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EXISTENCE RESULTS FOR FUZZY DIFFERENTIAL EQUATION WITH ψ-HILFER FRACTIONAL DERIVATIVE

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ABSTRACT. This manuscript concerns the fuzzy differential equation involving ψ-Hilfer type fractional derivative with nonlocal condition. By using successive approximation, we obtain the existence, uniqueness results of solution for ψ-Hilfer fuzzy differential equation. Further, nonlocal conditions are extended to the existence results. Furthermore, an application is shown to demonstrate the theoretical conclusions utility.

1. INTRODUCTION

Consider the ψ-Hilfer fuzzy fractional differential equation of the kind

$$\begin{cases} D_{a+}^{\alpha,\beta,\psi}x(t) = f(t,x(t)), \quad \text{for all} \quad t \in [a,b], \\ I_{a+}^{1-\gamma,\psi}x(a) = x_0 = \sum_{i=1}^{m} C_i x(t_i), \quad \gamma = \alpha + \beta(1-\alpha), \end{cases} \tag{1.1}$$

where $x \in \mathbb{R}$, $0 < \alpha < 1$, $\beta \in [0,1]$, $f : [a,b] \times E \to E$ is a fuzzy function. Moreover, $I_{a+}^{1-\gamma,\psi}$, $D_{a+}^{\alpha,\beta,\psi}$ are the ψ-Hilfer fractional integral and derivative, which will be given in the next section. $t_i(i = 1,2,\dots,m)$ satifies $a < t_1 \leq t_2 \leq \dots < b$ and C_i is a real number, $x_0 \in \mathbb{R}$. Here nonlocal conditions are more effective than the initial conditions $I_{a+}^{1-\gamma}x(0) = x_0$ in terms of physical problems. x is said to be a solution of (1.1).

The fundamental concept of theory of differential equation is a rich and beautiful field of pure and applied mathematics which deals with many disciplines including engineering, physics, economics, biology. There are many branches of theory of differential equation is fuzzy fractional differential equations that in recent year. The theoretical development of fractional differential equation is the Riemann-Liouville's or Caputo sense have been excellently given in [1, 2, 3, 4, 5, 6], it has gathered significant not only in mathematical research but also in other applied sciences. In this way of fractional derivative concept that we should considered depends on the experimental data that best fits in the theoretical model. Hilfer has suggest a new generalized form of the fractional derivative, the so-called Hilfer fractional derivative(HFD) that merge the wide number of definition of fractional differential operators. For many definition on HFD and interesting applications, one can

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refer to [7, 8, 21]. Inspired by the definitions of the HFD and the concepts of fractional derivative of a function with respect to the another ψ kernal function suggest a new idea of fractional derivative,the so-called ψ-HFD.

Recently, the topics of existence and uniqueness for the solution to the linear and nonlinear fuzzy differential equations with ψ-HFD has been further investigated and discuss by many researches in various aspects. In [9] the existence and uniqueness of Riemann-Liouville fuzzy fractional differential equation has been demonstrate by Arshad and the concept of fuzzy type Riemann-Liouville differentiabilty based on Hukuhara differentiability in [15] by using the Hausdroff measure of noncompactness. Furthermore, the existence and uniqueness for fuzzy fractional differential equation with ψ-Hilfer under Liouville-Caputo generalized Hukuhara differentiability has been investigated in [10], and further see [11, 12, 13, 14, 19, 22]. In [16], the existence results for extremal solutions of interval fractional function integro-differential equation by using the monotone iteration approaches associated with the method of upper and lower solution was investigated.

This paper is organized as follows: In Section 2, we give some preliminary facts that we need in what follows. In section 3, we present our main results on the existence results of solution by using successive approximation method. An illustrative example is given to show the practical usefulness of the analytical results. Conclusion is given in section 4.

2. Preliminaries

Now in this section we give some definitions and lemmas useful in our subsequent discussion. We denote by E the sapce of all fuzzy numbers on $\mathbb{R}$. For $c \in \mathbb{R}$, $p \in [1, \infty]$, Let $X_c^p(a,b)$ denote the space of all complex-valued Lebesgue measurable functions f on a finite interval $[a,b]$ for which

$$\|f\|_{X_c^p} < \infty$$

with the norm

$$\|f\|_{X_c^p} = \left(\int_a^b \left|f(t)\right|^p \frac{dt}{t}\right)^{1/p} < \infty.$$

Definition 2.1. [20] A fuzzy number is a fuzzy set $x : \mathbb{R} \to [0,1]$ which satisfies the following conditions:

(i) x is normal, that is, there exists $t_0 \in \mathbb{R}$ such that $x(t_0) = 1$;

(ii) x is fuzzy convex in $\mathbb{R}$, that is, for $\lambda \in [0,1]$,

$$x\big(\lambda t_1 + (1-\lambda)t_2\big) \geq \min\big\{x(t_1), x(t_2)\big\}, \quad \text{for any } t_1, t_2 \in \mathbb{R};$$

(iii) x is a upper semicontinuous on $\mathbb{R}$;

(iv) $[x]^0 = \text{cl}\{z \in \mathbb{R} \mid x(z) > 0\}$ is compact.

Denote by $C([a,b], E)$ the set of all continuous fuzzy function and by $\text{AC}([a,b], E)$ the set of all absolutely continuous fuzzy functions on the intervals $[a,b]$ with values in E. Let $\gamma \in (0,1)$, by $C_{\gamma,\psi}[a,b]$ we denote the space of continuous functions defined by $C_{\gamma,\psi}[a,b] = \{f : (a,b] \to E : (\psi(t) - \psi(a))^{1-\gamma} f(t) \in C[a,b]\}$. Let $L([a,b], E)$ be the set of all fuzzy functions $x : [a,b] \to E$ such that the functions $t \mapsto D_0[x(t), \widehat{0}]$ belongs to $L^1[a,b]$.

If x is a fuzzy numbers on $\mathbb{R}$, we define $[x]^r = \{z \in \mathbb{R} \mid x(z) \geq r\}$ the r-level of x, with $r \in (0,1]$. From condition (i) and (iv), it follows that the r-level set of $x \in E$, $[x]^r$, is a nonempty compact interval for any $r \in [0,1]$. We denote by $[\underline{x}(r), \overline{x}(r)]$ the r-level of a fuzzy number x. For $x_1, x_2 \in E$, and $\lambda \in \mathbb{R}$, the sum $x_1 + x_2$ and the product $\lambda \cdot x_1$

are defined by $[x_1 + x_2]^r = [x_1]^r + [x_2]^r$, $[\lambda \cdot x_1]^r = \lambda[x_1]^r$, for all $r \in [0,1]$, where $[x_1]^r + [x_2]^r$ means the usual addition of two intervals of $\mathbb{R}$ and $\lambda[x_1]^r$ means the usual scalar product between λ and an real interval. For $x \in E$, we define the diameter of the r-level set of x as $\mathrm{diam}[u]^r = \overline{u}(r) - \underline{u}(r)$.

Definition 2.2. [19] Let $x_1, x_2 \in E$. If there exists $x_3 \in E$ such that $x_1 = x_2 + x_3$, then x_3 is called the Hukuhara difference of x_1 and x_2 and it is denoted by $x_1 \ominus x_2$. We note that $x_1 \ominus x_2 \neq x_1 + (-)x_2$.

Definition 2.3. [19] The distance $D_0[x_1, x_2]$ between two fuzzy numbers is defined as

$$D_0[x_1, x_2] = \sup_{r\in[0,1]} H\big([x_1]^r, [x_2]^r\big), \quad \text{for all } x_1, x_2 \in E,$$

where $H([x_1]^r, [x_2]^r) = \max\{|\underline{u}_1(r) - \underline{u}_1(r)|, |\overline{u}_1(r) - \overline{u}_1(r)|\}$ is a Hausdorff distance between $[x_1]^r$ and $[x_2]^r$.

Triangular fuzzy numbers are defined as a fuzzy set in E that is specifed by an ordered triple $x = (a, b, c) \in \mathbb{R}^3$ with $c \in [a, b]$ such that $[x]^r = [\underline{x}(r), \overline{x}(r)]$ are the end points of r-level sets for all $r \in [0, 1]$, where $\underline{x}(r) = a + (b - a)r$ and $\overline{x}(r) = c - (c - b)r$. In general, the parametric form of a fuzzy number x is a pair $[x]^r = [\underline{x}(r), \overline{x}(r)]$ of function $\underline{x}(r)$, $\overline{x}(r)$, $r \in [0, 1]$, which satisfy the following conditions: $\underline{u}(r)$ is a monotonically increasing left-continuous function, $\overline{u}(r)$ is a monotonically decreasing left-continuous function, and $\underline{u}(r) \leq \overline{u}(r)$, $r \in [0, 1]$.

Definition 2.4. [17] The generalized Hukuhara difference of two fuzzy numbers $x, y \in E$ (gH-difference for short) is defined as follows:

$$x \ominus_{gH} y = \omega \quad \Leftrightarrow \quad x = y + \omega, \quad \text{or} \quad y = x + (-1)\omega.$$

A function $x : [a, b] \to E$ is called d-increasing $(d - decreasing)$ on $[a, b]$ if for every $r \in [0, 1]$ the function $t \mapsto \mathrm{diam}[x(t)]^r$ is nondecreasing $(nonincreasing)$ on $[a, b]$. If x is a d-increasing or d- decreasing on $[a, b]$, then we say that x is d-monotone on $[a, b]$.

Definition 2.5. [7, 8] The left-sided ψ-fractional integral of order $\alpha > 0$, $x \in X^p_c(a, b)$ for -$\infty < a < t < \infty$ is defined by

$$\big(I^{\alpha,\psi}_{a+} x\big)(t) = \frac{1}{\Gamma(\alpha)} \int_a^t \psi^{'}(\tau)(\psi(t) - \psi(\tau))^{\alpha-1} x(\tau) d\tau. \tag{2.1}$$

Definition 2.6. [7, 8] The ψ-fractional derivative associated with the generalized fractional integrals (2) are defined, for $0 \leq a < t < \infty$, $n = [\alpha] + 1$, by

$$\begin{aligned}\big(D^{\alpha,\psi}_{a+} x\big)(t) =& \left(\frac{1}{\psi'(t)} \frac{d}{dt}\right)^n \big(I^{n-\alpha,\psi}_{a+} x\big)(t) \\ =& \frac{1}{\Gamma(n-\alpha)} \left(\frac{1}{\psi'(t)} \frac{d}{dt}\right)^n \int_a^t \psi^{'}(\tau)(\psi(t) - \psi(\tau))^{n-\alpha-1} x(\tau) d\tau. \end{aligned} \tag{2.2}$$

Let $x \in L([a, b], E)$, then the ψ-Hilfer fractional integral of order α of the fuzzy function x is defined as follows:

$$x_{\alpha,\psi}(t) = \big(I^{\alpha,\psi}_{a+} x\big)(t) = \frac{1}{\Gamma(\alpha)} \int_a^t \psi^{'}(\tau)(\psi(t) - \psi(\tau))^{\alpha-1} x(\tau) d\tau, t \geq a.$$

Since $[x(t)]^r = [\underline{x}(r, t), \overline{x}(r, t)]$ and $0 < \alpha < 1$, we can considered the fuzzy ψ-fractional integral of the fuzzy function x based on lower and upper functions, that is,

$$[\big(I^{\alpha,\psi}_{a+} x\big)(t)]^r = [\big(I^{\alpha,\psi}_{a+} \underline{x}\big)(r, t), \big(I^{\alpha,\psi}_{a+} \overline{x}\big)(r, t)], \quad t \geq a,$$

where

$\big(I_{a+}^{\alpha,\psi}\underline{x}\big)(r,t)=\frac{1}{\Gamma(\alpha)}\int_a^t\psi^{'}(\tau)(\psi(t)-\psi(\tau))^{\alpha-1}\underline{x}(r,\tau)(\tau)d\tau,$

and

$\big(I_{a+}^{\alpha,\psi}\overline{x}\big)(r,t)=\frac{1}{\Gamma(\alpha)}\int_a^t\psi^{'}(\tau)(\psi(t)-\psi(\tau))^{\alpha-1}\overline{x}(r,\tau)(\tau)d\tau.$

In addition, it follows that the opeartor $x_{\alpha,\psi}(t)$ is linear and bounded from $C([a,b],E)$ to $C([a,b],E)$. Indeed, we have

$$c\le\|x\|_0\frac{1}{\Gamma(\alpha)}\int_a^t\psi^{'}(\tau)(\psi(t)-\psi(\tau))^{\alpha-1}d\tau=\frac{\|x\|_0}{\Gamma(\alpha+1)}\big(\psi(t)-\psi(a)\big)^{\alpha},$$

where $\|z\|_0=\sup_{t\in[a,b]}D_0[z(t),\widehat{0}]$.

Definition 2.7. [7, 8] Let order α and type β satisfy $n-1<\alpha\le n$ and $0\le\beta\le1$, with $n\in N$. The fuzzy ψ-Hilfer generalized Hukuhara fractional derivative(or ψ-Hilfer gH-fractional derivative) (left-sided/right-sided), with respect to t, with a function $t\in C_{1-\gamma,\psi}[a,b]$, is defined by

$$\begin{aligned}\big(D_{a+}^{\alpha,\beta,\psi}x\big)(t)=&\big(I_{a+}^{\beta(1-\alpha),\psi}\big)\left(\frac{1}{\psi'(t)}\frac{d}{dt}\right)\big(I_{a^+}^{(1-\beta)(1-\alpha),\psi}x\big)(t)\\=&\big(I_{a+}^{\beta(1-\alpha),\psi}f^{\psi}I_{a+}^{(1-\beta)(1-\alpha),\psi}x\big)(t),\end{aligned}$$

if the gH-derivative $x^{'}_{(1-\alpha),\psi}(t)$ exists for $t\in[a,b]$, where

$$x_{(1-\alpha),\psi}(t):=\big(I_{a+}^{(1-\alpha),\psi}x\big)(t)=\frac{1}{\Gamma(1-\alpha)}\int_a^t\psi^{'}(\tau)[\psi(t)-\psi(\tau)]^{-\alpha}x(\tau)d\tau.$$

Lemma 2.1. [7, 8] *Let $I_{a+}^{\alpha,\psi}$ according to Eqs(2.1). Then*

$$I_{a+}^{\alpha,\psi}(\psi(t)-\psi(a))^{\beta-1}(t)=\frac{\Gamma(\beta)}{\Gamma(\alpha+\beta)}\big(\psi(t)-\psi(a)\big)^{\alpha+\beta-1},\quad\alpha\ge0,\beta>0.$$

Lemma 2.2. [7, 8] *Let $\alpha>0$, $0\le\gamma<1$. If $x\in C_{\gamma,\psi}[a,b]$ and $I_{a+}^{1-\alpha,\psi}x\in C^1_{\gamma,\psi}[a,b]$, then*

$$\big(I_{a+}^{\alpha,\psi}D_{a+}^{\alpha,\psi}x\big)(t)=x(t)-\frac{(I_{a+}^{1-\alpha,\psi}x)(a)}{\Gamma(\alpha)}\big(\psi(t)-\psi(a)\big)^{\alpha-1}.$$

Lemma 2.3. [7, 8] *Let $x\in L^1(a,b)$. If $D_{a+}^{\beta(1-\alpha),\psi}x$ exists on $L^1(a,b)$, then*

$D_{a+}^{\alpha,\beta,\psi}I_{a+}^{\alpha,\psi}x=I_{a+}^{\beta(1-\alpha),\psi}D_{a+}^{\beta(1-\alpha),\psi}x,$ *for all $t\in(a,b]$.*

Lemma 2.4. [20] *If $x\in AC([a,b],E)$ is a d-monotone fuzzy function, where $[x(t)]^r=[\underline{x}(r,t),\overline{x}(r,t)]$ for $0\le r\le1$, $a\le t\le b$, then for $0<a<1$, we have that*

(i) $[(D_{a+}^{\alpha,\beta,\psi}x)(t)]^r=[D_{a+}^{\alpha,\beta,\psi}\underline{x}(r,t),D_{a+}^{\alpha,\beta,\psi}\overline{x}(r,t)]$ *for $t\in[a,b]$, if x is d-increasing*

(ii) $[(D_{a+}^{\alpha,\beta,\psi}x)(t)]^r=[D_{a+}^{\alpha,\beta,\psi}\overline{x}(r,t),D_{a+}^{\alpha,\beta,\psi}\underline{x}(r,t)]$ *for $t\in[a,b]$, if x is d-decreasing*

Proof. Let $x \in AC([a,b],E)$ be a d-monotone fuzzy function, then $[x(t)]^r = [\underline{x}(r,t), \overline{x}(r,t)]$.
If x is d-monotone then either x is d-increasing or d-decreasing, for any $r \in [0,1]$
To prove(i):
Assume that x is d-increasing,

$$[x^{'}(t)] = [\tfrac{d}{dt}\underline{x}(r,t), \tfrac{d}{dt}\overline{x}(r,t)],$$

by definition of fuzzy ψ-Hilfer gH-fractional derivative

$$\begin{aligned}\big[\big(D^{\alpha,\beta,\psi}_{a+}x\big)(t)\big]^r =& \big[\big(I^{\beta(1-\alpha),\psi}_{a+} f^{1,\psi}_{\psi} I^{(1-\beta)(1-\alpha),\psi}_{a+}\underline{x}\big)(r,t), \big(I^{\beta(1-\alpha),\psi}_{a+} f^{1,\psi}_{\psi} I^{(1-\beta)(1-\alpha),\psi}_{a+}\overline{x}\big)(r,t)\big]\\ =& \big[\big(I^{\beta(1-\alpha),\psi}_{a+}\left(\frac{1}{\psi'(t)}\frac{d}{dt}\right)^{1,\psi} I^{(1-\beta)(1-\alpha),\psi}_{a+}\underline{x}\big)(r,t),\\ & \big(I^{\beta(1-\alpha),\psi}_{a+}\left(\frac{1}{\psi'(t)}\frac{d}{dt}\right)^{1,\psi} I^{(1-\beta)(1-\alpha),\psi}_{a+}\overline{x}\big)(r,t)\big]\\ =& \big[D^{\alpha,\beta,\psi}_{a+}\underline{x}(r,t), D^{\alpha,\beta,\psi}_{a+}\overline{x}(r,t)\big].\end{aligned}$$

To prove(ii):
Assume that x is d-decreasing,

$$[x^{'}(t)] = [\tfrac{d}{dt}\overline{x}(r,t), \tfrac{d}{dt}\underline{x}(r,t)],$$

by definition of fuzzy ψ-Hilfer gH-fractional derivative

$$\begin{aligned}\big[\big(D^{\alpha,\beta,\psi}_{a+}x\big)(t)\big]^r =& \big[\big(I^{\beta(1-\alpha)'\psi}_{a+} f^{1,\psi}_{\psi} I^{(1-\beta)(1-\alpha),\psi}_{a+}\overline{x}\big)(r,t), \big(I^{\beta(1-\alpha),\psi}_{a+} f^{1,\psi}_{\psi} I^{(1-\beta)(1-\alpha),\psi}_{a+}\underline{x}\big)(r,t)\big]\\ =& \big[\big(I^{\beta(1-\alpha)}_{a+}\left(\frac{1}{\psi'(t)}\frac{d}{dt}\right)^{1,\psi} I^{(1-\beta)(1-\alpha),\psi}_{a+}\overline{x}\big)(r,t),\\ & \big(I^{\beta(1-\alpha),\psi}_{a+}\left(\frac{1}{\psi'(t)}\frac{d}{dt}\right)^{1,\psi} I^{(1-\beta)(1-\alpha),\psi}_{a+}\underline{x}\big)(r,t)\big]\\ =& \big[D^{\alpha,\beta,\psi}_{a+}\overline{x}(r,t), D^{\alpha,\beta,\psi}_{a+}\underline{x}(r,t)\big].\end{aligned}$$

This completes the proof. □

Lemma 2.5. *If $x \in AC([a,b],E)$ is a d-monotone fuzzy function $t \in (a,b]$ and $\alpha \in (0,1)$, we set $z(t) := I^{\alpha,\psi}_{a+}$ and $z_{(1-\alpha),\psi}(t)$ is d-increasing on $(a,b]$ then*

$$\big(I^{\alpha,\psi}_{a+}D^{\alpha,\beta,\psi}_{a+}x\big)(t) = x(t) \ominus \frac{\sum_{i=1}^{m} C_i x(t_i)}{\Gamma(\gamma)}\big([\psi(t)-\psi(a)]\big)^{1-\gamma}$$

and

$$\big(D^{\alpha,\beta,\psi}_{a+}I^{\alpha,\psi}_{a+}x\big)(t) = x(t).$$

Proof. Let $x \in AC([a,b],E)$ be a d-monotone fuzzy function then by using ψ-HFD, we have,

$$\big(D^{\alpha,\beta,\psi}_{a+}x\big)(t) = \big(I^{\beta(1-\alpha),\psi}_{a+}\big)\left(\tfrac{1}{\psi'(t)}\tfrac{d}{dt}\right)\big(I^{(1-\beta)(1-\alpha),\psi}_{a+}\big)x(t).$$

By applying $I_{a+}^{\alpha,\psi}$ on the both sides, we get

$$\begin{aligned}\big(I_{a+}^{\alpha,\psi}D_{a+}^{\alpha,\beta,\psi}\big)x(t) =&\big(I_{a+}^{\alpha,\psi}I_{a+}^{\beta(1-\alpha),\psi}\big)\left(\frac{1}{\psi'(t)}\frac{d}{dt}\right)\big(I_{a+}^{(1-\beta)(1-\alpha),\psi}\big)x(t)\\ =&\big(I_{a+}^{\alpha+\beta(1-\alpha),\psi}\big)\left(\frac{1}{\psi'(t)}\frac{d}{dt}\right)\big(I_{a+}^{(1-\beta)(1-\alpha),\psi}\big)x(t)\\ =&\big(I_{a+}^{\gamma,\psi}\big)\left(\frac{1}{\psi'(t)}\frac{d}{dt}\right)\big(I_{a+}^{(1-\gamma),\psi}\big)x(t)\\ =&\big(I_{a+}^{\gamma,\psi}D_{a+}^{\gamma,\psi}\big)x(t),\end{aligned}$$

where $D_{a+}^{\gamma,\psi}x(t) = \left(\frac{1}{\psi'(t)}\frac{d}{dt}\right)I_{a+}^{1-\gamma,\psi}x(t)$,

and we get $\big(I_{a+}^{\alpha,\psi}D_{a+}^{\alpha,\psi}x\big)(t) = x(t) \ominus \frac{I_{a+}^{(1-\gamma),\psi}}{\Gamma(\alpha)}x(a)[\psi(t)-\psi(a)]^{\alpha-1}$

$\big(I_{a+}^{\alpha,\psi}D_{a+}^{\alpha,\beta,\psi}x\big)(t) = x(t) \ominus \frac{I_{a+}^{(1-\gamma),\psi}}{\Gamma(\alpha)}x(a)[\psi(t)-\psi(a)]^{\alpha-1}$.

Applying initial condition, we get

$$I_{a+}^{(1-\gamma),\psi}x(a) = x_0 = \sum_{i=1}^{m} C_i x(t_i). \tag{2.3}$$

That is, $\big(I_{a+}^{\alpha,\psi}D_{a+}^{\alpha,\beta,\psi}x\big)(t) = x(t) \ominus_{gH} \frac{\sum_{i=1}^{m} C_i x(t_i)}{\Gamma(\gamma)}[\psi(t)-\psi(a)]^{\gamma-1}$,
if $z(t)$ is d-increasing on $[a,b]$ or $z(t)$ is d- decreasing on $[a,b]$ and $z_{(1-\alpha),\psi}(t)$ is d-increasing on $(a,b]$.
In similar, $\big(I_{a+}^{\alpha,\psi}D_{a+}^{\alpha,\beta,\psi}x\big)(t) = x(t) + (-1)\frac{\sum_{i=1}^{m} C_i x(t_i)}{\Gamma(\gamma)}[\psi(t)-\psi(a)]^{\gamma-1}$.
Next we have, to prove that $D_{a+}^{\alpha,\beta,\psi}I_{a+}^{\alpha,\psi}x(t) = x(t)$.
Let $x \in L^1(a,b)$,

$$\begin{aligned}D_{a+}^{\alpha,\beta,\psi}I_{a+}^{\alpha,\psi}x(t) &= I_{a+}^{\beta(1-\alpha),\psi}DI_{a+}^{(1-\beta)(1-\alpha),\psi}x(t)\\ &= I_{a+}^{\beta(1-\alpha),\psi}DI_{a+}^{1-\beta(1-\alpha),\psi}x(t)\\ &= I_{a+}^{\beta(1-\alpha),\psi}D_{a+}^{\beta(1-\alpha),\psi}x(t)\end{aligned}$$

$$D_{a+}^{\alpha,\beta,\psi}I_{a+}^{\alpha,\psi}x(t) = x(t) \ominus \frac{I_{a+}^{1-\beta(1-\alpha),\psi}}{\Gamma(\beta(1-\alpha))}x(a)[\psi(t)-\psi(a)]^{\beta(1-\alpha)-1} = x(t).$$

On the other hand, since $x \in AC([a,b],E)$,there exists a constant K such that $K = \sup_{t\in[a,b]} D_0[x(t),\widehat{0}]$. Then

$$\begin{aligned}D_0[I_{a+}^{\alpha,\psi}x(t),\widehat{0}] &= \frac{1}{\Gamma(\alpha)}\int_a^t \psi^{'}(\tau)(\psi(t)-\psi(\tau))^{\alpha-1}x(\tau)ds\\ \sup_{t\in[a,b]} D_0[I_{a+}^{\alpha,\psi}x(t),\widehat{0}] &\le \frac{1}{\Gamma(\alpha)}\int_a^t \psi^{'}(\tau)(\psi(t)-\psi(\tau))^{\alpha-1}|x(\tau)|d\tau\\ D_0[I_{a+}^{\alpha,\psi}x(t),\widehat{0}] &\le \frac{K}{\Gamma(\alpha)}\int_a^t \psi^{'}(\tau)(\psi(t)-\psi(\tau))^{\alpha-1}d\tau\\ D_0[I_{a+}^{\alpha,\psi}x(t),\widehat{0}] &\le \frac{K}{\Gamma(\alpha+1)}(\psi(t)-\psi(a))^{\alpha},\end{aligned}$$

and $I_{a+}^{\alpha,\psi}x(t) = 0$ at $t = a$. This completes the proof. □

Lemma 2.6. *Let $\chi : [a,b] \to \mathbb{R}^+$ be a continuous function on the interval $[a,b]$ and satisfy $D_{a^+}^{\alpha,\beta,\psi}\chi(t) \leq g(t,\chi(t))$, $t \leq a$, where $g \in C([a,b] \times \mathbb{R}^+, \mathbb{R}^+)$. Assume that $m(t) = m(t,a,\xi_0)$ is the maximal solution of the initial value problem*

$$D_{a+}^{\alpha,\beta,\psi}\xi(t) = g(t,\xi), \quad \big(I_{a+}^{1-\gamma,\psi}\xi\big)(a) = \xi_0 \geq 0, \quad t \in [a,b]. \tag{2.4}$$

Then, if $\chi(a) \leq \xi_0$, we have $\chi(t) \leq m(t)$, $\quad t \in [a,b]$.

Lemma 2.7. *Consider the initial value problem as follows:*

$$D_{a+}^{\alpha,\beta,\psi} = g(t,\chi(t)), \quad \big(I_{a+}^{1-\gamma}\chi\big)(a) = \chi_0 = 0, \quad \textit{for all} \quad t \in [a,b] \tag{2.5}$$

Let $\eta > 0$ be a given constant and $B(\chi_0,\eta) = \{\chi \in \mathbb{R} : |\chi - \chi_0| \leq \eta\}$. Assume that the real-valued function $g : [a,b] \times [0,\eta] \to \mathbb{R}^+$ satisfies the following conditions:

- (i) *$g \in C([a,b] \times [0,\eta], \mathbb{R}^+)$, $g(t,0) = 0$, $0 \leq g(t,\chi) \leq M_g$, for all $(t,x) \in [a,b] \times [0,\eta]$;*
- (ii) *$g(t,\chi)$ is nondecreasing in χ for every $t \in [a,b]$. Then problem (6) has at least one solution defined on $[a,b]$ and $\chi(t) \in B(\chi_0,\eta)$.*

Proof. The problem (2.5) is equivalent to the following fractional integral equation:

$$\chi(t) = \chi_0 + \tfrac{1}{\Gamma(\alpha)}\int_a^t \psi'(s)(\psi(t)-\psi(s))^{\alpha-1}g(s,\chi(s))ds.$$

Given: $\chi(a) = \chi_0 = 0$

$$\chi(t) = 0 + \tfrac{1}{\Gamma(\alpha)}\int_a^t \psi'(s)(\psi(t)-\psi(s))^{\alpha-1}g(s,\chi(s))ds.$$

Choose $t^* > a$ such that $t^* \leq \left[\left(\frac{\eta\Gamma(1+\alpha)}{M_g}\right)^{1/\alpha} + a\right]$, and put $b^* = min\{t^*, b\}$. Let us define a sequence $\{x\}_{n=0}^{\infty}$ of successive approximation of problem (2.5) on [a,b] as follows:

$$\chi^0(t) = \frac{M_g}{\Gamma(\alpha+1)}(\psi(t)-\psi(a))^{\alpha}, \quad \chi^{n+1}(t) = \frac{1}{\Gamma(\alpha)} + \int_a^t \psi'(s)(\psi(t)-\psi(s))^{\alpha-1}g(s,\chi^n(s))ds$$

Then, for n=0, we have

$$\begin{aligned}
\chi^1(t) =& \frac{1}{\Gamma(\alpha)}\int_a^t \psi'(s)(\psi(t)-\psi(s))^{\alpha-1}g(s,\chi^0(s))ds\\
\chi^1(t) \leq& \frac{M_g}{\Gamma(\alpha+1)}(\psi(t)-\psi(a))^{\alpha}\\
\chi^1(t) \leq& \chi^0(t) \leq \eta, \quad t \in [a,b].
\end{aligned}$$

Hence $g(t,\eta)$ is nondecreasing in χ for every $t \in ([a,b^*])$ and proceeding recursively, we find that,

$$0 \leq \chi^{n+1}(t) \leq \chi^n(t) \leq ..... \leq \chi^0(t) \leq \eta, \quad n = 0,2,3,\ldots,$$

it follows that, the sequence $\{\chi^n\}_{n=0}^{\infty}$ is uniformly bounded for all $n \geq 0$. Moreover., $D_{a^+}^{\alpha,\beta,\psi}\chi(t) = g(t,\chi^n(t)) \leq M$, we get the equicontinuity of the sequence $\{\chi^n\}$. Indeed, for $a \leq t_1 \leq t_2 \leq b^*$ and by using Mean-Value Theorem, we have

$$|\chi^n(t_2) - \chi^n(t_1)| \leq \frac{2M_2}{\Gamma(\alpha+1)}(t_2-t_1)^{\alpha} \quad \leq \frac{2M_2}{\Gamma(\alpha+1)}(t_2-t_1)^{\alpha}\tau^{\alpha,\psi}, \quad \forall\tau \in [t_1,t_2] \subseteq [a,b^*].$$

Thus, if $|t_2 - t_1| \leq \delta$, we have $|\chi^n(t_2) - \chi^n(t_1)| \leq \epsilon$, where $\delta = \left(\frac{\epsilon}{2M_g}\Gamma(1+\alpha)\tau^\alpha\right)^{1/\alpha}$. Hence by using Arzela-Ascoli Theorem and the monotonicity of the sequence $\{\chi^n\}$. Therefore $\lim_{n\to\infty} \chi^n(t) = \chi(t)$ is uniformly on $[a, b^*]$. Thus, $\chi \in C([a, b^*], [0, \eta])$ and $\chi(t)$ is a solution of the problem (2.5).
This completes the proof. □

3. Main results

In this section, we discuss the existence and uniqueness of solution of problem (1.1) to initial value problem by using successive approximation method under generalized lipschitz condition of the right-hand side.

Lemma 3.1. *Let $\gamma = \alpha + \beta(1-\alpha)$, where $0 < \alpha < 1$, $0 \leq \beta \leq 1$, let $f : (a, b] \times E \to E$ be a fuzzy function such that $t \longmapsto f(t, x)$ belongs to $C_{\gamma,\psi}([a, b], E)$ for any $x \in E$. Then a d-monotone fuzzy function $x \in C([a, b], E)$ is a solution of problem (1.1) if and only if x satisfies the integral equation*

$$x(t) \ominus_{gH} \frac{\sum_{i=1}^m C_i x(t_i)}{\Gamma(\gamma)}\big(\psi(t) - \psi(a)\big)^{\gamma-1}$$
$$= \frac{1}{\Gamma(\alpha)} \int_a^t \psi'(\tau)(\psi(t) - \psi(\tau))^{\alpha-1} f(\tau, x(\tau))d\tau, \quad t \in [a, b] \tag{3.1}$$

and the fuzzy function $t \longmapsto I_{a+}^{1-\gamma,\psi} f(t, x)$ is d-increasing on $(a, b]$.

Proof. First, we have to prove the necessary condition.
Let $x \in C([a, b], E)$ be a d-monotone solution of problem (1.1), and
let $z(t) := x(t) \ominus_{gH} (I_{a+}^{1-\gamma,\psi} x(a))$, $t \in [a, b]$. Because x is d-monotone on $[a, b]$, it follows that $t \longmapsto z(t)$ is d-increasing on $[a, b]$. From (1.1) and Lemma 2.12 we have that

$$\big(I_{a+}^{\alpha,\psi} D_{a+}^{\alpha,\beta,\psi} x\big)(t) = x(t) \ominus_{gH} \frac{\sum_{i=1}^m C_i x(t_i)}{\Gamma(\gamma)}\big([\psi(t) - \psi(a)]\big)^{1-\gamma} \quad t \in [a, b]. \tag{3.2}$$

Since $f(t, x) \in C_{\gamma,\psi}([a, b], E)$ for any $x \in E$, and from (1.1), it follows that

$$\big(I_{a+}^{\alpha,\psi} D_{a+}^{\alpha,\beta,\psi} x\big)(t) = I_{a+}^{\alpha,\psi} f(t, x(t))$$
$$= \frac{1}{\Gamma(\alpha)} \int_a^t \psi'(\tau)(\psi(t) - \psi(\tau))^{\alpha-1} f(s, x(\tau))d\tau, \quad for \quad t \in [a, b]. \tag{3.3}$$

In addition, since $z(t)$ is d-increasing on $(a, b]$, it follows that $t \longmapsto f_{\alpha,\psi}(t, x)$ is also d-increasing on $(a, b]$. Consequently, combining (3.2) and (3.3) proves the necessity condition.
Next, we prove that the sufficiency. Let $x \in C([a, b], E)$ be a d-monotone fuzzy function x satisfies the integral equation and such that $t \longmapsto f_{\alpha,\psi}(t, x)$ is d-increasing on $(a, b]$. Because of the continuity of the fuzzy function f, the fuzzy function $t \longmapsto f_{\alpha,\psi}(t, x)$ is continuous on $(a, b]$ and $f_{\alpha,\psi}(a, x(a)) = \lim_{t\to a^+} f_{\alpha,\psi}(t, x) = 0$. Then

$$x(t) = \frac{\sum_{i=1}^m C_i x(t_i)}{\Gamma(\gamma)}\big(\psi(t) - \psi(a)\big)^{1-\gamma} + I_{a+}^{\alpha,\psi} f(t, x(t))(t),$$
$$I_{a+}^{1-\gamma,\psi} x(t) = \sum_{i=1}^m C_i x(t_i) + I_{a+}^{1-\beta(1-\alpha),\psi} f(t, x(t))(t),$$

and

$$I_{a+}^{1-\gamma,\psi}x(0)=\sum_{i=1}^{m}C_ix(t_i).$$

In addition, since $t\longmapsto f_{\alpha,\psi}(t,x)$ is d-increasing on $(a,b]$, by applying $D_{a+}^{\alpha,\beta}$ on both sides, we obtain that

$$\begin{aligned}
&D_{a+}^{\alpha,\beta,\psi}\left[x(t)\ominus_{gH}\frac{\sum_{i=1}^{m}C_ix(t_i)}{\Gamma(\gamma)}\big([\psi(t)-\psi(a)]\big)^{\gamma-1}\right]\\
&=D_{a+}^{\alpha,\beta,\psi}\left[\frac{1}{\Gamma(\alpha)}\int_a^t\psi^{'}(\tau)(\psi(t)-\psi(\tau))^{\alpha-1}f(\tau,x(\tau))d\tau\right]\\
&=D_{a+}^{\alpha,\beta,\psi}I_{a^+}^{\alpha,\psi}f(t,x(t)).
\end{aligned}$$

Thus,

$$\begin{aligned}
&D_{a+}^{\alpha,\beta,\psi}\left[I_{a+}^{\alpha,\psi}D_{a+}^{\alpha,\beta,\psi}x(t)\right]=D_{a+}^{\alpha,\beta,\psi}I_{a+}^{\alpha,\psi}f(t,x(t))\\
&D_{a+}^{\alpha,\beta,\psi}I_{a+}^{\alpha,\psi}D_{a+}^{\alpha,\beta,\psi}x(t)=D_{a+}^{\alpha,\beta,\psi}I_{a+}^{\alpha,\psi}f(t,x(t))\\
&D_{a+}^{\alpha,\beta,\psi}x(t)=f(t,x(t)).
\end{aligned}$$

This completes the proof. □

Theorem 3.2. *Let $f\in C([a,b]\times\mathbb{B}(x_0,h),E)$ and assume that the following conditions hold:*

(i) *There exists a positive constant M_f such that $D_0[f(t,z),\widehat{0}]\le M_f$, for all $(t,z)\in[a,b]\times\mathbb{B}(x_0,h)$;*

(ii) *For every $t\in[a,b]$ and every z, $\omega\in\mathbb{B}(x_0,h)$,*

$$D_0[f(t,z),f(t,\omega)]\le g(t,D_0[z,\omega]),$$

where $g(t,\cdot)\in C([a,b]\times[0,\psi],\mathbb{R}^+)$ satisfies the condition in Lemma 2.14 provided that the problem (2.5) has only the solution $\chi(t)=0$ on $[a,b]$. Then, the following successive approximations given by $x^0(t)=x_0$ and for $n=1,2,\dots,$

$$\begin{aligned}
&x^n(t)\ominus_{gH}\frac{\sum_{i=1}^{m}C_ix(t_i)}{\Gamma(\gamma)}\big(\psi(t)-\psi(a)\big)^{\gamma-1}\\
&=\frac{1}{\Gamma(\alpha)}\int_a^t\psi^{'}(\tau)(\psi(t)-\psi(\tau))^{\alpha-1}f(\tau,x^{n-1}(\tau))d\tau
\end{aligned}\tag{3.4}$$

converge uniformly to a unique solution of problem (1.1) on some intervals $[a,T]$ for some $T\in(a,b]$ provided that the function $t\longmapsto I_{a+}^{\alpha,\psi}f(t,x^n(t))$ is d-increasing on $[a,T]$.

Proof. Choose $t^*>a$ such that $t^*\le\left[\left(\frac{h\Gamma(1+\alpha)}{M}\right)^{1/\alpha}\right]$, where $M=max\{M_g,M_f\}$, and setting $T=min\{t^*,b\}$. Let $\mathbb{S}=\{x:\omega(a)=x_0$ and $\omega(t)\in\mathbb{B}(x_0,h)$, for all $t\in[a,T]\}$, clearly S is a set of continuous fuzzy functions x.

Next, we consider the sequence of continuous fuzzy function $\{x^n\}_{n=0}^{\infty}$ given by:
$x^0(t)=x_0$ for all $t\in[a,T]$, and for $n=1,2,\dots$

$$\begin{aligned}
&x^n(t)\ominus_{gH}\frac{\sum_{i=1}^{m}C_ix(t_i)}{\Gamma(\gamma)}\big(\psi(t)-\psi(a)\big)^{\gamma-1}\\
&=\frac{1}{\Gamma(\alpha)}\int_a^t\psi^{'}(\tau)(\psi(t)-\psi(\tau))^{\alpha-1}f(\tau,x^{n-1}(\tau))d\tau\quad[a,T].
\end{aligned}\tag{3.5}$$

Step 1: First of all, we prove that $x^n(t)\in C([a,T],B(x_0,h))$. For $n\ge1$ and for any $t_1,t_2\in[a,T]$ with $t_1<t_2$, we have

$$D_0\Big[x^n(t_1)\ominus_{gH}\frac{\sum_{i=1}^m C_i x(t_i)}{\Gamma(\gamma)}\big(\psi(t)-\psi(a)\big)^{\gamma-1}, x^n(t_2)\ominus_{gH}\frac{\sum_{i=1}^m C_i x(t_i)}{\Gamma(\gamma)}\big(\psi(t)-\psi(a)\big)^{\gamma-1}\Big]$$
$$\le \frac{1}{\Gamma(\alpha)}\int_a^{t_1}\psi^{'}(\tau)[(\psi(t_1)-\psi(\tau))^{\alpha-1}-(\psi(t_2)-\psi(\tau))^{\alpha-1}]D_0[f(\tau,x^n(\tau),\widehat{0}]d\tau$$
$$+\frac{1}{\Gamma(\alpha)}\int_{t_1}^{t_2}\psi^{'}(\tau)[(\psi(t_2)-\psi(\tau))^{\alpha-1}]D_0[f(\tau,x^n(\tau),\widehat{0}]d\tau.$$

The second integral on right-hand side of the last inequality has the value $\frac{1}{\Gamma(\alpha+1)}(\psi(t_2)-\psi(t_1))^\alpha$. For the first integral, it has the value $\frac{1}{\Gamma(\alpha+1)}[(\psi(t_1)-\psi(a))^\alpha-(\psi(t_2)-\psi(a))^\alpha]$. Hence, we get

$$D_0[x^n(t_1),x^n(t_2)]\le\frac{M_f}{\Gamma(\alpha+1)}[(\psi(t_2)-\psi(t_1))^\alpha+(\psi(t_2)-\psi(t_1))^\alpha-(\psi(t_2)-\psi(a))^\alpha]$$
$$\le\frac{2M_f}{\Gamma(\alpha+1)}(\psi(t_2)-\psi(t_1))^\alpha,$$

and it follows that the last expression converges to 0 as $t_1\to t_2$, which proves that x^n is a continuous function on $[a,T]$ for all $n\ge 0$. In addition, it follows that $x^n(t)\in\mathbb{B}(x_0,h)$ for all $t\in[a,T]$ if and only if

$$x^n(t)\ominus_{gH}\frac{\sum_{i=1}^m C_i x(t_i)}{\Gamma(\gamma)}\big(\psi(t)-\psi(a)\big)^{\gamma-1}\in\mathbb{B}(0,h), forall\quad t\in[a,T].$$

Indeed, if we suppose that $x^{n-1}(t)\in\mathbb{S},\quad$ for all $t\in[a,T]$ and for $n\ge 2$, then from

$$D_0\Big[x^n(t)\ominus_{gH}\frac{\sum_{i=1}^m C_i x(t_i)}{\Gamma(\gamma)}\big(\psi(t)-\psi(a)\big)^{\gamma-1},\widehat{0}\Big]$$
$$\le\frac{1}{\Gamma(\alpha)}\int_a^t\psi^{'}(\tau)(\psi(t)-\psi(\tau))^{\alpha-1}D_0[f(\tau,x^{n-1}(\tau)),\widehat{0}]d\tau$$
$$\le\frac{M_f}{\Gamma(\alpha+1)}[\psi(t)-\psi(a)]^\alpha\le h,$$

it follows that $x^n(t)\in S$, forall $t\in[a,T]$. Hence by mathematical induction, $x^n(t)\in S$ for all $t\in[a,T]$ and for $n\ge 1$. Next, we have to prove that the sequence $x^n(t)$ converges uniformly to a continuous function $x\in C([a,T],\mathbb{B}(x_0,h))$.

By assumption (ii) and mathematical induction, we have for $t\in[a,T]$

$$D_0\Big[x^{n+1}(t)\ominus_{gH}\frac{\sum_{i=1}^m C_i x(t_i)}{\Gamma(\gamma)}\big(\psi(t)-\psi(a)\big)^{\gamma-1}, x^n(t)\ominus_{gH}\frac{\sum_{i=1}^m C_i x(t_i)}{\Gamma(\gamma)}\big(\psi(t)-\psi(a)\big)^{\gamma-1}\Big]$$
$$\le\frac{1}{\Gamma(\alpha)}\int_a^t\psi^{'}(\tau)[\psi(t)-\psi(\tau)]^{\alpha-1}g(\tau,\psi^{n-1}(\tau))d\tau$$
$$\le\psi^n(t),\quad n=0,1,2,\ldots, \tag{3.6}$$

where $\psi^n(t)$ is defined as follows:

$$\psi^n(t)=\frac{1}{\Gamma(\alpha)}\int_a^t\psi^{'}(\tau)(\psi(t)-\psi(\tau))^{\alpha-1}g(\tau,\psi^{n-1}(\tau))d\tau$$

and $\psi^0(t)=\frac{M}{\Gamma(\alpha+1)}[\psi(t)-\psi(a)]^\alpha$. Thus, we have, for $t\in[a,T]$ and for $n=0,1,2,\ldots$,

$$\begin{aligned}D_0[D_{a+}^{\alpha,\beta,\psi}x^{n+1}(t), D_{a+}^{\alpha,\beta,\psi}x^{n}(t)] &\leq D_0[f(t,x^n(t)), f(t,x^{n-1}(t))]\\ &\leq g(t, D_0[x^n(t), x^{n-1}(t)])\\ &\leq g(t,\chi^{n-1}(t)).\end{aligned}$$

Let $m \geq n$ and $t \in [a,T]$, then we can obtain

$$\begin{aligned}D_{a+}^{\alpha,\beta,\psi}D_0[x^n(t), x^m(t)] &\leq D_0[D_{a+}^{\alpha,\beta,\psi}x^{n}(t), D_{a+}^{\alpha,\beta,\psi}x^{m}(t)]\\ &\leq D_0[D_{a+}^{\alpha,\beta,\psi}x^{n}(t), D_{a+}^{\alpha,\beta,\psi}x^{n+1}(t)]\\ &\quad + D_0[D_{a+}^{\alpha,\beta,\psi}x^{n+1}(t), D_{a+}^{\alpha,\beta,\psi}x^{m+1}(t)]\\ &\quad + D_0[D_{a+}^{\alpha,\beta,\psi}x^{m+1}(t), D_{a+}^{\alpha,\beta,\psi}x^{m}(t)]\\ &\leq g(t,\chi^{n-1}(t)) + g(t,\chi^{n-1}(t)) + g(t, D_0[x^n, x^m(t)])\\ &\leq 2g(t,\chi^{n-1}(t)) + g(t, D_0[x^n, x^m(t)]).\end{aligned}$$

From (ii), because we have that the solution $\chi(t) = 0$ is a unique solution of problem (2.5)

$$D_{a+}^{\alpha,\beta,\psi}\chi(t) = g(t,\chi(t)), \quad I_{a+}^{1-\gamma,\psi}\chi(a) = \chi_0 = 0 \quad \text{for all } t \in [a,b].$$

That is, $g(\cdot,\chi^{n-1}(.)) : [a,T] \to [0, M_g]$ uniformly converges to 0, for every $\epsilon > 0$, there exists a natural numbers n_0 such that

$$D_{a+}^{\alpha,\beta,\psi}D_0[x^n(t), x^m(t)] \leq g(t, D_0[x^n(t), x^m(t)]) + \epsilon, \text{ for } m \geq n \geq n_0.$$

Now, we consider $D_0[x^n(a), x^m(a)] = 0 < \epsilon$, it follows that, we have for $t \in [a,T]$,

$$D_0[x^n(t), x^m(t)] \leq \lambda_\epsilon(t), \quad m \geq n \geq n_0, \tag{3.7}$$

where $\lambda_\epsilon(t)$ is the maximal solution to the following problem

$$D_{a+}^{\alpha,\beta,\psi}\lambda_\epsilon(t) = g(t,\lambda_\epsilon(t)) + \epsilon, \quad (I_{a+}^{1-\gamma,\psi})\lambda_\epsilon(a) = \epsilon.$$

It follows that, $\{\chi_\epsilon(\cdot,\omega)\}$ converges uniformly to the maximal solution $\chi(t) = 0$ of problem (2.5) on $[a,T]$ as $\epsilon \to 0$.

From (3.7), we can find $n_0 \in N$ large enough such that, for $n, m > n_0$,

$$\sup_{t\in[a,T]} D_0\Bigg[x^n(t) \ominus_{gH} \frac{\sum_{i=1}^m C_i x(t_i)}{\Gamma(\gamma)}\big([\psi(t)-\psi(a)]\big)^{\gamma-1}, \\ x^m(t) \ominus_{gH} \frac{\sum_{i=1}^m C_i x(t_i)}{\Gamma(\gamma)}\big([\psi(t)-\psi(a)]\big)^{\gamma-1}\Bigg] \leq \epsilon. \tag{3.8}$$

Since (E, D_0) is a complete metric space and (3.8) holds, it follows that $\{x^n(t)\}$ converges uniformly to $x \in C([a,b], \mathbb{B}(x_0,h))$. Hence, we obtain

$$\begin{aligned}x(t) \ominus_{gH} \frac{\sum_{i=1}^m C_i x(t_i)}{\Gamma(\gamma)}\big([\psi(t)-\psi(a)]\big)^{\gamma-1}\big) &= \lim_{n\to\infty}\left(x^n(t) \ominus_{gH} \frac{\sum_{i=1}^m C_i x(t_i)}{\Gamma(\gamma)}\big([\psi(t)-\psi(a)]\big)^{\gamma-1}\right)\\ &= \frac{1}{\Gamma(\alpha)}\int_a^t \psi^{'}(\tau)[\psi(t)-\psi(\tau)]f(\tau,x(\tau))d\tau, \quad \text{for all} \tau \in [a,T]\end{aligned}$$

Due to Lemma 3.1 the function $x(t)$ is a solution to (1.1) on $[a,T]$.

Step 2: To show that the solution x is uniqueness, assume that $y : [a,T] \to E$ is another solution of problem (1.1) on $[a,T]$. Denote $k(t) = D_0[x(t), y(t)]$. Then $k(a) = 0$ and for every $t \in [a,T]$ we have

$$D_{a^+}^{\alpha,\beta,\psi} k(t) \leq D_0[f(t,x(t)), f(t,y(t))]$$
$$D_{a^+}^{\alpha,\beta,\psi} k(t) \leq g(t, D_0(x(t),y(t)))$$
$$D_{a^+}^{\alpha,\beta,\psi} k(t) \leq g(t,k(t))$$

It follows that, we obtain $k(t) \leq m(t)$, if $k(a) \leq \xi_0 \; \forall t \in [a,T]$, where m is a maximal solution of the problem

$$D_{a^+}^{\alpha,\beta,\psi} k(t) \leq g(t,k(t))$$
$$D_{a^+}^{\alpha,\beta,\psi} m(t) \leq g(t,m(t)),$$
$$I_{a^+}^{1-\gamma,\psi} k(a) = 0$$
$$I_{a^+}^{1-\gamma.\psi} m(a) = 0.$$

Clearly, $m(t) = 0$. Therefore $x(t) = y(t), \quad$ for all $t \in [a,T]$. Hence x is a solution of uniqueness.
This completes the proof. □

Corollary 3.3. *Let $f \in C([a,b],E)$. Assume that there exists positive constants L, M_f such that, for every $z,\omega \in E$,*

$$D_0\big[f(t,z), f(t,\omega)\big] \leq LD_0[z,\omega], \qquad D_0[f(t,z),\widehat{0}] \leq M_f.$$

Then the following successive approximations given by $x^0(t) = x_0$ and for $n = 1,2,3,\dots$

$$x^n(t) \ominus_{gH} \frac{\sum_{i=1}^m C_i x(t_i)}{\Gamma(\gamma)} \big(\psi(t)-\psi(a)\big)^{\gamma-1} = \frac{1}{\Gamma(\alpha)} \int_a^t \psi^{'}(\tau)(\psi(t)-\psi(\tau))^{\alpha-1} f(\tau, x^{n-1}(\tau)) d\tau$$

converge uniformly to a unique solution of problem (1.1) on some intervals $[a,T]$ for some $T \in (a,b]$ provided that the function $t \longmapsto I_{a^+}^{\alpha,\psi} f(t,x(t^n))$ is d-increasing on $[a,T]$.

Example 3.1. Let $\gamma = \alpha + \beta(1-\alpha)$, where $0 < \alpha < 1$, $0 \leq \beta \leq 1$, and $\lambda \in \mathbb{R}$. We consider the linear fuzzy fractional differential equation under ψ-HFD and assume that the following conditions hold:

$$\begin{cases} D_{a+}^{\alpha,\beta,\psi} x(t) = \lambda x(t) + p(t), & t \in (a,b] \\ I_{a+}^{1-\gamma,\psi} x(a) = x_0 = \sum_{i=1}^m C_i x(t_i), & \gamma = \alpha + \beta(1-\alpha). \end{cases} \tag{3.9}$$

Then x satiefies the integral equations

$$\begin{aligned} & x(t) \ominus_{gH} \frac{\sum_{i=1}^m C_i x(t_i)}{\Gamma(\gamma)} \big(\psi(t)-\psi(a)\big)^{\gamma-1} \\ =& \frac{\lambda}{\Gamma(\alpha)} \int_a^t \psi^{'}(\tau)(\psi(t)-\psi(\tau))^{\alpha-1} x(\tau) d\tau + \frac{1}{\Gamma(\alpha)} \int_a^t \psi^{'}(\tau)(\psi(t)-\psi(\tau))^{\alpha-1} p(\tau) d\tau \\ =& \lambda I_{a+}^{\alpha,\psi} x(t) + I_{a+}^{\alpha,\psi} p(t), \end{aligned}$$

where $p \in C([a,b],E)$ and we also assume that the right-hand side of the above integral equation of diameter is increasing . We see that $f(t,x) = \lambda x + p$ satisfies the assumption of Corollary 3.3. To find the explicit solution of (3.9), we apply the method of successive approximations. Setting $u^0(t) = u_0$ and

$$x^n(t) \ominus_{gH} \frac{\sum_{i=1}^m C_i x(t_i)}{\Gamma(\gamma)} \big(\psi(t)-\psi(a)\big)^{\gamma-1} = \lambda I_{a+}^{\alpha,\psi} x^{n-1}(t) + I_{a+}^{\alpha,\psi} p(t), \quad \text{for } n = 1, 2, 3, \ldots.$$

For $n = 1$ and $\lambda > 0$, if we assume that x is d-increasing, then it follows that

$$x^1(t) \ominus_{gH} \frac{\sum_{i=1}^m C_i x(t_i)}{\Gamma(\gamma)} \big(\psi(t)-\psi(a)\big)^{\gamma-1} = \sum_{i=1}^m C_i x(t_i) \frac{\lambda(\psi(t)-\psi(a))^{\alpha}}{\Gamma(\alpha+1)} + I_{a+}^{\alpha,\psi} p(t), \quad t \in [a,b].$$

On the other hand, if we assume that $\lambda < 0$ and x is d-decreasing, then it follows that

$$(-1)\bigg(\frac{\sum_{i=1}^m C_i x(t_i)}{\Gamma(\gamma)} \big(\psi(t)-\psi(a)\big)^{\gamma-1} \ominus_{gH} x^1(t)\bigg) = \sum_{i=1}^m C_i x(t_i) \frac{\lambda(\psi(t)-\psi(a))^{\alpha}}{\Gamma(\alpha+1)} + I_{a+}^{\alpha,\psi} p(t).$$

For $n = 2$, we also see that

$$x^2(t) \ominus_{gH} \frac{\sum_{i=1}^m C_i x(t_i)}{\Gamma(\gamma)} \big(\psi(t)-\psi(a)\big)^{\gamma-1} = \sum_{i=1}^m C_i x(t_i) \bigg[\frac{\lambda[\psi(t)-\psi(a)]^{\alpha}}{\Gamma(\alpha+1)} + \frac{\lambda^2[\psi(t)-\psi(a)]^{2\alpha}}{\Gamma(2\alpha+1)}\bigg] + I_{a+}^{\alpha,\psi} p(t) + I_{a+}^{2\alpha,\psi} p(t).$$

Suppose $\lambda < 0$ and x is d-decreasing such that

$$\begin{aligned}(-1)&\bigg(\frac{\sum_{i=1}^m C_i x(t_i)}{\Gamma(\gamma)} \big(\psi(t)-\psi(a)\big)^{\gamma-1} \ominus_{gH} x^2(t)\bigg) \\ &= \sum_{i=1}^m C_i x(t_i) \bigg(\frac{\lambda(\psi(t)-\psi(a))^{\alpha}}{\Gamma(\alpha+1)} + \frac{\lambda^2(\psi(t)-\psi(a))^{2\alpha}}{\Gamma(2\alpha+1)}\bigg) + I_{a+}^{\alpha,\psi} p(t) + I_{a+}^{2\alpha,\psi} p(t).\end{aligned}$$

If we proceed inductively and let $n \to \infty$, we obtain the

$$\begin{aligned}x(t) &\ominus_{gH} \frac{\sum_{i=1}^m C_i x(t_i)}{\Gamma(\gamma)} \big(\psi(t)-\psi(a)\big)^{\gamma-1} \\ &= \sum_{i=1}^m C_i x(t_i) \sum_{j=1}^{\infty} \frac{\lambda^j \big(\psi(t)-\psi(a)\big)^{j\alpha}}{\Gamma(j\alpha+1)} + \int_a^t \sum_{j=1}^{\infty} \frac{\lambda^{j-1}\psi^{'}(\tau)(\psi(t)-\psi(\tau))}{\Gamma(j\alpha)}^{j\alpha-1} p(\tau)d\tau \\ &= \sum_{i=1}^m C_i x(t_i) \sum_{j=1}^{\infty} \frac{\lambda^j \big(\psi(t)-\psi(a)\big)^{j\alpha}}{\Gamma(j\alpha+1)} + \int_a^t \sum_{j=0}^{\infty} \frac{\lambda^{j}\psi^{'}(\tau)(\psi(t)-\psi(\tau))}{\Gamma(j\alpha+\alpha)}^{j\alpha+(\alpha-1)} p(\tau)d\tau \\ &= \sum_{i=1}^m C_i x(t_i) \sum_{j=1}^{\infty} \frac{\lambda^j \big(\psi(t)-\psi(a)\big)^{j\alpha}}{\Gamma(j\alpha+1)} + \int_a^t \sum_{j=0}^{\infty} \frac{\lambda^{j}(\psi(t)-\psi(\tau))^{j\alpha}}{\Gamma(j\alpha+\alpha)} \psi^{'}(\tau)(\psi(t)-\psi(\tau))^{\alpha-1} p(\tau)d\tau.\end{aligned}$$

We see that, $\lambda > 0$ and x is d-increasing or $\lambda < 0$ and x is d-decreasing, respectively. Then, by applying definition of Mittag-Leffler function $E_{\alpha,\beta}(x) = \sum_{j=1}^{\infty} \frac{x^k}{\Gamma(j\alpha+\beta)}, \quad \alpha, \beta > 0,$ if $\lambda > 0$ and x is d-increasing then the solution of problem (3.9) is given by

$$
\begin{aligned}
&x(t) \ominus_{gH} \frac{\sum_{i=1}^{m} C_i x(t_i)}{\Gamma(\gamma)} \big(\psi(t)-\psi(a)\big)^{\gamma-1} \\
&= \sum_{i=1}^{m} C_i x(t_i) E_{\alpha,1}(\lambda(\psi(t)-\psi(a))^{\alpha}) \\
&\quad + \int_a^t \psi^{'}(\tau)(\psi(t)-\psi(\tau))^{\alpha-1} E_{\alpha,\alpha}(\lambda(\psi(t)-\psi(a))^{\alpha}) p(\tau) d\tau.
\end{aligned}
$$

On the other hand, if $\lambda < 0$ and x is d-decreasing, then we obtain the solution of problem (3.9) is given by

$$
\begin{aligned}
&x(t) \ominus_{gH} \frac{\sum_{i=1}^{m} C_i x(t_i)}{\Gamma(\gamma)} \big(\psi(t)-\psi(a)\big)^{\gamma-1} \\
&= \sum_{i=1}^{m} C_i x(t_i) E_{\alpha,1}(\lambda(\psi(t)-\psi(a))^{\alpha}) \\
&\quad \ominus (-1) \int_a^t \psi^{'}(\tau)(\psi(t)-\psi(\tau))^{\alpha-1} E_{\alpha,\alpha}(\lambda(\psi(t)-\psi(a))^{\alpha}) p(\tau) d\tau
\end{aligned}
$$

Remark. *In problem (3.9), suppose that $\lambda > 0$ and the solution of (3.9) is d-increasing. We observe that the solution of problem (3.9) admit particular cases as follows: if $\beta = 0$, then we obtain the solution of problem (3.9) with the ψ-HFD as follows:*

$$
\begin{aligned}
&x(t) \ominus_{gH} \frac{\sum_{i=1}^{m} C_i x(t_i)}{\Gamma(\alpha)} \big(\psi(t)-\psi(a)\big)^{\alpha-1} \\
&= \sum_{i=1}^{m} C_i x(t_i) E_{\alpha,1}(\lambda(\psi(t)-\psi(a))^{\alpha}) \\
&\quad + \int_a^t \psi^{'}(\tau)(\psi(t)-\psi(\tau))^{\alpha-1} E_{\alpha,\alpha}(\lambda(\psi(t)-\psi(a))^{\alpha}) p(\tau) d\tau.
\end{aligned}
$$

If the value of $\psi(x) = x$ and taking $\beta = 0$, then we obtain the solution of the problem (3.9) with the Caputo fractional derivative as follows:

$$
\begin{aligned}
&x(t) \ominus_{gH} \frac{\sum_{i=1}^{m} C_i x(t_i)}{\Gamma(\alpha)} \big(t-a\big)^{\alpha-1} \\
&= \sum_{i=1}^{m} C_i x(t_i) E_{\alpha,1}(\lambda(t-a)^{\alpha}) \\
&\quad + \int_a^t (t-\tau)^{\alpha-1} E_{\alpha,\alpha}(\lambda(t-a)^{\alpha}) p(\tau) d\tau.
\end{aligned}
$$

In addition, if the value of $\psi(x) = \log x$ and taking $\beta = 0$, then we obtain the following solution of problem (3.9) with the Caputo-Hadamard fractional derivative:

$$\begin{aligned} x(t) \ominus_{gH} \frac{\sum_{i=1}^{m} C_i x(t_i)}{\Gamma(\alpha)} \big(\big(\log \frac{t}{a}\big)\big)^{\alpha-1} &= \sum_{i=1}^{m} C_i x(t_i) E_{\alpha,1}(\lambda\big(\log \frac{t}{a}\big)^{\alpha}) \\ &+ \int_a^t \frac{1}{\tau}\big(\log \frac{t}{\tau}\big)^{\alpha-1} E_{\alpha,\alpha}(\lambda\big(\log \frac{t}{a}\big)^{\alpha}) p(\tau) d\tau. \end{aligned}$$

Remark. *Suppose that $\lambda < 0$ and the solution of (3.9) is d-decreasing. We observe that the solution of problem (3.9) admit the following cases: if $\beta = 0$ then the solution (3.9) with the ψ-type Caputo fractional derivative as follows:*

$$\begin{aligned} x(t) \ominus_{gH} \frac{\sum_{i=1}^{m} C_i x(t_i)}{\Gamma(\alpha)} \big(\psi(t) - \psi(a)\big)^{\alpha-1} &= \sum_{i=1}^{m} C_i x(t_i) E_{\alpha,1}(\lambda(\psi(t) - \psi(\tau))^{\alpha}) \\ &\ominus (-1) \int_a^t \psi^{'}(\tau)(\psi(t) - \psi(\tau))^{\alpha-1} E_{\alpha,\alpha}(\lambda(\psi(t) - \psi(\tau))^{\alpha}) p(\tau) d\tau. \end{aligned}$$

If the value of $\psi(x) = x$ and taking $\beta = 0$, then we obtain the following solution of problem (3.9) with the Riemann-Liouville fractional derivative as follows:

$$\begin{aligned} x(t) \ominus_{gH} \frac{\sum_{i=1}^{m} C_i x(t_i)}{\Gamma(\alpha)} \big((t-a)\big)^{\alpha-1} &= \sum_{i=1}^{m} C_i x(t_i) E_{\alpha,1}(\lambda(t-a)^{\alpha}) \\ &\ominus (-1) \int_a^t (t-\tau)^{\alpha-1} E_{\alpha,\alpha}(\lambda(t-\tau)^{\alpha}) p(\tau) d\tau. \end{aligned}$$

In addition, if the value of $\psi(x) = \log x$ and taking $\beta = 0$, then we obtain the following solution of problem (3.9) with Riemann-Hadamard fractional derivative as follows,

$$\begin{aligned} x(t) \ominus_{gH} \frac{\sum_{i=1}^{m} C_i x(t_i)}{\Gamma(\alpha)} \big(\big(\log \frac{t}{a}\big)\big)^{\alpha-1} &= \sum_{i=1}^{m} C_i x(t_i) E_{\alpha,1}(\lambda\big(\log \frac{t}{a}\big)^{\alpha}) \\ &\ominus (-1) \int_a^t \frac{1}{\tau}\big(\log \frac{t}{\tau}\big)^{\alpha-1} E_{\alpha,\alpha}(\lambda\big(\log \frac{t}{a}\big)^{\alpha}) p(\tau) d\tau. \end{aligned}$$

4. Concluding remarks

The existence and uniqueness of solutions for a fuzzy differential equations of ψ-Hilfer fractional derivative with nonlocal condition have obtained. Our investigation based on the successive approximation. The acquired results in this paper are more general and cover many of the parallel problems that contain special cases of function ψ, because our proposed system contains a global fractional derivative that integrates many classic fractional derivatives.

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AUTHORS CONTRIBUTIONS

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