ u \longmapsto K_A^+(u) \cap L_A^+(u); \\ K_A^- \cup L_A^- &: A \longrightarrow P(U),\ u \longmapsto K_A^-(u) \cup L_A^-(u). \end{aligned}$$

It is denoted by $\langle (K_A^+, K_A^-); A \rangle \sqcap \langle (L_A^+, L_A^-); A \rangle = \langle (K_A^+ \cap L_A^+, K_A^- \cup L_A^-); A \rangle$.

• A double framed soft set (briefly, $DFSS$) $K_A = \langle (K_A^+, K_A^-); A \rangle$ of S over U is known as:

(i) a DFS-sub-AG–Groupoid if it holds:
$K_A^+(uv) \supseteq K_A^+(u) \cap K_A^+(v)$ and $K_A^-(uv) \subseteq K_A^-(u) \cup K_A^-(v)$, $\forall\, u, v \in S$.

(ii) a DFS *left ideal,* (briefly, DFS-LI) if it holds:
$K_A^+(uv) \supseteq K_A^+(v)$ and $K_A^-(uv) \subseteq K_A^-(v)$, $\forall\, u, v \in S$.

(iii) a DFS *right ideal* (briefly, DFS-RI) if it holds:
$K_A^+(uv) \supseteq K_A^+(u)$ and $K_A^-(uv) \subseteq K_A^-(u)$, $\forall\, u, v \in S$.

(iv) a DFS *two-sided ideal* (briefly, DFS-$2SI$) of S over U, if it is both DFS-LI and DFS-RI of S over U.

(v) a DFS *interior ideal* (briefly, DFS-II) if it holds:
$K_A^+((uv)w) \supseteq K_A^+(v)$ and $K_A^-((uv)w) \subseteq K_A^-(v)$, $\forall\, u, v, w \in S$.

(vi) a DFS *generalized bi-ideal* (briefly, DFS-GBI) if it holds:
$K_A^+((uv)w) \supseteq K_A^+(u) \cap K_A^+(w)$ and $K_A^-((uv)w) \subseteq K_A^-(u) \cup K_A^-(w)$, $\forall\, u, v, w \in S$.

(vii) a DFS *bi-ideal* (briefly, DFS-BI) if it holds:
(a) $K_A^+(uv) \supseteq K_A^+(u) \cap K_A^+(v)$ and $K_A^-(uv) \subseteq K_A^-(u) \cup K_A^-(v)$;
(b) $K_A^+((uv)w) \supseteq K_A^+(u) \cap K_A^+(w)$ and $K_A^-((uv)w) \subseteq K_A^-(u) \cup K_A^-(w)$, $\forall\, u, v, w \in S$.

$viii)$ a DFS *(1,2)-ideal* (briefly, DFS-*(1,2)*-I) if it holds:
(a) $K_A^+(uv) \supseteq K_A^+(u) \cap K_A^+(v)$ and $K_A^-(uv) \subseteq K_A^-(u) \cup K_A^-(v)$;
(b) $K_A^+((ua)(vw)) \supseteq K_A^+(u) \cap K_A^+(v) \cap K_A^+(w)$ and $K_A^-((ua)(vw) \subseteq K_A^-(u) \cup K_A^-(v) \cup K_A^-(w)$, $\forall\, u, v, w \in S$.

(iu) a DFS *idempotent* if it holds: $K_A \diamond K_A = K_A$ i.e., $K_A^+ \widetilde{\circ} K_A^+ = K_A^+$ and $K_A^- \widetilde{*} K_A^- = K_A^-$.

(u) a DFS *semiprime* if it holds: $a \le a^2 \Longrightarrow K_A^+(a) \supseteq K_A^+(a^2)$ and $K_A^-(a) \subseteq K_A^-(a^2)$ $\forall\, a \in S$.

• Let $\emptyset \neq A \subseteq S$, where S is an AG–Groupoid. The *characteristic* DFS*-function* of A, represented by $\langle (C_A^+, C_A^-); A \rangle = C_A$ is described to be a $DFSS$, in which C_A^+ and C_A^- are soft functions over U, given as follows:

$$C_A^+ \;:\; S \longrightarrow P(V), w \longmapsto \begin{cases} U & \text{if } w \in A \\ \emptyset & \text{if } w \notin A, \end{cases}$$

$$C_A^- \;:\; S \longrightarrow P(V), w \longmapsto \begin{cases} \emptyset & \text{if } w \in A \\ U & \text{if } w \notin A. \end{cases}$$

Clearly the characteristic function of the whole set S, represented as $C_S = \langle (C_S^+, C_S^-); S \rangle$, is known as the *identity* DFS*-function*, where $C_S^+(w) = U$ and $C_S^-(w) = \emptyset, \forall\, w \in S$.

• Recall that an AG**–Groupoids is an AG-Groupoid in which $a(bc) = b(ac)$, $\forall$ $a, b, c \in S$.

• Note that an AG**–Groupoid also satisfies the paramedial law as well.

Now let us introduce the concept of an AG***–Groupoid as follows:

• An AG**-groupoid S is called an AG***–Groupoid if $S = S^2$.

Lemma 2.1. [42]*For* $DFSS$ $K_A = \langle (K_A^+, K_A^-); A \rangle$ *of* S *over* U, *the given statements are valid.*

(i) $K_A = \langle (K_A^+, K_A^-); A \rangle$ *is a* DFS*-*LI*-(*DFS*-*RI*) of* S *over* $U \Longleftrightarrow$ *if* $C_A \diamond K_A \sqsubseteq K_A$ $(K_A \diamond C_A \sqsubseteq K_A)$.

(ii) $K_A = \langle (K_A^+, K_A^-); A \rangle$ *is a* DFS*-sub-AG–Groupoid of* S *over* $U \Longleftrightarrow$ *if* $K_A \diamond K_A \sqsubseteq K_A$.

Lemma 2.2. [42]*For an AG–Groupoid* S *over* V, *the following conditions holds.*

(i) *Let* A *is an sub-AG–Groupoid of* $S \Longleftrightarrow C_A = \langle (C_A^+, C_A^-); A \rangle$ *is a* DFS*-sub-AG–Groupoid of* S *over* V.

(ii) A *be right* (*left, two-sided, bi-, interior*) *ideal of* $S \Longleftrightarrow C_A = \langle (C_A^+, C_A^-); A \rangle$ *is a* DFS*-*RI*-* (LI, $2SI$, BI, II) *of* S *over* V.

(iii) *For* $\emptyset \neq A, B \subseteq S$, *where* S *is an AG–Groupoid*, $C_A \diamond C_A = C_{AB}$ *and* $C_A \sqcap C_B = C_{A\cap B}$.

Throughout this paper, suppose $E = S$, where S is an AG–Groupoid, otherwise stated. By a unitary AG–Groupoid mean an AG–Groupoid having left identity.

3. DFS IDEALS IN (2,2)-REGULAR AG-GROUPOIDS

A very major conclusion from this section is that DFS (LIs, RIs, $2SI$s, GBIs, BIs, IIs, $(1,2)$-Is) need not to be coincide in an AG-groupoid (an AG***–Groupoid) S even if S has a unitary AG-groupoid (an AG***–Groupoid), but they will coincide in a (2,2)-regular class of an unitary AG-groupoid (an AG***–Groupoid) S.

Definition 3.1. An element a of an ordered AG–Groupoid S is called a (2, 2)-regular element of S, if there exists some x in S such that $a = (a^2x)a^2$.

Theorem 3.1. [40]*Let* S *be a unitary AG-groupoid (an AG***–Groupoid). An element* a *of* S *is (2, 2)-regular if and only if for all* $a \in S$, $a = (ay)(az)$ *for some* $y, z \in S$ *(*$a = (at)a$, $at = ta$ *for some* $t \in S$*).*

Theorem 3.2. *For a (2,2)-regular unitary AG–Groupoid (an AG***–Groupoid)* S,*the following conditions are equivalent* :

(i) $K_A = \langle (K_A^+,\ K_A^-); A\rangle$ *is a* DFS-$2SI$ *of* S over $U \Leftrightarrow C_A \diamond K_A = K_A = K_A \diamond C_A$.

(ii) $K_A = \langle (K_A^+,\ K_A^-); A\rangle$ is a DFS-BI (GBI) *of* S over $U \Leftrightarrow (K_A \diamond C_A) \diamond K_A = K_A = K_A \diamond K_A$

(iii) $K_A = \langle (K_A^+,\ K_A^-); A\rangle$ *is a* DFS-II *of* S over $U \Leftrightarrow (C_A \diamond K_A) \diamond C_A = K_A$

(iv) $K_A = \langle (K_A^+,\ K_A^-); A\rangle$ is a DFS-$(1,2)$-I *of* S over $U \Leftrightarrow (K_A \diamond C_A) \diamond (K_A \diamond K_A) = K_A = K_A \diamond K_A$.

Proof. Let S is (2,2)-regular unitary AG–Groupoid.

$(i) \Rightarrow$ Let $K_A = \langle (K_A^+,\ K_A^-); A\rangle$ be a DFS-$2SI$ of S over U. For $a \in S$, there exists $u \in S$ such that $a = (ax)(ay)$, we have

$$\begin{aligned}
(K_A^+ \tilde{\circ} C_A^+)(a) &= \bigcup_{a=(ax)(ay)} \{K_A^+(ua) \cap C_A^+(a)\} \\
&\supseteq K_A^+(ax) \cap C_A^+(ay) \supseteq K_A^+(ax) \cap U \\
&= K_A^+(ax) \supseteq K_A^+(a),
\end{aligned}$$

and

$$\begin{aligned}
(K_A^- \tilde{*} C_A^-)(a) &= \bigcap_{a=(ax)(ay)} \{K_A^-(ax) \cup C_A^-(a)\} \\
&\subseteq K_A^-(ax) \cup C_A^-(a) \subseteq K_A^-(a^2) \cup \emptyset \\
&= K_A^-(ax) \subseteq K_A^-(a),
\end{aligned}$$

therefore $C_A^+ \tilde{\circ} K_A^+ = K_A^+$ and $C_A^- \tilde{*} K_A^- = K_A^-$, that is $C_A \diamond K_A = K_A$.

$\Leftarrow$ The converse is obvious.

(ii). $\Rightarrow$ Let $K_A = \langle (K_A^+,\ K_A^-); A\rangle$ be a DFS-BI of S over U. For $a \in S$, there exists $x, y \in S$ such that $a = (ax)(ay)$, we have

$$\begin{aligned}
a &= (ax)(ay) = (ax)(ay) = (aa)(xy) = ((xy)a)\,a = ((xy)\,((ax)(ay)))\,a \\
&= ((xy)\,((aa)(xy)))\,a = ((aa)\,((xy)(xy)))\,a = \left((aa)\left(x^2y^2\right)\right)a = \left(\left(y^2x^2\right)(aa)\right)a \\
&= \left(\left(a\left(y^2x^2\right)\right)a\right)a = \left(\left(((ax)(ay))\left(y^2x^2\right)\right)a\right)a = \left(\left(((aa)(xy))\left(y^2x^2\right)\right)a\right)a \\
&= \left(\left(\left(\left(x^2y^2\right)(xy)\right)(aa)\,a\right)a\right)a = \left(\left(\left(\left(x^3y^3\right)(aa)\right)a\right)a\right)a = \left(\left(\left(a\left(\left(x^3y^3\right)a\right)\right)a\right)a\right)a
\end{aligned}$$

where $p = \left(\left(a\left(\left(x^3y^3\right)a\right)\right)a\right)a$ and $p = qa$ where $q = \left(a\left(\left(x^3y^3\right)a\right)\right)a$, therefore

$$\begin{aligned}
((K_A^+ \tilde{\circ} C_A^+) \tilde{\circ} K_A^+)(a) &= \bigcup_{a=pa} \left\{(K_A^+ \tilde{\circ} C_A^+)(p) \cap K_A^+(a)\right\} \\
&\supseteq (K_A^+ \tilde{\circ} C_A^+)(p) \cap K_A^+(a) \\
&= \left(\bigcup_{p=qa} \{K_A^+(q) \cap C_A^+(u)\}\right) \cap K_A^+(a) \\
&\supseteq K_A^+\left(\left(a\left(\left(x^3y^3\right)a\right)\right)a\right) \cap U \cap K_A^+(a) \\
&\supseteq K_A^+(a) \cap K_A^+(a) \cap K_A^+(a) = K_A^+(a)
\end{aligned}$$

and

$$\begin{aligned}
((K_A^- \,\tilde{*}\, C_A^-) \,\tilde{*}\, K_A^-)(a) &= \bigcup_{a=pa} \left\{ (K_A^- \,\tilde{*}\, C_A^-)(p) \cup K_A^-(a) \right\} \\
&\subseteq (K_A^- \,\tilde{*}\, C_A^-)(p) \cup K_A^-(a) \\
&= \left(\bigcup_{p=qa} \{ K_A^-(q) \cup C_A^-(u) \} \right) \cup K_A^-(a) \\
&\subseteq K_A^- \left(\left(a \left(\left(x^3 y^3 \right) a \right) \right) a \right) \cup \emptyset \cup K_A^-(a) \\
&\subseteq K_A^-(a) \cup K_A^-(a) \cup K_A^-(a) = K_A^-(a)
\end{aligned}$$

Again, we have

$$\begin{aligned}
a &= (ax)(ay) = (aa)(xy) = ((xy)a)a = ((xy)((ax)(ay)))a \\
&= ((xy)((aa)(xy)))a = ((aa)((xy)(xy)))a = \left((aa)\left(x^2 y^2 \right) \right) a \\
&= \left(\left(y^2 x^2 \right)(aa) \right) a = \left(a \left(\left(y^2 x^2 \right) a \right) \right) a = pa,
\end{aligned}$$

where $p = a\left(\left(y^2 x^2 \right) a \right)$ and $p = aq$, where $q = \left(y^2 x^2 \right) a$, therefore

$$\begin{aligned}
((K_A^+ \,\tilde{\circ}\, C_A^+) \,\tilde{\circ}\, K_A^+)(a) &= \bigcup_{a=pa} \left\{ (K_A^+ \,\tilde{\circ}\, C_A^+)(p) \cap K_A^+(a) \right\} \\
&= \bigcup_{a=pa} \left(\bigcup_{p=aq} \{ K_A^+(a) \cap C_A^+(q) \} \cap K_A^+(a) \right) \\
&= \bigcup_{a=pa} \left(\bigcup_{p=aq} K_A^+(a) \cap U \cap K_A^+(a) \right) \\
&= \bigcup_{a=pa} \left(\bigcup_{p=aq} K_A^+(a) \cap K_A^+(a) \right) \\
&= \bigcup_{a=pa} \left(K_A^+(a) \cap K_A^+(a) \right) \\
&\subseteq \bigcup_{a=pa} \left\{ K_A^+(\left(a \left(\left(y^2 x^2 \right) a \right) \right) a) \right\} = K_A^+(a).
\end{aligned}$$

and

$$\begin{aligned}((K_A^- \,\widetilde{*}\, C_A^-) \,\widetilde{*}\, K_A^-)(a) &= \bigcap_{a=pa} \left\{(K_A^- \,\widetilde{*}\, C_A^-)(p) \cup K_A^-(a)\right\} \\ &= \bigcap_{a=pa} \left(\bigcap_{p=aq} \{K_A^-(a) \cup C_A^-(q)\} \cup K_A^-(a)\right) \\ &= \bigcap_{a=pa} \left(\bigcap_{p=aq} K_A^-(a) \cup \emptyset \cup K_A^-(a)\right) \\ &= \bigcap_{a=pa} \left(\bigcap_{p=aq} K_A^-(a) \cup K_A^-(a)\right) \\ &= \bigcap_{a=pa} \left(K_A^-(a) \cup K_A^-(a)\right) \\ &\supseteq \bigcap_{a=pa} \left\{K_A^-(\left(a\left(\left(y^2x^2\right)a\right)\right)a)\right\} = K_A^-(a).\end{aligned}$$

Therefore $(K_A^+ \,\widetilde{\circ}\, C_A^+) \,\widetilde{\circ}\, K_A^+ = K_A^+$ and $(K_A^- \,\widetilde{*}\, C_A^-) \,\widetilde{*}\, K_A^- = K_A^-$. Thus $(K_A \diamond C_A^+) \diamond K_A = K_A$.

Now, we have

$$\begin{aligned} a &= (ax)(ay) = (ax)(ay) = (aa)(xy) = ((xy)a)\,a = ((xy)\,((ax)(ay)))\,a \\ &= ((xy)\,((aa)(xy)))\,a = ((aa)\,((xy)(xy)))\,a = \left((aa)\left(x^2y^2\right)\right)a \\ &= \left(\left(y^2x^2\right)(aa)\right)a = \left(\left(a\left(y^2x^2\right)\right)a\right)a \end{aligned}$$

where $p = \left(a\left(y^2x^2\right)\right)a$, therefore

$$\begin{aligned}(K_A^+ \,\widetilde{\circ}\, K_A^+)(a) &= \bigcup_{a=pa} \{K_A^+(p) \cap K_A^+(a)\} \\ &\supseteq K_A^+(\left(a\left(y^2x^2\right)\right)a) \cap K_A^+(a) \\ &\supseteq K_A^+(a) \cap K_A^+(a) \cap K_A^+(a) = K_A^+(a).,\end{aligned}$$

and

$$\begin{aligned}(K_A^- \,\widetilde{\circ}\, K_A^-)(a) &= \bigcup_{a=pa} \{K_A^-(p) \cap K_A^-(a)\} \\ &\supseteq K_A^-(\left(a\left(y^2x^2\right)\right)a) \cap K_A^-(a) \\ &\supseteq K_A^-(a) \cap K_A^-(a) \cap K_A^-(a) = K_A^-(a),\end{aligned}$$

so by Lemma 2.1, $K_A^+ \,\widetilde{\circ}\, K_A^+ = K_A^+$ and $K_A^- \,\widetilde{*}\, K_A^- = K_A^-$, Thus $K_A \diamond K_A = K_A$.

$\Longleftarrow$ Assume that $(K_A \diamond C_A) \diamond K_A = K_A = K_A \diamond K_A$. Since $K_A \diamond K_A = K_A$, so by Lemma 2.1, it follows that K_A^- is a DFS-sub-AG–Groupoid of S over U. Also

$$\begin{aligned}K_A^+((uv)w) &= ((K_A^+ \,\widetilde{\circ}\, C_A^+) \,\widetilde{\circ}\, K_A^+)((uv)w) = \bigcup_{(uv)w=(uv)w} \left\{(K_A^+ \,\widetilde{\circ}\, C_A^+)(uv) \cap K_A^+(w)\right\} \\ &= \bigcup_{(uv)w=(uv)w} \left(\left\{\bigcup_{uv=uv} K_A^+(u) \cap C_A^+(v)\right\} \cap K_A^+(w)\right) \\ &\supseteq K_A^+(u) \cap U \cap K_A^+(w) = K_A^+(u) \cap K_A^+(w),\end{aligned}$$

and

$$\begin{aligned} K_A^-((uv)w) &= ((K_A^- \,\widetilde{*}\, C_A^-) \,\widetilde{*}\, K_A^-)((uv)w) = \bigcap_{(uv)w=(uv)w} \left\{(K_A^- \,\widetilde{*}\, C_A^-)(uv) \cup K_A^-(w)\right\} \\ &= \bigcap_{(uv)w=(uv)w} \left(\left\{\bigcap_{uv=uv} K_A^-(u) \cup C_A^-(v)\right\} \cup K_A^-(w)\right) \\ &\subseteq K_A^-(u) \cup \emptyset \cup K_A^-(w) = K_A^-(u) \cup K_A^-(w). \end{aligned}$$

Thus by $K_A \diamond K_A = K_A$ and Lemma 2.1, it follows that K_A is a DFS-sub-AG–Groupoid of S over U. Hence K_A is a DFS-BI of S over U.

(iii). It is immediate.

(iv). $\Rightarrow$ Let $K_A = \left\langle (K_A^+,\ K_A^-); A\right\rangle$ be a DFS-$(1,2)$-I of S over U. Now for $a \in S$, there exists some $x, y \in S$ such that $a = (ax)(ay)$, we have

$$\begin{aligned} a &= (ax)(ay) = (ax)(ay) = (aa)(xy) = ((xy)a)\,a = ((xy)\,((ax)(ay)))\,a \\ &= ((xy)\,((aa)(xy)))\,a = ((aa)\,((xy)(xy)))\,a = \left((aa)\left(x^2y^2\right)\right)a = \left(a\left(y^2x^2\right)\right)(aa) = p(aa) \end{aligned}$$

where $p = a\left(y^2x^2\right) = aq$, where $q = y^2x^2$, therefore

$$\begin{aligned} ((K_A^+ \,\widetilde{\circ}\, C_A^+) \,\widetilde{\circ}\, (K_A^+ \,\widetilde{\circ}\, K_A^+))(a) &= \bigcup_{a=p(aa)} \left\{(K_A^+ \,\widetilde{\circ}\, C_A^+)(p) \cap (K_A^+ \,\widetilde{\circ}\, K_A^+)(aa)\right\} \\ &\supseteq (K_A^+ \,\widetilde{\circ}\, C_A^+)(p) \cap (K_A^+ \,\widetilde{\circ}\, K_A^+)(aa) \\ &= \bigcup_{p=aq} \left\{K_A^+(a) \cap C_A^+(q)\right\} \cap \bigcup_{aa=aa} \left\{K_A^+(a) \cap K_A^+(a)\right\} \\ &\supseteq K_A^+(a) \cap U \cap K_A^+(a) \cap K_A^+(a) = K_A^+(a) \end{aligned}$$

and

$$\begin{aligned} ((K_A^- \,\widetilde{*}\, C_A^-) \,\widetilde{*}\, (K_A^- \,\widetilde{*}\, K_A^-))(a) &= \bigcap_{a=p(aa)} \left\{(K_A^- \,\widetilde{*}\, C_A^-)(p) \cup (K_A^- \,\widetilde{*}\, K_A^-)(aa)\right\} \\ &\subseteq (K_A^- \,\widetilde{*}\, C_A^-)(p) \cup (K_A^- \,\widetilde{*}\, K_A^-)(aa) \\ &= \bigcap_{p=aq} \left\{K_A^-(a) \cup C_A^-(q)\right\} \cup \bigcap_{aa=aa} \left\{K_A^-(a) \cup K_A^-(a)\right\} \\ &\subseteq K_A^-(a) \cup \emptyset \cup K_A^-(a) \cup K_A^-(a) = K_A^-(a) \end{aligned}$$

Again, we have

$$\begin{aligned} a &= (ax)(ay) = (ax)(ay) = (aa)(xy) = ((xy)a)\,a = ((xy)\,((ax)(ay)))\,a \\ &= ((xy)\,((aa)(xy)))\,a = ((aa)\,((xy)(xy)))\,a = \left((aa)\left(x^2y^2\right)\right)a \\ &= \left(a\left(x^2y^2\right)\right)(aa) = p(aa) \end{aligned}$$

where $p = a\left(x^2y^2\right) = aq$, where $q = x^2y^2$, therefore

$$
\begin{aligned}
((K_A^+ \tilde{\circ}\, C_A^+) \tilde{\circ}\, (K_A^+ \tilde{\circ}\, K_A^+))(a) &= \bigcup_{a=p(aa)} \left\{(K_A^+ \tilde{\circ}\, C_A^+)(p) \cap (K_A^+ \tilde{\circ}\, K_A^+)(aa)\right\} \\
&= \bigcup_{a=p(aa)} \left\{\left(\bigcup_{p=aq} \{K_A^+(a) \cap C_A^+(q)\}\right) \cap \left(\bigcup_{aa=aa} \{K_A^+(a) \cap K_A^+)(a)\}\right)\right\} \\
&= \bigcup_{a=p(aa)} \left\{\left(\bigcup_{p=aq} \{K_A^+(a) \cap U\}\right) \cap \left(\bigcup_{aa=aa} \{K_A^+(a) \cap K_A^+)(a)\}\right)\right\} \\
&= \bigcup_{a=p(aa)} \left\{\left(\bigcup_{p=aq} K_A^+(a)\right) \cap \left(\bigcup_{aa=aa} K_A^+(a)\right)\right\} \\
&= \bigcup_{a=p(aa)} \{K_A^+(a) \cap K_A^+(a)\} = \bigcup_{a=p(aa)} K_A^+(a) \\
&\subseteq \bigcup_{a=p(aa)} \left\{K_A^+(\left(a\left(x^2y^2\right)\right)(aa))\right\} = K_A^+(a),
\end{aligned}
$$

and

$$
\begin{aligned}
((K_A^- \tilde{*}\, C_A^-) \tilde{*}\, (K_A^- \tilde{*}\, K_A^-))(a) &= \bigcap_{a=p(aa)} \left\{(K_A^- \tilde{*}\, C_A^-)(p) \cup (K_A^- \tilde{*}\, K_A^-)(aa)\right\} \\
&= \bigcap_{a=p(aa)} \left\{\left(\bigcap_{p=aq} \{K_A^-(a) \cup C_A^-(q)\}\right) \cup \left(\bigcap_{aa=aa} \{K_A^-(a) \cup K_A^-)(a)\}\right)\right\} \\
&= \bigcap_{a=p(aa)} \left\{\left(\bigcap_{p=aq} \{K_A^-(a) \cup \emptyset\}\right) \cup \left(\bigcap_{aa=aa} \{K_A^-(a) \cup K_A^-)(a)\}\right)\right\} \\
&= \bigcap_{(p,(aa))\in A_a} \left\{\left(\bigcap_{p=aq} K_A^-(a)\right) \cup \left(\bigcap_{aa=aa} K_A^-(a)\right)\right\} \\
&= \bigcap_{a=p(aa)} \{K_A^-(a) \cup K_A^-(a)\} \\
&\supseteq \bigcap_{a=p(aa)} \left\{K_A^-(\left(a\left(x^2y^2\right)\right)(aa))\right\} = K_A^-(a),
\end{aligned}
$$

which implies that $(K_A^+ \tilde{\circ}\, C_A^+) \tilde{\circ}\, (K_A^+ \tilde{\circ}\, K_A^+) = K_A^+$ and $(K_A^- \tilde{*}\, C_A^-) \tilde{*}\, (K_A^- \tilde{*}\, K_A^-) = K_A^-$. Thus $(K_A \diamond C_A^+) \diamond K_A = K_A = K_A \diamond K_A$. Now

$$
\begin{aligned}
a &= (ax)(ay) = a((ax)y) = a((yx)a) = a((yx)((ax)(ay))) = a((yx)((aa)(xy))) \\
&= a((aa)((yx)(xy))) = a((a((ax)(ay)))((yx)(xy))) = a((a((aa)(xy)))((yx)(xy))) \\
&= a(((aa)(a(xy)))((yx)(xy))) = a(((yx)(xy)(a(xy)))(aa)) \\
&= a(a((yx)(xy)((xy)))(aa)) = ap,
\end{aligned}
$$

where $p = a\left((yx)(xy)((xy))\right)(aa)$, therefore

$$\begin{aligned}(K_A^+ \,\tilde{\circ}\, K_A^+)(a) &= \bigcup_{a=ap}\left\{K_A^+(a) \cap K_A^+(p)\right\} \\ &\supseteq K_A^+(a) \cap K_A^+(p) \\ &= K_A^+(a) \cap K_A^+(a\left((yx)(xy)((xy))\right)(aa)) \supseteq K_A^+(a)\end{aligned}$$

and

$$\begin{aligned}(K_A^- \,\tilde{*}\, K_A^-)(a) &= \bigcap_{a=pa}\left\{K_A^-(a) \cup K_A^-(p)\right\} \subseteq K_A^-(a) \cup K_A^-(p) \\ &= K_A^-(a) \cup K_A^-(a\left((yx)(xy)((xy))\right)(aa)) \subseteq K_A^-(a).\end{aligned}$$

By Lemma 2.1, $K_A^+ \,\tilde{\circ}\, K_A^+ = K_A^+$ and $K_A^- \,\tilde{*}\, K_A^- = K_A^-$. Thus $K_A \diamond K_A = K_A$.

$\Leftarrow$ Assume that $(K_A \diamond C_A) \diamond (K_A \diamond K_A) = K_A = K_A \diamond K_A$. Since $K_A \diamond K_A = K_A$, so by Lemma 2.1, it follows that K_A^- is a DFS-sub-AG–Groupoid of S over U, we have

$$\begin{aligned}K_A^+((ua)(vw)) &= ((K_A^+ \,\tilde{\circ}\, C_A^+) \,\tilde{\circ}\, (K_A^+ \,\tilde{\circ}\, K_A^+))((ua)(vw)) \\ &= ((K_A^+ \,\tilde{\circ}\, C_A^+) \,\tilde{\circ}\, K_A^+)((ua)(vw)) \\ &= \bigcup_{(ua)(vw)=(ua)(vw)}\left\{(K_A^+ \,\tilde{\circ}\, C_A^+)(ua) \cap K_A^+(vw)\right\} \\ &\supseteq (K_A^+ \,\tilde{\circ}\, C_A^+)(ua) \cap K_A^+(vw) \\ &= \left(\bigcup_{(u,a)\in A_{ua}}\{K_A^+(u) \cap C_A^+(a)\}\right) \cap K_A^+(vw) \\ &\supseteq K_A^+(u) \cap U \cap K_A^+(v) \cap K_A^+(w) = K_A^+(u) \cap K_A^+(v) \cap K_A^+(w).\end{aligned}$$

and

$$\begin{aligned}K_A^-((ua)(vw)) &= ((K_A^- \,\tilde{*}\, C_A^-) \,\tilde{*}\, (K_A^- \,\tilde{*}\, K_A^-))((ua)(vw)) \\ &= ((K_A^- \,\tilde{*}\, C_A^-) \,\tilde{*}\, K_A^-)((ua)(vw)) \\ &= \bigcap_{(ua)(vw)=(ua)(vw)}\left\{(K_A^- \,\tilde{*}\, C_A^-)(ua) \cup K_A^-(vw)\right\} \\ &\subseteq (K_A^- \,\tilde{*}\, C_A^-)(ua) \cup K_A^-(vw) \\ &= \left(\bigcap_{(u,a)\in A_{ua}}\{K_A^-(u) \cup C_A^-(a)\}\right) \cup K_A^-(vw) \\ &\subseteq K_A^-(u) \cup \emptyset \cup K_A^-(v) \cup K_A^-(w) = K_A^-(u) \cup K_A^-(v) \cup K_A^-(w).\end{aligned}$$

Hence K_A is a DFS-$(1,2)$-I of S over U. □

Theorem 3.3. *In a (2,2)-regular unitary AG–Groupoid (an AG***–Groupoid) S over U, the DFS-LIs (RIs, 2SIs, GBIs, BIs, IIs, (1,2)-Is) are coincides.*

Proof. Let S is (2,2)-regular unitary AG–Groupoid.

For $a, b \in S$, there exists $u \in S$ such that $a = (ax)(ay)$. Let $K_A = \left\langle (K_A^+,\ K_A^-); A\right\rangle$ be a DFS-LI of S, we have

$$\begin{aligned}K_A^+(ab) &= K_A^+(((ax)(ay))b) = K_A^+(((aa)(xy))b) \\ &= K_A^+(((xy))b)(aa)) \supseteq K_A^+(aa) \supseteq K_A^+(a),\end{aligned}$$

and

$$\begin{aligned} K_A^-(ab) &= K_A^-(((ax)(ay))b) = K_A^-(((aa)(xy))b) \\ &= K_A^-(((xy))b)(aa)) \subseteq K_A^-(aa) \subseteq K_A^-(a). \end{aligned}$$

Hence K_A is a DFS-RI of S over U. Similarly, every DFS-RI of S is a DFS-LI of S over U.

Clearly a DFS-BI of S is a DFS-GBI of S. For $a, b \in S$, there exists $x, y \in S$ such that $a = (ax)(ay)$. Let $K_A = \langle (K_A^+,\ K_A^-); A \rangle$ be a DFS-GBI of S, we have

$$\begin{aligned} K_A^+(ab) &= K_A^+(((ax)(ay))b) = K_A^+(((aa)(xy))b) = K_A^+((((xy)a)a)b) \\ &= K_A^+((((xy)(ax)(ay))a)b) = K_A^+((((xy)(aa)(xy))a)b \\ &= K_A^+((((aa)((xy)(xy)))a)b = K_A^+\left(\left(\left((aa)\left(\left(x^2y^2\right)\right)\right)a\right)b\right. \\ &= K_A^+\left(\left(\left(\left(y^2x^2\right)(aa)\right)a\right)b = K_A^+\left(\left(\left(a\left(\left(y^2x^2\right)a\right)\right)a\right)b \supseteq K_A^+(a) \cap K_A^+(b),\right.\right. \end{aligned}$$

and

$$\begin{aligned} K_A^-(ab) &= K_A^-(((ax)(ay))b) = K_A^-(((aa)(xy))b) = K_A^-((((xy)a)a)b) \\ &= K_A^-((((xy)(ax)(ay))a)b) = K_A^-((((xy)(aa)(xy))a)b \\ &= K_A^-((((aa)((xy)(xy)))a)b = K_A^-\left(\left(\left((aa)\left(\left(x^2y^2\right)\right)\right)a\right)b\right. \\ &= K_A^-\left(\left(\left(\left(y^2x^2\right)(aa)\right)a\right)b = K_A^-\left(\left(\left(a\left(\left(y^2x^2\right)a\right)\right)a\right)b \supseteq K_A^-(a) \cap K_A^-(b),\right.\right. \end{aligned}$$

Hence K_A is a DFS-BI of S over U.

It is easy to see that a DFS-$2SI$ of S is a DFS-BI (GBI) of S over U. For $a, b \in S$ there exist $p, q, x, y \in S$ such that $a = (ap)(aq)$ and $b = (bx)(by)$. Let $K_A = \langle (K_A^+,\ K_A^-); A \rangle$ is a DFS-BI of S over U, we have

$$\begin{aligned} K_A^+(ab) &= K_A^+(a((bx)(by))) = K_A^+(a((bb)(xy))) = K_A^+((bb)(a(xy))) \\ &= K_A^+(((a(xy))b)b) = K_A^+(((a(xy))((bx)(by)))b) \\ &= K_A^+(((a(xy))((bb)(xy)))b) = K_A^+((bb)(((a(xy))(xy)))b) \\ &= K_A^+(((xy)(a(xy))(bb))b) = K_A^+((b(((xy)(a(xy))b))b) \\ &\supseteq K_A^+(b) \cap K_A^+(b) = K_A^+(b), \end{aligned}$$

and

$$\begin{aligned} K_A^-(ab) &= K_A^-(a((bx)(by))) = K_A^-(a((bb)(xy))) = K_A^-((bb)(a(xy))) \\ &= K_A^-(((a(xy))b)b) = K_A^-(((a(xy))((bx)(by)))b) \\ &= K_A^-(((a(xy))((bb)(xy)))b) = K_A^-((bb)(((a(xy))(xy)))b) \\ &= K_A^-(((xy)(a(xy))(bb))b) = K_A^-((b(((xy)(a(xy))b))b) \\ &\supseteq K_A^-(b) \cap K_A^-(b) = K_A^-(b), \end{aligned}$$

which shows that K_A is a DFS-LI of S over U. Similarly K_A is a DFS-RI of S over U. Hence K_A is a DFS-$2SI$ of S over U.

It is easy to see that a DFS-$2SI$ of S is a DFS-$(1,2)$-I of S over U. Now for $a, b \in S$, there exists some $x, y \in S$ such that $a = (ax)(ay)$. Let K_A be a DFS-$(1,2)$-I of S we have

$$\begin{aligned} K_A^+(ab) &= K_A^+\left(((ax)(ay))\,b\right) = K_A^+\left(((aa)(xy))\,b\right) = K_A^+\left(((((ax)(ay))\,a)(xy))\,b\right) \\ &= K_A^+\left(((((aa)(xy))\,a)(xy))\,b\right) = K_A^+\left(((xy)a)\,((aa)(xy))\,b\right) \\ &= K_A^+\left((aa)\,(((xy)a)\,(xy))\,b\right) = K_A^+\left(b\,(((xy)a)\,(xy))\,(aa)\right) \\ &= K_A^+\left(((xy)a)\,(b(xy))\,(aa)\right) = K_A^+\left(((xy)b)\,(a(xy))\,(aa)\right) \\ &= K_A^+\left(a\,(((xy)b)\,(xy))\,(aa)\right) \supseteq K_A^+(a) \cap K_A^+(a) \cap K_A^+(a) = K_A^+(a), \end{aligned}$$

and

$$\begin{aligned} K_A^-(ab) &= K_A^-\left(((ax)(ay))\,b\right) = K_A^-\left(((aa)(xy))\,b\right) = K_A^-\left(((((ax)(ay))\,a)(xy))\,b\right) \\ &= K_A^-\left(((((aa)(xy))\,a)(xy))\,b\right) = K_A^-\left(((xy)a)\,((aa)(xy))\,b\right) = K_A^-\left((aa)\,(((xy)a)\,(xy))\,b\right) \\ &= K_A^-\left(b\,(((xy)a)\,(xy))\,(aa)\right) = K_A^-\left(((xy)a)\,(b(xy))\,(aa)\right) \\ &= K_A^-\left(((xy)b)\,(a(xy))\,(aa)\right) = K_A^-\left(a\,(((xy)b)\,(xy))\,(aa)\right) \\ &\supseteq K_A^+(a) \cap K_A^+(a) \cap K_A^+(a) = K_A^+(a), \end{aligned}$$

Thus K_A is a DFS-RI of S over U. Similarly K_A is a DFS-LI of S over U. Hence K_A be a $DFS\ 2SI$ of S over U.

It is easy to see that a DFS-II of S is a DFS-$(1,2)$-I of S over U. Now let $a, u, v \in S$, then there exists $x, y \in S$ such that $a = (ax)(ay)$. Let K_A is a $DFS\ (1,2)$-I of S over U, we have

$$\begin{aligned} K_A^+((ua)v) &= K_A^+((ua)v) = K_A^+((u\,((ax)(ay)))v) = K_A^+((u\,((((ax)(ay))\,x)(ay)))v) \\ &= K_A^+((u\,((((aa)(xy))\,x)(ay)))v) = K_A^+((((aa)(xy))\,x)(u(ay))v) \\ &= K_A^+(((x(xy))\,(aa))(u(ay))v) = K_A^+((((ay)u)\,((aa)\,((x(xy))))\,v) \\ &= K_A^+((aa)\,(((ay)u)((x(xy)))\,v) = K_A^+((v\,(((ay)u)(x(xy))))\,(aa)) \\ &= K_A^+((((ay)u)\,(v(x(xy))))\,(aa)) = K_A^+((((yu)a)\,(v(x(xy))))\,(aa)) \\ &= K_A^+((((x(xy)v))\,(a(yu)))\,(aa)) = K_A^+(((a(((x(xy)v))\,(yu)))\,(aa)) \\ &\supseteq K_A^+(a) \cap K_A^+(a) \cap K_A^+(a) = K_A^+(a), \end{aligned}$$

similarly $K_A^-((ua)v) \subseteq K_A^-(a)$. Thus K_A is a DFS-II of S over U. Again let $a, u, v, w \in S$, then there exist p, q and $r, s \in S$ such that $u = (up)(uq)$ and $w = (wr)(ws)$. Let K_A be a DFS-II of S over U, we have

$$K_A^+((ua)(vw)) = K_A^+((wv)(au)) \supseteq K_A^+(v)$$

and

$$\begin{aligned} K_A^+((ua)(vw)) &= K_A^+((((up)(uq))\,(vw)) = K_A^+((((uu)(pq))\,(vw)) \\ &= K_A^+((((uu)v)\,((pq)\,(vw))) = K_A^+((((vu)u)\,((pq)\,(vw))) \supseteq K_A^+(u). \end{aligned}$$

and

$$\begin{aligned} K_A^-((ua)(vw)) &= K_A^-((ua)(v((wr)(ws))) = K_A^-((ua)(v((ww)(rs))) \\ &= K_A^-((ua)((ww)(v(rs))) = K_A^-((ww)((ua)(v(rs))) \subseteq K_A^-(w). \end{aligned}$$

Therefore, $K_A^+((ua)(vw)) \supseteq K_A^+(u) \cap K_A^+(v) \cap K_A^+(w)$, similarly $K_A^-((ua)(vw)) \subseteq K_A^-(u) \cup K_A^-(v) \cup K_A^-(w)$. If $a, b \in S$, then there exist $p., q$ and $r, s \in S$ such that $a = (ap)\,(aq)$ and $b = (br)(bs)$, we have

$$\begin{aligned} K_A^+(ab) &= K_A^+(((ap)(aq))b) = K_A^+(((aa)(pq))b) = K_A^+(((b(pq))\,(aa))) \\ &= K_A^+(((ba)\,((pq)a))) \supseteq K_A^+(a), \end{aligned}$$

and

$$K_A^+(ab) = K_A^+(a((br)(bs))) = K_A^+(a((bb)(rs))) = K_A^+((bb)(a(rs))) \supseteq K_A^+(b).$$

Thus, $K_A^+(ab) \supseteq K_A^+(a) \cap K_A^+(b)$, similarly $K_A^-(ab) \subseteq K_A^-(a) \cup K_A^-(b)$. Hence K_A is a DFS-$(1,2)$-I of S over U. □

Example 3.2. [10] Suppose there are twelve houses over a universal set U given by

$$U := \{s_1, s_2, s_3, s_4, s_5, s_6, s_7, s_8, s_9, s_{10}, s_{11}, s_{12}\}.$$

Let $S = \{e_1, e_2, e_3, e_4\}$ be a set of parameters, shows status of houses in which
e_0 stands for "in green surroundings",
e_1 stands for "beautiful",
e_2 stands for "in good location",
e_3 stands for "cheap,
with the following binary operation.

$*$	e_0	e_1	e_2	e_3	e_4
e_0	e_0	e_0	e_0	e_0	e_0
e_1	e_0	e_4	e_4	e_2	e_4
e_2	e_0	e_4	e_4	e_1	e_4
e_3	e_0	e_1	e_2	e_3	e_4
e_4	e_0	e_4	e_4	e_4	e_4

Clearly $(S, *)$ is a unitary AG-groupoid having left identity d. Note that S is not (2,2)-regular because for $e_2 \in S$ there do not exists $u \in S$ such that $e_2 = ue_2^2$.

If we define $DFSS$ $\langle (K_A^+, K_A^-); A \rangle$ of S over U as follows:

$$K_A^+(u) : S \longrightarrow P(U), u \mapsto \left\{ \begin{array}{c} U \text{ if } u = e_0 \\ \{s_2, s_3, s_4, s_5, s_6, s_7, s_8\} \text{ if } u = e_1 \\ \{s_2, s_3, s_4, s_5, s_6\} \text{ if } u = e_2 \\ \{s_2, s_3, s_4, s_5\} \text{ if } u = e_3 \\ \{s_1, s_2, s_3, s_4, s_5, s_6, s_7, s_8, s_9, s_{10}\} \text{ if } u = e_4 \end{array} \right\} \text{ and}$$

$$K_A^-(u) : S \longrightarrow P(U), u \mapsto \left\{ \begin{array}{c} \{s_2, s_3, s_4\} \text{ if } u = e_0 \\ \{s_2, s_3, s_4, s_6, s_7\} \text{ if } u = e_1 \\ \{s_1, s_2, s_3, s_4, s_5, s_6, s_7, s_8\} \text{ if } u = e_2 \\ U \text{ if } u = e_3 \\ \{s_2, s_3, s_4, s_6\} \text{ if } u = e_4 \end{array} \right\}.$$

Then it is simple to verify that $\langle (K_A^+,\ K_A^-); A \rangle$ *is a* DFS-LI *of* S *over* U, but $\langle (K_A^+,\ K_A^-); A \rangle$ is not a DFS-RI of S *over* U, because

$$K_A^+(e_1 e_3) \not\supseteq K_A^+(e_1) \text{ and } K_A^-(e_1 e_3) \subsetneq K_A^-(e_1)$$

It is simple to see that $\langle (K_A^+, K_A^-); A \rangle$ is a DFS-II of S over U but it is not a DFS-$2SI$ of S *over* U. On the other hand it is easy to see that every DFS-$2SI$ of S is a DFS-II (BI) of S over U.

4. DFS LEFT (RIGHT) IDEALS IN AG–GROUPOIDS

In this section, we characterize a (2,2)-regular AG–Groupoid by using the properties of DFS-LIs (RIs). We also provide few counter examples to discuss the converse part of given problem.

Lemma 4.1. *If $\langle (K_A^+,\ K_A^-); A\rangle$ is a $DFSS$ of S over U, then $\langle (K_A^+,\ K_A^-); A\rangle$ is a DFS semiprime $\Longleftrightarrow$ if $K_A^+(u) \supseteq K_A^+(u^2)$ and $K_A^-(u) \subseteq K_A^-(u^2)$, for all $u \in S$.*

Proof. It is immediate. □

Example 4.1. Let us define a $DFSS$ $\langle (K_A^+, K_A^-); A\rangle$ of $S = \{e_1,e_2,e_3,e_4\}$ over $U := \{h_1,h_2,h_3,h_4,h,h_6,h_7,h_8,h_9,h_{10}\}$ given in Example 3.2 as follows:

$$K_A^+(u) : S \longrightarrow P(U), u \mapsto \left\{ \begin{array}{c} \{h_1,h_2,h_3,h_4\} \text{ if } u = e_0 \\ \{h_1,h_2,h_3,h_{4,}h_{5,}h_{6,}h_7,h_8\} \text{ if } u = e_1 \\ U \text{ if } u = e_2 \\ \{h_2,h_3,h_{4,}h_5\} \text{ if } u = e_3 \\ \{h_2,h_3,h_4\} \text{ if } u = e_4 \end{array} \right\} \text{ and}$$

$$K_A^-(u) : S \longrightarrow P(U), u \mapsto \left\{ \begin{array}{c} U \text{ if } u = e_0 \\ \{h_1,h_2,h_3,h_{4,}h_{5,}h_{6,}h_7\} \text{ if } u = e_1 \\ \{h_1,h_2,h_3,h_{4,}h_{5,}h_{6,}h_{7,}h_8\} \text{ if } u = e_2 \\ \{h_{7,}h_8,h_9,h_{10}\} \text{ if } u = e_3 \\ \{h_1,h_2,h_3,h_{4,}h_{5,}h_{6,}h_{7,}h_8,h_9\} \text{ if } u = e_4 \end{array} \right\}$$

Then it is easy to verify that a $DFSS$ $\langle (K_A^+, K_A^-); A\rangle$ is a DFS semiprime.

Theorem 4.2. *A right (left, two-sided) ideal of S over U is semiprime $\Longleftrightarrow C_R = \langle (C_R^+,\ C_R^-); R\rangle$ is DFS semiprime.*

Proof. $\Longrightarrow$ Let A be a right (left, two-sided) ideal of S. By Lemma 2.2, $C_A = \langle (C_A^+,\ C_A^-); A\rangle$ is a DFS-RI (LI, $2SI$) of S over U. Now let $a \in S$, if $a^2 \in A$, since S semiprime, then $a \in A$.Hence $(C_A^+)(a) = U = (C_A^+)(a^2)$ and $(C_A^{--})(a) = \emptyset = (C_A^-)(a^2)$. If $a^2 \notin A$, then $(C_A^+)(a) \supseteq \emptyset = (C_A^+)(a^2)$ and $(C_A^-)(a) \subseteq U \subseteq (C_A^-)(a^2)$. Thus in both cases $C_A(a) \sqsupseteq C_A(a^2)$.Hence $C_A = \langle (C_A^+,\ C_A^-); A\rangle$ DFS semiprime.

$\Longleftarrow$ It is immediate. □

Corollary 4.3. *If any DFS-RI (LI, $2SI$) of S is DFS semiprime, then any right (left, two-sided) ideal of S is semiprime.*

Lemma 4.4. *For a (2,2)-regular unitary AG–Groupoid (an AG***–Groupoid) S over U, the following assertions hold.*

(i) All DFS-RIs *of* S are DFS semiprime.

(ii) All DFS-LIs *of* S are DFS semiprime in a unitary AG–Groupoid S over U.

Proof. (i) : It is immediate.

(ii) : If $K_A = \langle (K_A^+,\ K_A^-); A\rangle$ is an DFS-LI of S over U and $a \in S$, then there exists $x, y \in S$ such that $a = (ax)(ay)$, we have $K_A^+(a) = K_A^+((ax)(ay)) = K_A^+((aa)(xy)) = K_A^+((yx)(aa)) \supseteq K_A^+(a^2)$ and $K_A^-(a) = K_A^-((ax)(ay)) = K_A^-((aa)(xy)) = K_A^-((yx)(aa)) \supseteq K_A^-(a^2)$, which shows that K_A is DFS semiprime. □

Theorem 4.5. *For a unitary AG–Groupoid (an AG***–Groupoid) S, the following conditions are equivalent .*

(i) *S is (2,2)-regular.*
(ii) All DFS-RIs, (LIs, $2SIs$) *of* S are DFS semiprime.

Proof. $(i) \Longrightarrow (ii)$: It follows from Lemma 4.4.

$(ii) \Longrightarrow (i)$: Since a^2S [24] is a right and also a left ideal of S, so by Corollary 4.3, $C_{a^2S} = \left\langle (C^+_{a^2S},\ C^-_{a^2S}); a^2S \right\rangle$ is DFS semiprime. Now clearly $a^2 \in a^2S$, implies $a \in a^2S$, hence S is (2,2)-regular. □

Lemma 4.6. *Every DFS-RI of a unitary AG–Groupoid (an AG***–Groupoid) S becomes a DFS-LI of S.*

Proof. Let $K_A = \left\langle (K^+_A,\ K^-_A); A \right\rangle$ be a DFS-RI of S. Then for any $a, b \in S$, we have $K^+_A(ab) = K^+_A((ea)b) = K^+_A((ba)e) \supseteq K^+_A(b)$ and $K^-_A(ab) = K^-_A((ea)b) = K^-_A((ba)e) \subseteq K^-_A(b)$. hence K_A is a DFS-LI of S. □

Theorem 4.7. *The following conditions are equivalent for a unitary AG–Groupoid (an AG***–Groupoid) S.*

(i) *S is (2,2)-regular.*
(ii) Each DFS-RIs *of* S are DFS semiprime.
(iii) Each DFS-LIs *of* S are DFS semiprime.

Proof. $(i) \Longrightarrow (iii)$ and $(ii) \Longrightarrow (i)$ follows from Theorem 4.5.

$(iii) \Longrightarrow (ii)$: If $K_A = \left\langle (K^+_A,\ K^-_A); A \right\rangle$ is a DFS-RI of S over U, then by Lemma 4.6, K_A is a DFS-LI of S over U, therefore K_A is a DFS semiprime. □

Lemma 4.8. *For a unitary AG–Groupoid (an AG***–Groupoid) S, the following conditions are equivalent:*

(i) *S is (2,2)-regular.*
(ii) $K_A \diamond K_A = K_A$, *for each DFS-LI (RI, $2SI$) ideal of* S over U.

Proof. $(i) \Longrightarrow (ii)$: Let $K_A = \left\langle (K^+_A,\ K^-_A); A \right\rangle$ be a DFS-LI of S over U, then $K^+_A \,\tilde{\circ}\, K^+_A \subseteq K^+_A$ and.$K^-_A \,\tilde{*}\, K^-_A \supseteq K^-_A$ Let $a \in S$, then there exists $u \in S$ such that $a = (ax)(ay) = (ya)(xa)$, we have

$$\begin{aligned}(K^+_A \,\tilde{\circ}\, K^+_A)(a) &= \bigcup_{a=(ya)(xa)} \{K^+_A(ya) \cap K^+_A(xa)\} \supseteq K^+_A(xa) \cap K^+_A(ya) \\ &\supseteq K^+_A(a) \cap K^+_A(a) = K^+_A(a),\end{aligned}$$

and

$$\begin{aligned}(K^-_A \,\tilde{*}\, K^-_A)(a) &= \bigcap_{a=(ya)(xa)} \{K^-_A(ya) \cup K^-_A(xa)\} \subseteq K^-_A(ya) \cup K^-_A(xa) \\ &\subseteq K^-_A(a) \cup K^-_A(a) = K^-_A(a).\end{aligned}$$

Hence $K_A \diamond K_A = K_A$.

$(ii) \Longrightarrow (i)$: Let $K_A \diamond K_A = K_A$ holds for each DFS-LI of S over U. Since Sa [24] is left ideal of S, so by Lemma 2.2, $C_{Sa} = \left\langle (C^+_{Sa},\ C^-_{Sa}); Sa \right\rangle$ is a DFS-LI of S over U. Since $a \in Sa$, therefore $(C^+_{Sa})(a) = U$ and $(C^-_{Sa})(a) = \emptyset$. By Lemma 2.2 and hypothesis, we have $(C_{Sa}) \diamond (C_{Sa}) = C_{Sa}$ and $(C_{Sa}) \diamond (C_{Sa}) = C_{(Sa)(Sa)}$. Therefore $\left(C^+_{((Sa)(Sa))}\right)(a) = C^+_{[Sa]}(a) = U$ and $C^-_{((Sa)(Sa))}(a) = C^-_{[Sa]}(a) = \emptyset$, which implies that $a \in ((Sa)(Sa)) = (aS)(aS)$. Hence S is (2,2)-regular. □

Theorem 4.9. *For a unitary AG–Groupoid (an AG***–Groupoid) S, the following conditions are equivalent:*

(i) *S is (2,2)-regular.*
(ii) $K_A = (C_A^+ \diamond K_A) \diamond (C_A^+ \diamond K_A)$, *where* $K_A = \big\langle (K_A^+,\ K_A^-); A\big\rangle$ is an arbitrary DFS-LI $(RI, 2SI)$ *ideal of* S over U.

Proof. $(i) \Longrightarrow (ii)$: Let S is *(2,2)*-regular and $K_A = \big\langle (K_A^+,\ K_A^-); A\big\rangle$ is a DFS-LI of S over U. It is simple to see that $C_A \diamond K_A$ is also a DFS-LI of S over U. By Lemma 4.8, we obtain $(C_A^+ \mathbin{\widetilde{\circ}} K_A^+) \mathbin{\widetilde{\circ}} (C_A^+ \mathbin{\widetilde{\circ}} K_A^+) = (C_A^+ \mathbin{\widetilde{\circ}} K_A^+) \subseteq K_A^+$ and $(C_A^- \mathbin{\widetilde{*}} K_A^-) \mathbin{\widetilde{*}} (C_A^- \mathbin{\widetilde{*}} K_A^-) = (C_A^- \mathbin{\widetilde{*}} K_A^-) \supseteq K_A^-$. Now let $a \in S$, then there exists $u \in S$ such that $a = (ax)(ay) = (ya)(xa)$, we have

$$\begin{aligned}
\Big((C_A^+ \mathbin{\widetilde{\circ}} K_A^+) \mathbin{\widetilde{\circ}} (C_A^+ \mathbin{\widetilde{\circ}} K_A^+)\Big)(a) &= \bigcup_{a=(ya)(xa)} \{(C_A^+ \mathbin{\widetilde{\circ}} K_A^+)(ya) \cap (C_A^+ \mathbin{\widetilde{\circ}} K_A^+)(xa)\} \\
&\supseteq (C_A^+ \mathbin{\widetilde{\circ}} K_A^+)(ya) \cap (C_A^+ \mathbin{\widetilde{\circ}} K_A^+)(xa) \\
&= \left(\bigcup_{ya=ya} \{C_A^+(y) \cap K_A^+(a)\}\right) \cap \left(\bigcup_{xa=xa} \{C_A^+(x) \cap K_A^+(a)\right) \\
&\supseteq C_A^+(y) \cap K_A^+(a) \cap C_A^+(x) \cap K_A^+(a) \\
&= U \cap K_A^+(a) \cap U \cap K_A^+(a) = K_A^+(a),
\end{aligned}$$

and

$$\begin{aligned}
\Big((C_A^- \mathbin{\widetilde{*}} K_A^-) \mathbin{\widetilde{*}} (C_A^- \mathbin{\widetilde{*}} K_A^-)\Big)(a) &= \bigcap_{a=(ya)(xa)} \{(C_A^- \mathbin{\widetilde{*}} K_A^-)(ya) \cap (C_A^- \mathbin{\widetilde{*}} K_A^-)(xa)\} \\
&\subseteq (C_A^- \mathbin{\widetilde{*}} K_A^-)(ya) \cup (C_A^- \mathbin{\widetilde{*}} K_A^-)(xa) \\
&= \left(\bigcap_{ya=ya} \{C_A^-(y) \cup K_A^-(a)\}\right) \cup \left(\bigcap_{xa=xa} \{C_A^-(x) \cup K_A^-(a)\right) \\
&\subseteq C_A^-(y) \cup K_A^-(a) \cup C_A^-(x) \cup K_A^-(a) \\
&= \emptyset \cup K_A^-(a) \cup \emptyset \cup K_A^-(a) = K_A^-(a).
\end{aligned}$$

Hence $K_A = (C_A \diamond K_A) \diamond (C_A \diamond K_A)$.

$(ii) \Longrightarrow (i)$: Let $K_A = (C_A \diamond K_A) \diamond (C_A \diamond K_A)$ holds for all DFS-LI K_A of S over U. Then $K_A = (C_A \diamond K_A) \diamond (C_A \diamond K_A) \sqsubseteq K_A \diamond K_A \sqsubseteq C_A \diamond K_A \sqsubseteq K_A$. Thus by Lemma 4.8, S is (2,2)-regular. □

5. Duo and DFS duo AG–Groupoids

Definition 5.1. An AG–Groupoid (an AG***–Groupoid) S is called a left (right) duo if every left (right) ideal of S is an ideal of S and is called a duo if it is both left and right duo.

Lemma 5.1. *If each DFS-LI of a unitary AG–Groupoid (an AG***–Groupoid) S is a DFS-II of S over U, then S is a left duo.*

Proof. Let I is a left ideal of unitary AG–Groupoid S. By Lemma 2.2, $X_I = \big\langle (X_I^+,\ X_I^-); I\big\rangle$ is a DFS-LI of S over U. So by assumption, X_I is a DFS-II of S over U and by Lemma 2.2, I is an interior ideal of S. Now

$$IS = (eI)S \subseteq (SI)S \subseteq I.$$

Hence S is left duo. □

Corollary 5.2. *Every interior ideal of unitary AG–Groupoid (an AG***–Groupoid) S is right ideal of S.*

Theorem 5.3. *For a (2,2)-regular unitary AG-groupoid (an AG***–Groupoid) S, the following conditions are equivalent:*
(i) S is left duo.
(ii) Every DFS-LI of S is a DFS-II of S over U.

Proof. $(i) \Rightarrow (ii)$ Let a (2,2)-regular unitary AG–Groupoid S is left duo and $K_A = \langle (K_A^+,\ K_A^-); A \rangle$ is a DFS-LI of S over U. If $a, b, c \in S$, then $b = (bx)(by)$ for some $x, y \in S$. Since Sa is a left and also a right ideal of S [24], so Sa is an ideal of S, we have

$$\begin{aligned} (ab)c &= (a((bx)(by)))c = (a((bb)(xy)))c = ((bb)(a(xy)))c \\ &= (c(a(xy)))(bb) \in (S(a(SS)))(Sb) \subseteq (S(aS))b \\ &= ((eS)(aS))b = ((Sa)(Se))b \subseteq ((Sa)(SS))b \\ &\subseteq ((Sa)S)b \subseteq (Sa)b. \end{aligned}$$

Thus $(ab)c = (ta)b$, for some $t \in S$. Now $K_A^+((ab)c) = K_A^+((ta)b) \supseteq K_A^+(b)$ and $K_A^-((ab)c) = K_A^-((ta)b) \subseteq K_A^-(b)$. Hence $K_A = \langle (K_A^+,\ K_A^-); A \rangle$ is a $DFS\ II$ of S over U.

$(ii) \Rightarrow (i)$ is follows from Lemma 5.1. □

Definition 5.2. An AG–Groupoid (an AG***–Groupoid) S is called a DFS left duo (right duo) (briefly, DFS-L-duo (DFS-R-duo)) if every DFS-LI (RI) of S is a DFS-$2SI$ of S and is called a DFS-duo if it is both DFS-L-duo and DFS-R-duo.

Remark. *Every DFS-L-duo or DFS-R-duo is a DFS-duo in a (2,2)-regular unitary AG–Groupoid (an AG***-Groupoid).*

Lemma 5.4. *Every left ideal of a unitary AG–Groupoid (an AG***–Groupoid) S an interior ideal of S over U if S is a DFS-L-duo.*

Proof. It is immediate. □

Theorem 5.5. *For a (2,2)-regular unitary AG–Groupoid (an AG***–Groupoid) S, the following conditions are equivalent:*
(i) S is a DFS-L-duo.
(ii) Every left ideal of S over U is an interior ideal of S over U.

Proof. $(i) \Rightarrow (ii)$ cab be followed from Lemma 5.4.
$(ii) \Rightarrow (i)$ cab be followed from Theorem 3.3. □

6. Some Applications of Double-framed soft expert sets

In this section we utilize DFS expert sets to solve some real world problem, specifically stated, to decision making.

In real life, there are many cases where the properties of the universal set $\mathbb{U}$, called parameters are multi-values rather than a single value. For example, if we take a collection of books on the subject of , "Calculus & Analytic Geometry" available in the market. One of the parameters may be the content of the book. This parameter is two-value. Calculus content means single variable calculus and multi-variable calculus. Other parameters may be the geometric content of the book. This parameter is also two-value. Geometric material means 2-dimensional geometry and 3-dimensional geometry. In the case where the parameters involved are two-value, the $DFSS$s concept plays a useful role. This concept naturally extends to n-framed software sets when the parameters are n-valued.

Suppose we have a team of experts and their views are recorded in case of agreement or disagreement on a particular estimate. We will develop a decision algorithm based on DFS expert sets.

We will be making use of some preliminary concepts from Graph Theory which we present here for completion. A graph $L = (V, E)$ consists of two sets, V called the set of vertices and E called the set of edges. V represents some entities and E represents the relationships between the elements of V. The degree of a vertex is defined to be the number of edges connected to it or in real world, number of relationships it has with other elements. for a vertex $v \in V(L)$, we will denote the degree by $d_L(v)$. Reader is referred to any book on graph theory for further readings of this topic.

Let $\mathbb{U}$ be a universe, E a set of parameters, and X a set of experts (agents). Let $O = \{\text{disagree}= 0, \text{agree}= 1\}$ be a set of opinions, $Z = E \times X \times O$ and $A \subseteq Z$. A $DFSS$ $\langle(\Gamma, \Psi)\,; A\rangle$ is then known as a DFS expert-set over calU, where Γ and Ψ are mappings from A to $P(\text{calU})$ (power set of $\mathscr{U}$).

Example 6.1. Assume that $\mathscr{U} = \{t_1, t_2, t_3, t_4, t_5\}$ is a set of five newspapers under consideration and $A = \{m_1, m_2, m_3\} = \{\text{coverage, Calls, Data}\}$ is a set of parameters. The parameters involved here are two-valued. m_1 stands for coverage which includes urban and rural coverage, m_2 stands for Calss which includes low rate and call packages and m_3 stands for Data which includes low rates and data packages. Suppose a best performance award is to be announced by some agency. Let $Z = \{z_1, z_2, z_3\}$ be a set of experts. According to the data collected, the $DFSS$ $\langle(\Gamma_A, \Psi_A)\,; A\rangle$ can be viewed as the collection of the following approximations:

$$(\Gamma_A, Z) = \left\{\begin{array}{l} ((m_1, z_1, 1)\,, \{t_2, t_3, t_4\})\,, ((m_1, z_2, 1)\,, \{t_1, t_3, t_4\})\,, ((m_1, z_3, 1)\,, \{t_1, t_2, t_4\})\,, \\ ((m_2, z_1, 1)\,, \{t_1, t_2, t_3\})\,, ((m_2, z_2, 1)\,, \{t_3, t_4\})\,, ((m_2, z_3, 1)\,, \{t_2, t_4\})\,, \\ ((m_3, z_1, 1)\,, \{t_2, t_3\})\,, ((m_3, z_2, 1)\,, \{t_1, t_4\})\,, ((m_3, z_3, 1)\,, \{t_1, t_3\})\,, \\ ((m_1, z_1, 0)\,, \{t_4, t_5\})\,, ((m_1, z_2, 0)\,, \{t_3, t_5\})\,, ((m_1, z_3, 0)\,, \{t_3, t_4\})\,, \\ ((m_2, z_1, 0)\,, \{t_2, t_5\})\,, ((m_2, z_2, 0)\,, \{t_2, t_4\})\,, ((m_2, z_3, 0)\,, \{t_3, t_4\})\,, \\ ((m_3, z_1, 0)\,, \{t_2, t_3\})\,, ((m_3, z_2, 0)\,, \{t_1, t_5\})\,, ((m_3, z_3, 0)\,, \{t_1, t_4\}) \end{array}\right\}$$

$$(\Psi_A, Z) = \left\{\begin{array}{l} ((m_1, z_1, 1)\,, \{t_1\})\,, ((m_1, z_2, 1)\,, \{t_2\})\,, ((m_1, z_3, 1)\,, \{t_3\})\,, \\ ((m_2, z_1, 1)\,, \{t_4\})\,, ((m_2, z_2, 1)\,, \{t_5\})\,, ((m_2, z_3, 1)\,, \{t_1, t_2, t_5\})\,, \\ ((m_3, z_1, 1)\,, \{t_2, t_4, t_5\})\,, ((m_3, z_2, 1)\,, \{t_1, t_3, t_5\})\,, ((m_3, z_3, 1)\,, \{t_1, t_4, t_5\})\,, \\ ((m_1, z_1, 0)\,, \{t_2, t_3, t_4\})\,, ((m_1, z_2, 0)\,, \{t_2, t_4, t_5\})\,, ((m_1, z_3, 0)\,, \{t_3, t_4, t_5\})\,, \\ ((m_2, z_1, 0)\,, \{t_1, t_2, t_3, t_4\})\,, ((m_2, z_2, 0)\,, \{t_1, t_2, t_3, t_5\})\,, ((m_2, z_3, 0)\,, \{t_1, t_2, t_4, t_5\})\,, \\ ((m_3, z_1, 0)\,, \{t_1, t_3, t_4, t_5\})\,, ((m_3, z_2, 0)\,, \{t_2, t_3, t_4, t_5\})\,, ((m_3, z_3, 0)\,, U) \end{array}\right\}$$

Let us define the agree-soft expert graph $L(\Gamma_1(Z))$ henceforth denoted by L_1 as follows:

The vertex set $V(L_1) = (A \times Z) \cup \mathbb{U}$ and the edge set $E(L_1) = \{\{(m_i, z_j), t_k\}$ when $t_k \in \Gamma_A(m_i, z_j, 1)\}$.

The disagree-soft expert graph $L(\Gamma_0(Z))$ henceforth denoted by L_0 has the same vertex set and the edge set is defined as $E(L_0) = \{\{(m_i, z_j), t_k\}$ when $t_k \in \Gamma_A(m_i, z_j, 0)\}$.

These two graphs L_1 and L_0 are given in Fig. 1 and Fig. 2 respectively.

Using the same path of definitions for the graphs but using (Ψ_1, Z) and (Ψ_0, Z), we get two more graphs, the agree-soft expert graph $L(\Psi_1(Z))$ henceforth denoted by L_1' and The disagree-soft expert graph $L(\Psi_0(Z))$ henceforth denoted by L_0'. These graphs are given in Fig. 3 and Fig. 4 respectively.

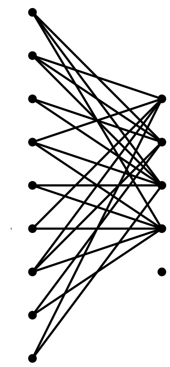

FIGURE 1. Agree-soft expert graph $G(\Gamma_1(Z))$

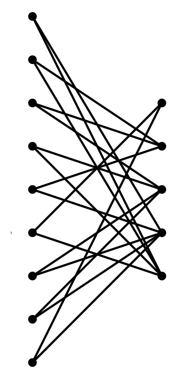

FIGURE 2. Disgree-soft expert graph $G(\Gamma_0(Z))$

These graphs enable us to see the above mentioned detailed information in a pictorial way and enable us to see the opinions in a clear and concise way. We also use these graphs to develop the following algorithm for choosing the best option based on expert opinions.

Algorithm

(1) Input the DFS expert-set $\langle (\Gamma_A, \Psi_A) ; Z \rangle$.
(2) Determine agree as well as disagree expert sets for (Γ, Z) and (Ψ, Z) .
(3) Determine the agree-soft expert graph $L(\Gamma_1(Z))$, disagree-soft expert graph $L(\Gamma_0(Z))$, agree-soft expert graph $L(\Psi_1(Z))$ and disagree-soft expert graph $L(\Psi_0(Z))$.
(4) Calculate $d_L(t_i)$ for agree-soft expert graph, similarly $d_{L_0}(t_i)$ for disagree-soft expert graph of (Γ, Z).
(5) Calculate $d'_L(t_i)$ for agree-soft expert graph, similarly $d'_{L_0}(t_i)$ for disagree-soft expert graph of (Ψ, Z).
(6) Calculte $a_i = d_L(t_i) - d_{L_0}(t_i)$ and $a'_i = d'_L(t_i) - d'_{L_0}(t_i)$.
(7) Calculate $c_i = \frac{a_i + a'_i}{2}$
(8) Choose j for which $c_j = \max c_i$.

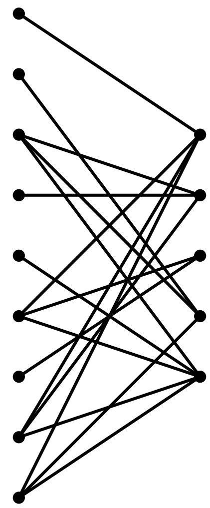

FIGURE 3. Agree-soft expert graph $G(\Gamma_1(Z))$

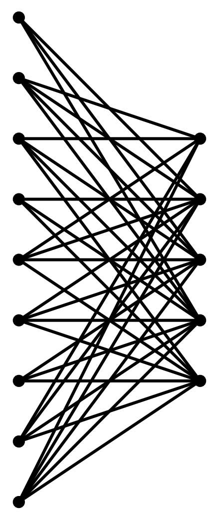

FIGURE 4. Disgree-soft expert graph $G(\Gamma_0(Z))$

Applying the steps 4-8 of above algorithm to Example No. 6.1, we get the following table.

	d_L	d_{L_0}	d'_L	d'_{L_0}	$a_i = d_L - d_{L_0}$	$a'_i = d'_L - d'_{L_0}$	$c_i = \frac{a_i + a'_i}{2}$
t_1	5	2	4	5	3	-1	1
t_2	5	3	3	7	2	-4	-1
t_3	6	4	2	7	2	-5	-1.5
t_4	6	5	3	8	1	-5	-2
t_5	0	4	5	7	-4	-2	-2

Thus the optimal choice is 1 or t_1.

Conclusion

We have considered the following problems in detail:

i) Study and compare DFS left (right, two-sided) ideals, DFS (generalized) bi-ideals, DFS interior ideals and DFS (1, 2)-ideals of AG–Groupoid over an initial universe set U.

ii) Discuss the structural properties of a (2,2) regular AG-groupoid in terms of DFS ideals.

iii) Compare a (2,2) regular class of an AG-groupoid with other important classes of an AG-groupoid, which will provide us a way to study DFS-sets in more generalized form in future.

iv) Apply DFS expert sets to develop a decision making scheme for everyday problems.

Some important issues for future work are:

i) To develop strategies for obtaining more valuable results in related areas.

ii) To apply these notions and results for studying DFS ideals in LA-semihypergroups and LA-semihyperrings.

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Availability of data and materials

No data were used to support this study.

Conflicts of Interest

The authors declare that there is no conflict of interest regarding the publication of this paper.

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