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FUZZY CUT POINT SPACES

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ABSTRACT. By means of fuzzy connected topological space having no any fuzzy non-cut point, the concept of fuzzy cut-point space is introduced. We show that a fuzzy cut-point space is infinite. Moreover it is showed that a fuzzy cut-point space is non-compact. Finally we have studied fuzzy irreducible cut-point space.

1. INTRODUCTION

The idea of fuzzy set was initiated by Zadeh[4] in 1965. Chang[1] developed fuzzy topological spaces in 1968. A fuzzy set A is a function from X to $[0, 1]$, where $I = [0, 1]$. The family of all fuzzy sets on X is I^X consisiting all the mapping from X to I. I^X is called the fuzzy space which is infinite. The main aim of this paper is to extend and study fuzzy cut-point space and fuzzy irreducible space in terms of fuzzy connected topological spaces and which were already investigated by Honari and Bahrampour[2] in classical topology.

In this article, in section 2, first we define fuzzy cut-point spaces with example and we prove that every fuzzy cut-point space has an infinite number of fuzzy closed points. Further, it is showed that every fuzzy cut-point space is fuzzy non-compact. In section 3, we define fuzzy irreducible cut-point space as a fuzzy cut-point space whose proper fuzzy subsets are not fuzzy cut-point spaces.

For our convenience the phrases, fuzzy topological space and fuzzy connected topological space respectively abbreviated as FTS and FCTS throughout this paper.

2. FUZZY CUT-POINT SPACE AND ITS PROPERTIES

Definition 2.1. Let X be a nonzero FCTS. A fuzzy point $p_x^\alpha \in X$ is called fuzzy cut- point of X if $1_X \backslash \{p_x^\alpha\}$ is a fuzzy disconnected subset of X. A nonzero FCTS X is called a fuzzy cut- point space if every fuzzy point $p_x^\alpha \in X$ is a fuzzy cut-point of X.

Example 2.2. (Pao-Ming and Ying-Ming[3]) Let $X = \{y, z\}$ when $y \neq z$. Let δ be the fuzzy topology on X which has $\mathscr{B} = \left\{p_y^\lambda \,|\lambda \in (\frac{2}{3}, 1]\right\} \bigvee \left\{p_z^\lambda \,|\lambda \in (0, 1]\right\} \bigvee \{\phi\}$ as a base. It is observed that X is fuzzy connected but not a fuzzy cut-point FTS. Obviously, (X, δ) is a fuzzy T_2 space.

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Theorem 2.1. *Let X be a FCTS. Consider a fuzzy cut-point* $p_x^\alpha \in X$ *such that* $1_X \backslash \{p_x^\alpha\} = G|H$. *Then* $\{p_x^\alpha\}$ *is fuzzy open or fuzzy closed. If* $\{p_x^\alpha\}$ *is fuzzy open (resp.fuzzy closed), then G and H are fuzzy closed (resp.fuzzy open).*

Proof. As G is fuzzy clopen in $1_X \backslash \{p_x^\alpha\}$, we have a fuzzy open subset U of X such that $G = U \wedge (1_X \backslash \{p_x^\alpha\}) = U \backslash \{p_x^\alpha\}$ and we have a fuzzy closed subset E of X such that $G = E \wedge (1_X \backslash \{p_x^\alpha\}) = E \backslash \{p_x^\alpha\}$. As $G = U \backslash \{p_x^\alpha\} = E \backslash \{p_x^\alpha\}$, then $E = U$ contradicting the fuzzzy connectedness of X. We have either $\{p_x^\alpha\} = U \backslash E$ or $\{p_x^\alpha\} = E \backslash U$. If $\{p_x^\alpha\} = U \backslash E$, then $\{p_x^\alpha\}$ is fuzzy open and $G = E$ is fuzzy closed. If $\{p_x^\alpha\} = E \backslash U$, then $\{p_x^\alpha\}$ is fuzzy closed and $G = U$ is fuzzy open. □

Corollary 2.2. *Let X be a FCTS; let Y be the subset of all fuzzy cut-points of X. Then:*
(i) Each nonzero fuzzy connected subsets of Y having fuzzy non-singleton containing atleast one fuzzy closed point.
(ii) If $\{p_x^\alpha\} \in Y$ *is fuzzy open, then each fuzzy accumulation point of* $\{p_x^\alpha\} \in Y$ *is a fuzzy closed point.*

Lemma 2.3. *Let X be a FCTS and a fuzzy cut point* $p_x^\alpha \in X$. *If* $1_X \backslash \{p_x^\alpha\} = G|H$, *then* $G \vee \{p_x^\alpha\}$ *is fuzzy connected.*

Proof. Suppose $G \vee \{p_x^\alpha\}$ is not fuzzy connected, then $G \vee \{p_x^\alpha\} = K|M$ for any fuzzy open subsets K and M of X. Assume that $p_x^\alpha \in K$. Then $M \leq G$. Since $(Cl(H \vee K)) \wedge M = (Cl(H) \wedge M) \vee (Cl(K) \wedge M) = Cl(H) \wedge M \leq Cl(H) \wedge G = 0_X$, $Cl(H \vee K) \wedge M = 0_X$. Since, $(H \vee K) \wedge Cl(M) = (H \wedge Cl(M) \vee (K \wedge Cl(M)) = H \wedge Cl(M) \leq H \wedge Cl(G) = 0_X$, $(H \vee K) \wedge Cl(M) = 0_X$. Therefore $X = (H \vee K)|M$ which is contrary to fuzzy connectedness of X. □

Lemma 2.4. *Let a fuzzy cut point* $p_x^\alpha \in X$, *where X is a FCTS. If* $1_X \backslash \{p_x^\alpha\} = G|H$ *and if every fuzzy point of G is a fuzzy cut point of X, then G contains atleast one fuzzy closed point.*

Proof. Assume that G contains only of fuzzy open points. As by lemma 2.3, $G \vee \{p_x^\alpha\}$ is fuzzy connected , $\{p_x^\alpha\}$ is fuzzy closed and also by theorem 2.1 $G \vee \{p_x^\alpha\}$ is also fuzzy closed. So for each $p_y^\beta \in G$, $\{\overline{p_y^\beta}\} \leq G \vee \{p_x^\alpha\}$, by corollary 2.2(ii), p_x^α is possible fuzzy accumulation point of $\{p_y^\beta\}$. Since $\{p_y^\beta\}$ having a fuzzy accumulation point (it is clear that $\{p_y^\beta\}$ is fuzzy open and X is fuzzy connected), p_x^α is a fuzzy accumulation point of $\{p_y^\beta\}$. Therefore $\{p_x^\alpha, p_y^\beta\}$ is fuzzy connected for any $p_y^\beta \in G$. Let $p_y^{\beta_0} \in G$. Since $H \vee \{p_x^\alpha\}$ is fuzzy connected by lemma 2.3

$$1_X \backslash \{p_y^{\beta_0}\} = \bigvee_{p_y^\beta \in G, p_y^\beta \neq p_y^{\beta_0}} \{p_x^\alpha, p_y^\beta\} \vee (H \vee \{p_x^\alpha\}).$$

is also fuzzy connected. Hence it is contrary to our assumption that $p_y^{\beta_0}$ is a fuzzy cut-point of X. □

Lemma 2.5. *Let X be a FCTS; let two fuzzy cut points* $p_x^\alpha, p_y^\beta \in X$ *such that* $1_X \backslash \{p_x^\alpha\} = G|H$ *and* $1_X \backslash \{p_y^\beta\} = K|M$. *If* $p_x^\alpha \in K$ *and* $p_y^\beta \in G$, *then* $M \leq G$ *and* $H \leq K$.

Proof. As $M \vee \{p_y^\beta\}$ is fuzzy connected by lemma 2.3 and since $M \vee \{p_y^\beta\} \leq 1_X \backslash \{p_x^\alpha\}$, we have $M \vee \{p_y^\beta\} \leq G$ or $M \vee \{p_y^\beta\} \leq H$. Since $p_y^\beta \in G$, $M \vee \{p_y^\beta\} \leq H$ is not true. Therefore $M \leq G$. Similarly we can show that $H \leq K$. □

In the succeding theorem, we will prove that a finite FTS cannot be a fuzzy cut-point space.

Theorem 2.6. *Let X be a fuzzy cut-point space. Then the collection of fuzzy closed points of X is infinite.*

Proof. The proof is by mathematical induction: Let distinct fuzzy closed points in X form a sequence $p_x^{\alpha_1}, p_x^{\alpha_2}, p_x^{\alpha_3}$,.... Set $K_0 = X$. From corollary 3.3(a), we have a fuzzy closed point $p_x^{\alpha_1} \in K_0$. As a fuzzy cut-point $p_x^{\alpha_1} \in X$,then the fuzzy open subsets K_1 and M_1 of X such that $1_X \backslash \{p_x^{\alpha_1}\} = K_1|M_1$. Assume that the distinct fuzzy closed points $p_x^{\alpha_1}, p_x^{\alpha_2}, p_x^{\alpha_3}$.......,$p_x^{\alpha_m} \in X$ and the fuzzy open subsets K_1,....,K_m,M_1,.....,M_m of X are picked such that $1_X \backslash \{p_x^{\alpha_j}\} = K_j|M_j$, $p_x^{\alpha_j} \in K_{j-1}$ and $K_{j-1} \geq K_j$ for each $j, 1 \leq j \leq m$. As stated in lemma 2.4, there is a fuzzy closed point $p_x^{\alpha_{m+1}} \in K_m$. Then we have fuzzy open subsets K_{m+1} and M_{m+1} of X such that $1_X \backslash \{p_x^{\alpha_{m+1}}\} = K_{m+1}|M_{m+1}$. If needed K_{m+1} and M_{m+1} are interchanged, we may assume that $p_x^{\alpha_m} \in M_{m+1}$. Thus by Lemma 2.5, $K_m \geq K_{m+1}$. Since $p_x^{\alpha_j} \notin K_j, p_x^{\alpha_j} \notin K_m$ for any $j, 1 \leq j \leq m$. Now $p_x^{\alpha_{m+1}} \in K_m \Rightarrow p_x^{\alpha_{m+1}}$ differs from $p_x^{\alpha_1}, p_x^{\alpha_2}, ..., p_x^{\alpha_m}$. □

Corollary 2.7. *Let X be a fuzzy cut-point space.Then* $|X| = \infty$.

In fact, Theorem 2.6 is a generalisation of Corollary 2.7. Generalisation of corollary 2.7 is proved by means of fuzzy Hausdorff Maximal Principle, in the succeeding result.

Theorem 2.8. *Let X be a fuzzy compact connected topological space having more than one fuzzy point. Then X has atleast two fuzzy non-cut points.*

Proof. Let X have utmost one non fuzzy-cut point. Suppose a fuzzy cut point $p_x^{\alpha_0} \in X$ and $1_X - \{p_x^{\alpha_0}\} = G_0|H_0$. As X has utmost one non fuzzy cut-point, either G_0 or H_0 contains of fuzzy cut-points. By lemma 2.4, G_0 contains few fuzzy closed cut point of X say P_x^{α} . Let $1_X \backslash \{p_x^{\alpha}\} = G|H$ and consider $p_x^{\alpha_0} \in H$. From lemma 2.5, $G \subseteq G_0$. Set $F = \{T :$ T is a fuzzy open subset of $X, T \geq H, Cl(T) \backslash T$ is a fuzzy singleton,and $Cl(T) \neq 1_X\}$. As H is fuzzy open and $Cl(H) = H \vee \{p_x^{\alpha}\}$,$H \in F$. For each $T_i \in F$ and $T_j \in F$, write $T_i \leq T_j$ if $T_j = T_j$ or if $ClT_i \subseteq T_j$.$(F, \leq)$ is obviously a fuzzy partially ordered set, and by the fuzzy Hausdorff Maximal Principle, there is a fuzzy maximal chain C in F. Let $T_i \in F$ and let $\{p_x^{\alpha}\} = Cl\ T_i \backslash T_i$. As by lemma 2.4, $1_X \backslash \{p_x^{\alpha}\} = T_i|(1_X|Cl\ T_i)$, there is a fuzzy closed point $p_y^{\beta} \in 1_X \backslash T_i \leq G$. Let $1_X \backslash \{p_y^{\beta}\} = K|M$. Since by lemma 2.3, $Cl\ T_i$ is fuzzy connected, $Cl\ T_i \leq K$ or $Cl\ T_i \leq M$, i.e., $T_i < K$ or $T_i < M$. Since $Cl\ (T_i)$ was arbitrary in F, F not having a fuzzy maximal element. Thus $\bigvee_{T \in C} T = \bigvee_{T \in C} Cl\ T$. Let $W = \bigvee_{T \in C} T$. As $Cl\ T$ is fuzzy connected for each $T \in F$, W is also fuzzy connected. We claim that $Cl\ W = 1_X$. If not, then $1_X \backslash W$ is nonzero fuzzy closed subset of X. Since $1_X \backslash W \leq G$, every fuzzy point in $1_X \backslash W$ is a fuzzy cut-point of X and is either fuzzy open or fuzzy closed by Theorem 2.1. Since $1_X \backslash W$ is not fuzzy open, all the fuzzy points of $1_X \backslash W$ not be fuzzy open, then there is a fuzzy cut-point $p_x^{\alpha'} \in 1_X \backslash W$. Take $1_X \backslash \{p_x^{\alpha'}\} = E|F$. Since W is fuzzy connected,$W \leq E$ or $W \leq F$. Let $W \leq E$. Since $E \in F$,$T \leq E$ for any $T \in C$. As C is not containing a fuzzy maximal element,$E \notin C$ which is the contradiction to the the maximality of fuzzy chain C. Therefore $W = 1_X$ and hence C is an infinite fuzzy open covering of X. As C is a fuzzy chain except a fuzzy maximal element, we don't have a choice for finite fuzzy subcovering of C for X which is contrary to fuzzy compactness of X. □

Corollary 2.9. *Let X be a fuzzy cut-point space. Then X is non fuzzy compact.*

3. Irreducible fuzzy cut-point spaces and its properties

Definition 3.1. A fuzzy cut-point is called an irreducible fuzzy cut-point space if no proper fuzzy subset of it (with the fuzzy subspace topology)is a fuzzy cut-point space.

Lemma 3.1. *Let $p_x^\alpha \in X$, where X is a fuzzy cut-point space. Let $1_X\backslash\{p_x^\alpha\} = G|H$. If G is not fuzzy connected, then $G \vee \{p_x^\alpha\}$ is a fuzzy cut-point space.*

Proof. Set $Y = G \vee \{p_x^\alpha\}$. It is obvious that a fuzzy cut-point $p_x^\alpha \in Y$. Choose an arbitrary fuzzy point p_y^β in G. As $1_X\backslash\{p_y^\beta\} = (1_Y\backslash\{p_y^\beta\}) \vee (H \vee \{p_x^\alpha\})$ is not fuzzy connected and $p_x^\alpha \in (1_Y\backslash\{p_y^\beta\}) \wedge (H \vee \{p_x^\alpha\})$, either $1_Y\backslash\{p_y^\beta\}$ or $H \vee \{p_x^\alpha\}$ is fuzzy disconnected. Since Lemma 2.3, $H \vee \{p_x^\alpha\}$ is fuzzy connected. Thus $1_Y\backslash\{p_y^\beta\}$ is fuzzy disconnected. □

Corollary 3.2. *If X is a fuzzy irreducible cut-point space, then for each $p_x^\alpha \in X, 1_X\backslash\{p_x^\alpha\}$ having absolutely two fuzzy components.*

Proof. Let $1_X\backslash\{p_x^\alpha\} = G|H$. As X is fuzzy irreducible cut-point space, $G\vee\{p_x^\alpha\}$ and $H\vee\{p_x^\alpha\}$ are not fuzzy cut-point spaces. By Lemma 3.1, G and H are fuzzy connected. □

Lemma 3.3. *Let $p_x^\alpha \in X$ where X is a fuzzy irreducible cut-point space. Let $1_X\backslash\{p_x^\alpha\} = G|H$. Then there are exactly two fuzzy points $p_y^\beta \in G$ and $p_z^\gamma \in H$ such that $\{p_x^\alpha, p_y^\beta\}$ and $\{p_x^\alpha, p_z^\gamma\}$are fuzzy connected. Moreover if p_x^α is fuzzy closed, then p_y^β and p_z^γ are fuzzy open; ifp_x^α is fuzzy open, then p_y^β and p_z^γ are fuzzy closed.*

Proof. As G is fuzzy connected by Corollary 3.2 and as X is a fuzzy irreducible cut-point space, G has a fuzzy non cut-point p_y^β, that is, $G\backslash\{p_y^\beta\}$ is fuzzy connected. It is asserted that p_y^β is unique fuzzy point in G such that $\{p_x^\alpha, p_y^\beta\}$ is fuzzy connected. Initialy it is showed that if $\{p_x^\alpha, p_y^{\beta'}\}$ is fuzzy connected for some $p_y^{\beta'} \in G$, then $p_y^{\beta'} = p_y^\beta$. Let $p_y^{\beta'}$ be a fuzzy point in G such that $\{p_x^\alpha, p_y^{\beta'}\}$ is fuzzy connected. Assume that $p_y^{\beta'} \neq p_y^\beta$. Since $H \vee \{p_x^\alpha\}$ is fuzzy connected by Lemma 2.3 and since $1_X\backslash\{p_y^\beta\} = (G\backslash\{p_y^\beta\}) \vee (H \vee \{p_x^\alpha\})$, the fuzzy connectedness of $\{p_x^\alpha, p_y^{\beta'}\}$ gives the fuzzy connectedness of $1_X\backslash\{p_y^\beta\}$ (a contradiction). From the following two cases, we can show $\{p_x^\alpha, p_y^\beta\}$ is fuzzy connected.

(1)Case I:$\{p_x^\alpha\}$ is fuzzy closed. By theorem 2.3, G is not fuzzy closed but $G \vee \{p_x^\alpha\}$ is fuzzy closed. In this way fuzzy point p_x^α is a fuzzy accumulation point of G. As $1_X\backslash\{p_y^\beta\} = (G\backslash p_y^\beta) \vee (H \vee \{p_x^\alpha\})$ is not fuzzy connected, p_x^α is not a fuzzy accumulation point of $G\backslash\{p_y^\beta\}$. Therefore p_x^α is a fuzzy accumulation point of $\{p_y^\beta\}$.

(2)Case II:p_x^α is fuzzy open. By theorem 2.1, G is not fuzzy open but $G \vee \{p_x^\alpha\}$ is fuzzy open. Similarly, there exists a fuzzy point $p_y^{\beta'} \in G$ which is not fuzzy interior point of G. As $p_y^{\beta'}$ is a fuzzy interior point of$G \vee \{p_x^\alpha\}$, $p_y^{\beta'}$ is a fuzzy accumulation point of $\{p_x^\alpha\}$. Therefore $\{p_x^\alpha, p_y^{\beta'}\}$ is fuzzy connected. As proved $p_y^{\beta'} = p_y^\beta$, $\{p_x^\alpha, p_y^\beta\}$ is fuzzy connected.
From the above argument we can have unique fuzzy point p_z^γ in H such that $\{p_x^\alpha, p_z^\gamma\}$ is fuzzy connected. Since by Theorem 2.3, the final inclusion of this lemma and the fuzzy connectedness of $\{p_x^\alpha, p_y^\beta\}$ and $\{p_x^\alpha, p_z^\gamma\}$ is proved. □

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