ANNALS OF COMMUNICATIONS IN MATHEMATICS
Volume 3, Number 4 (2020), 273-284
ISSN: 2582-0818

r-FUZZY $\Re s$-COMPACTNESS AND r-FUZZY $\Re s$-CONNECTEDNESS IN THE SENSE OF $\check{S}$OSTAK'S

E. ELAVARASAN

ABSTRACT. The purpose of this paper is to introduce the concepts of fuzzy regular semi compactness, fuzzy regular semi connectedness, fuzzy regular semi strongly connectedness and fuzzy regular semi-C_5-connectedness. Some interesting properties of these notions are studied. In this connection, interrelations are discussed. Examples are provided wherever necessary.

1. INTRODUCTION

$\check{S}$ostak [11], introduced the concept of fuzzy topological spaces as an extension of Chang's fuzzy topological spaces [2]. It has been developed in many direction [4, 7, 9]. Mashhour et. al., [8], A. M. Zahran [13] and E. E. Kerre et. al., [6] introduced the notion of fuzzy regular semi open and regular semi closed sets and investigate the relationship among fuzzy regular semi continuity and fuzzy regular semi irresolute mappings. Recently, Vadivel and Elavarasan [12] introduce and study the concept of fuzzy regular semi open sets and fuzzy regular semi continuous functions in fuzzy topological spaces in the sense of $\check{S}$ostak's. In this paper, we introduce the concepts of r-fuzzy regular semi compactness, r-fuzzy regular semi connectedness, r-fuzzy regular semi strongly connectedness and r-fuzzy regular semi-C_5-connectedness in the sense of $\check{S}$ostak's. Some interesting properties of these notions are studied. In this connection, interrelations are discussed. Examples are provided wherever necessary.

2. PRELIMINARIES

Throughout this paper, let X be a non-empty set, $I = [0,1]$, $I_0 = (0,1]$. A fuzzy set λ of X is a mapping $\lambda : X \to I$, and I^X be the family of all fuzzy sets on X. The complement of a fuzzy set λ is denoted by $\overline{1} - \lambda$. For $\lambda \in I^X$, $\overline{\lambda}(x) = \lambda$ for all $x \in X$. For each $x \in X$ and $t \in I_0$, a fuzzy point x_t is defined by $x_t(y) = \begin{cases} t & \text{if } y = x \\ 0 & \text{if } y \neq x. \end{cases}$ Let $Pt(X)$

2010 *Mathematics Subject Classification.* 54A40, 03E72, 54D30.
Key words and phrases. r-fuzzy $\Re s$-compact, r-fuzzy $\Re s$-connected, r-fuzzy $\Re s$-strongly connected and r-fuzzy $\Re s$-C_5-connected.
Received: November 17, 2020. Accepted: December 14, 2020. Published: December 31, 2020.

be the family of all fuzzy points in X. All other notations and definitions are standard in the fuzzy set theory.

Definition 2.1. [11] A function $\tau : I^X \to I$ is called a fuzzy topology on X if it satisfies the following conditions:

(1) $\tau(\overline{0}) = \tau(\overline{1}) = 1$,
(2) $\tau(\bigvee_{i\in J} \mu_i) \geq \bigwedge_{i\in J} \tau(\mu_i)$, for any $\{\mu_i : i \in J\} \leq I^X$.
(3) $\tau(\mu_1 \wedge \mu_2) \geq \tau(\mu_1) \wedge \tau(\mu_2)$, for all $\mu_1,\ \mu_2 \in I^X$.

The pair (X, τ) is called a fuzzy topological space (for short, fts). A fuzzy set λ is called an r-fuzzy open (for short, r-fo) if $\tau(\lambda) \geq r$ and a fuzzy set λ is called an r-fuzzy closed (for short, r-fc) if $\tau(\overline{1} - \lambda) \geq r$.

Theorem 2.1. [3] *Let (X, τ) be a fts. Then for each $\lambda \in I^X$ and $r \in I_0$, we define an operator $C_\tau : I^X \times I_0 \to I^X$ as follows: $C_\tau(\lambda,\ r) = \bigwedge\{\mu \in I^X : \lambda \leq \mu,\ \tau(\overline{1}-\mu) \geq r\}$. For $\lambda,\ \mu \in I^X$ and $r, s \in I_0$, the operator C_τ satisfies the following statements:*

(C1) $C_\tau(\overline{0}, r) = \overline{0}$,
(C2) $\lambda \leq C_\tau(\lambda,\ r)$,
(C3) $C_\tau(\lambda,\ r) \vee C_\tau(\mu,\ r) = C_\tau(\lambda \vee \mu,\ r)$,
(C4) $C_\tau(\lambda,\ r) \leq C_\tau(\lambda,\ s)$ *if* $r \leq s$,
(C5) $C_\tau(C_\tau(\lambda,\ r),\ r) = C_\tau(\lambda,\ r)$.

Theorem 2.2. [3] *Let (X, τ) be a fts. Then for each $\lambda \in I^X$ and $r \in I_0$, we define an operator $I_\tau : I^X \times I_0 \to I^X$ as follows: $I_\tau(\lambda,\ r) = \bigvee\{\mu \in I^X : \mu \leq \lambda,\ \tau(\mu) \geq r\}$. For $\lambda,\ \mu \in I^X$ and $r, s \in I_0$, the operator I_τ satisfies the following statements:*

(I1) $I_\tau(\overline{1}, r) = \overline{1}$,
(I2) $I_\tau(\lambda,\ r) \leq \lambda$,
(I3) $I_\tau(\lambda,\ r) \wedge I_\tau(\mu,\ r) = I_\tau(\lambda \wedge \mu,\ r)$,
(I4) $I_\tau(\lambda,\ r) \leq I_\tau(\lambda,\ s)$ *if* $s \leq r$,
(I5) $I_\tau(I_\tau(\lambda,\ r),\ r) = I_\tau(\lambda,\ r)$.
(I6) $I_\tau(\overline{1} - \lambda, r) = \overline{1} - C_\tau(\lambda, r)$ *and* $C_\tau(\overline{1} - \lambda, r) = \overline{1} - I_\tau(\lambda,\ r)$

Definition 2.2. [10] Let $(X,\ \tau)$ be a fts, $\lambda \in I^X$ and $r \in I_0$. Then a fuzzy set λ is called

(1) r-fuzzy regular open (for short, r-fro) if $\lambda = I_\tau(C_\tau(\lambda, r), r)$.
(2) r-fuzzy regular closed (for short, r-frc) if $\lambda = C_\tau(I_\tau(\lambda, r), r)$.

Definition 2.3. [12] Let $(X,\ \tau)$ be a fts and $\lambda \in I^X,\ r \in I_0$. Then

(1) λ is called r-fuzzy regular semi open (for short, r-frso) if there exists r-fro set $\mu \in I^X$ and $\mu \leq \lambda \leq C_\tau(\mu, r)$.
(2) λ is called r-fuzzy regular semi closed (for short, r-frsc) if there exists r-frc set $\mu \in I^X$ and $I_\tau(\mu, r) \leq \lambda \leq \mu$.
(3) The r-fuzzy regular semi interior of λ, denoted by $RSI_\tau(\lambda, r)$, is defined by $RSI_\tau(\lambda, r) = \bigvee\{\mu \in I^X \mid \mu \leq \lambda,\ \mu \text{ is } r\text{-frso }\}$.
(4) The r-fuzzy regular semi closure of λ, denoted by $RSC_\tau(\lambda, r)$ is defined by $RSC_\tau(\lambda, r) = \bigwedge\{\mu \in I^X \mid \mu \geq \lambda,\ \mu \text{ is } r\text{-frsc }\}$.

We denote the set of all r-frso sets and r-frsc sets by $FRSO(X)$ and $FRSC(X)$.

Theorem 2.3. [12] *Let $(X,\ \tau)$ be a smooth topological space. For $\lambda \in I^X,\ r \in I_0$, the following statements are equivalent:*

(1) λ *is* r*-frso.*
(2) $\overline{1} - \lambda$ *is* r*-frso.*

(3) $I_\tau(\lambda, r) = I_\tau(C_\tau(\lambda, r), r)$.
(4) $C_\tau(\lambda, r) = C_\tau(I_\tau(\lambda, r), r)$.

Definition 2.4. [12] Let $(X,\ \tau)$ and $(Y,\ \eta)$ be fts's. Let $f : (X,\ \tau) \to (Y,\ \eta)$ be a mapping. Then f is said to be:

(1) fuzzy regular semi irresolute (resp. fuzzy regular semi continuous) iff $f^{-1}(\mu)$ is r-frso for each r-frso set $\mu \in I^Y$ (resp. $\mu \in I^Y,\ \eta(\mu) \geq r$).
(2) fuzzy regular semi irresolute open (resp. fuzzy regular semi open) iff $f(\lambda)$ is r-frso in Y for each r-frso set $\lambda \in I^X$ (resp. $\lambda \in I^X,\ \tau(\lambda) \geq r$).
(3) fuzzy regular semi irresolute closed (resp. fuzzy regular semi closed) iff $f(\lambda)$ is r-frsc in Y for each r-frsc set $\lambda \in I^X$ (resp. $\lambda \in I^X,\ \tau(\overline{1} - \lambda) \geq r$).

Definition 2.5. [1] Let $(X,\ \tau)$ and $(Y,\ \sigma)$ be a fts's. Let $f : (X,\ \tau) \to (Y,\ \sigma)$ be a function. Then f is called

(1) weakly continuous if for each $\mu \in I^Y$, where $\sigma(\mu) \geq r,\ r \in I_0,\ f^{-1}(\mu) \leq I_\tau(f^{-1}(C_\sigma(\mu, r)), r)$.
(2) weakly open if for each $\mu \in I^X$, where $\tau(\mu) \geq r,\ r \in I_0,\ f(\mu) \leq I_\sigma(f(C_\tau(\mu, r)), r)$.

Definition 2.6. [5] A fts (X, τ) is called an r-fuzzy compact (r-fuzzy nearly compact and r-fuzzy almost compact) if and only if for every family $\{\lambda_i | i \in J\}$ in $\{\lambda : \lambda \in I^X, \tau(\lambda) \geq r\}$ such that $\bigvee_{i \in J} \lambda_i = \overline{1}$, there exists a finite subset J_0 of J such that $\bigvee_{i \in J_0} \lambda_i = \overline{1}$ (resp. $\bigvee_{i \in J_0} I_\tau(C_\tau(\lambda_i, r), r) = \overline{1}$ and $\bigvee_{i \in J_0} C_\tau(\lambda_i, r) = \overline{1}$).

Theorem 2.4. *Let (X, τ) and (Y, σ) be two fts and $f : (X, \tau) \to (Y, \sigma)$ is fuzzy weakly open and fuzzy weakly continuous, then $f^{-1}(\lambda)$ is an r-fro (resp. r-frc) set for every r-fro $\lambda \in I^Y$, $r \in I_0$.*

Proof. Let λ be an r-fro set in Y, we have $\sigma(\lambda) \geq r$. Since f is fuzzy weakly continuous, $\tau(f^{-1}(\lambda) \geq r$. Hence $f^{-1}(\lambda) = I_\tau(f^{-1}(\lambda), r) \leq I_\tau(C_\tau(f^{-1}(\lambda), r), r)$. Since f is fuzzy weakly open, $f(I_\tau(C_\tau(f^{-1}(\lambda), r), r)) \leq I_\sigma(f(C_\tau(f^{-1}(\lambda), r), r))$. Since f is fuzzy weakly continuous, $I_\sigma(f(C_\tau(f^{-1}(\lambda), r), r)) \leq I_\sigma(ff^{-1}(C_\sigma(\lambda, r), r)) \leq I_\sigma(C_\sigma(\lambda, r), r)) = \lambda$. Hence $I_\tau(C_\tau(f^{-1}(\lambda), r), r) \leq f^{-1}(\lambda)$. Thus $f^1(\lambda)$ is r-fro. An r-frc case will be similar. □

3. r-FUZZY $\Re s$-COMPACTNESS

The most important of all covering properties is compactness. In this section, we introduce the concept of fuzzy $\Re s$-compactness and study some of its basic properties.

Definition 3.1. A fts (X, τ) is called

(1) r-fuzzy $\Re s$ compact if for every r-fuzzy regular semiopen cover $\{\lambda_i : i \in J\}$ of X, there exists a finite subset J_0 of J such that $\bigvee_{i \in J_0} \lambda_i = \overline{1}$.
(2) r-fuzzy weakly $\Re s$ compact if for every r-fuzzy regular semiopen cover $\{\lambda_i : i \in J\}$ of X, there exists a finite subset J_0 of J such that $\bigvee_{i \in J_0} I_\tau(\lambda_i, r) = \overline{1}$.
(3) r-fuzzy almost $\Re s$ compact if for every r-fuzzy regular semiopen cover $\{\lambda_i : i \in J\}$ of X, there exists a finite subset J_0 of J such that $\bigvee_{i \in J_0} C_\tau(\lambda_i, r) = \overline{1}$.

Remark. *(1) Every r-fuzzy weakly $\Re s$ compact is r-fuzzy $\Re s$ compact.*
(2) Every r-fuzzy $\Re s$ compact is r-fuzzy almost $\Re s$ compact.

From Theorem 2.3, we have the following theorem:

Theorem 3.1. *A fts (X, τ) is r-fuzzy $\Re s$-compact if and only if for each family $\{\lambda_i | i \in J\}$ of r-frso sets of X such that $\bigwedge_{i \in J} \lambda_i = \overline{0}$, there exists a finite subset J_0 of J such that $\bigwedge_{i \in J_0} \lambda_i = \overline{0}$.*

Theorem 3.2. *A fts (X, τ) is r-fuzzy weakly $\Re s$-compact if and only if for each family $\{\lambda_i | i \in J\}$ of r-frso sets of X such that $\bigwedge_{i \in J} \lambda_i = \overline{0}$, there exists a finite subset J_0 of J such that $\bigwedge_{i \in J_0} C_\tau(\lambda_i, r) = \overline{0}$.*

Proof. Suppose that (X, τ) is r-fuzzy weakly $\Re s$-compact. Let $\{\lambda_i | i \in J\}$ be a family of r-frso sets of X such that $\bigwedge_{i \in J} \lambda_i = \overline{0}$. Then by Theorem 2.3, $\{\overline{1} - \lambda_i | i \in J\}$ is a family of r-frso sets of X such that $\bigvee_{i \in J} \overline{1} - \lambda_i = \overline{1} - \bigwedge_{i \in J} \lambda_i = \overline{1}$. Since (X, τ) is r-fuzzy weakly $\Re s$ compact, there exists a finite subset J_0 of J such that $\bigvee_{i \in J_0} I_\tau(\overline{1} - \lambda_i, r) = \overline{1}$. Hence $\bigwedge_{i \in J_0} C_\tau(\lambda_i, r) = \overline{1} - (\bigvee_{i \in J_0} I_\tau(\overline{1} - \lambda_i, r) = \overline{0}$. □

Converse follows by reversing the previous arguments.

Theorem 3.3. *Let (X, τ) be a fts. Then the following are equivalent:*

(1) (X, τ) is r-fuzzy weakly $\Re s$-compact.
(2) For each family $\{\lambda_i | i \in J\}$ of r-frso sets of X such that $\bigwedge_{i \in J} \lambda_i = \overline{0}$, there exists a finite subset J_0 of J such that $\bigwedge_{i \in J_0} C_\tau(\lambda_i, r) = \overline{0}$.
(3) For each r-fuzzy regular closed cover $\{\lambda_i | i \in J\}$ of X, there exists a finite subset J_0 of J such that $\bigvee_{i \in J_0} I_\tau(\lambda_i, r) = \overline{1}$.

Proof. (1)⇒(2): Trivial.

(2)⇒(1): Let $\{\lambda_i | i \in J\}$ be a family of r-frso sets of X such that $\bigwedge_{i \in J} \lambda_i = \overline{0}$. Since λ_i is an r-frso set for each $i \in J$, $C_\tau(\lambda_i, r) = C_\tau(I_\tau(\lambda_i, r), r)$ for each $i \in J$. Since $\{I_\tau(\lambda_i, r) | i \in J\}$ is a family of r-fro sets of X such that $\bigwedge_{i \in J} I_\tau(\lambda_i, r) = \overline{0}$, by (2) there exists a finite subset J_0 of J such that $\bigwedge_{i \in J_0} C_\tau(\lambda_i, r) = \bigwedge_{i \in J_0} C_\tau(I_\tau(\lambda_i, r), r) = \overline{0}$. Thus (X, τ) is r-fuzzy weakly $\Re s$-compact.

(2)⇔(3): It is obvious. □

Theorem 3.4. *Let (X, τ) and (Y, σ) be two fts's and let $f : (X, \tau) \to (Y, \sigma)$ be surjective, fuzzy weakly open and fuzzy weakly continuous function. If (X, τ) is r-fuzzy weakly $\Re s$-compact, then so is (Y, σ).*

Proof. Let $\{\eta_i | i \in J\}$ be an r-fuzzy regular closed cover over Y. By Theorem 2.4, $\{f^1(\eta_i) | i \in J\}$ is an r-fuzzy regular closed cover of X. Since X is r- fuzzy weakly $\Re s$-compact, by Theorem 3.2, there exists a finite subset J_0 of J such that $\bigvee_{i \in J_0} I_\tau(f^{-1}(\eta_i), r) = \overline{1}$. From the surjectivity and fuzzy weakly openness of f, we have

$\overline{1} = f(\bigvee_{i \in J_0} (I_\tau(f^{-1}(\eta_i, r))))$
$= \bigvee_{i \in J_0} f(I_\tau(f^{-1}(\eta_i, r)))$
$\leq \bigvee_{i \in J_0} I_\sigma(f(C_\tau(I_\tau(f^{-1}(\eta_i), r), r)), r)$
$= \bigvee_{i \in J_0} (I_\sigma(f(f^{-1}(\eta_i)), r)$
$= \bigvee_{i \in J_0} I_\sigma(\eta_i, r)$.

Hence $\bigvee_{i \in J_0} I_\sigma(\eta_i, r) = \overline{1}$, and thus (Y, σ) is r-fuzzy weakly $\Re s$ compact. □

Theorem 3.5. *A fts (X, τ) is r-fuzzy almost $\Re s$ compact if and only if for each family $\{\lambda_i | i \in J\}$ of r-frso sets of X such that $\bigwedge_{i \in J} \lambda_i = \overline{0}$, there exists a finite subset J_0 of J such that $\bigwedge_{i \in J_0} I_\tau(\lambda_i, r) = \overline{0}$.*

Proof. Let (X, τ) be r-fuzzy almost $\Re s$-compact and let $\{\lambda_i | i \in J\}$ be a family of r-frso sets of X such that $\bigwedge_{i \in J} \lambda_i = \overline{0}$. Then $\{\overline{1} - \lambda_i | i \in J\}$ is a family of r-frso sets of X such that $\bigvee_{i\in J}\overline{1}-\lambda_i=\overline{1}-(\bigwedge_{i\in J}\lambda_i)=\overline{1}$. Since (X,τ) is r-fuzzy almost $\Re s$-compact, there exists a finite subset J_0 of J such that $\bigvee_{i\in J_0}C_\tau(\overline{1}-\lambda_i,r)=\overline{1}$. Hence $\bigwedge_{i\in J_0}I_\tau(\lambda_i,r)=\overline{1}-\bigvee_{i\in J_0}C_\tau(\overline{1}-\lambda_i,r)=\overline{0}$.

The converse can be proved similarly. □

Theorem 3.6. *Let (X,τ) be a fts. Then the following statements are equivalent:*

(1) (X,τ) is r-fuzzy almost $\Re s$-compact.
(2) For each family $\{\lambda_i|i\in J\}$ of r-fro sets of X such that $\bigwedge_{i\in J}\lambda_i=\overline{0}$, there exists a finite subset J_0 of J such that $\bigwedge_{i\in J_0}\lambda_i=\overline{0}$.
(3) For each r-fuzzy regular closed cover $\{\lambda_i|i\in J\}$ of X, there exists a finite subset J_0 of J such that $\bigvee_{i\in J_0}\lambda_i=\overline{1}$.

Proof. Straightforward. □

Definition 3.2. A fts (X,τ) is called an r-fuzzy S-closed if and only if for every an r-fuzzy semiopen cover $\{\lambda_i|i\in J\}$ of X, there exists a finite subset J_0 of J such that $\bigvee_{i\in J_0}C_\tau(\lambda_i,r)=\overline{1}$.

Theorem 3.7. *A fts (X,τ) is r-fuzzy almost $\Re s$-compact if and only if (X,τ) is r-fuzzy S-closed.*

Proof. Let (X,τ) be r-fuzzy S-closed. Since every r-frso set is r-fuzzy semiopen, (X,τ) is r-fuzzy almost $\Re s$-compact.

Conversely, suppose that (X,τ) is r-fuzzy almost $\Re s$-compact and let $\{\lambda_i|i\in J\}$ be an r-fuzzy semiopen cover of X. Then there exists $\mu_i\in I^X$ with $\tau(\mu_i)\geq r$, such that $\mu_i\leq\lambda_i\leq C_\tau(\mu_i,r)$, for each $i\in J$. We can easily show that $C_\tau(\mu_i,r)$ is an r-frc for each $i\in J$. Since $\mu_i\leq\lambda_i\leq C_\tau(\lambda_i,r)$, for each $i\in J$, $C_\tau(\mu_i,r)\leq C_\tau(\lambda_i,r)\leq C_\tau(C_\tau(\mu_i,r),r)$ for each $i\in J$. Thus $C_\tau(\lambda_i,r)=C_\tau(\mu_i,r)$ for each $i\in J$. Thus $\{C_\tau(\lambda_i,r)|i\in J\}$ is an r-fuzzy regular closed cover of X. Since (X,τ) is r-fuzzy almost $\Re s$-compact, there exists a finite subset J_0 of J such that $\bigvee_{i\in J_0}C_\tau(\lambda_i,r)=\overline{1}$. Hence (X,τ) is r-fuzzy S-closed. □

Theorem 3.8. *A fts (X,τ) is an r-fuzzy weakly $\Re s$-compact if and only if for every an r-fuzzy semiopen cover $\{\lambda_i|i\in J\}$ of X, there exists a finite subset J_0 of J such that $\bigvee_{i\in J_0}I_\tau(C_\tau(\lambda_i,r),r)=\overline{1}$.*

Proof. Similar to Theorem 3.7. □

Theorem 3.9. *Let (X,τ) and (Y,σ) be two fts's and let $f:(X,\tau)\to(Y,\sigma)$ be a surjective, fuzzy weakly open and fuzzy weakly continuous function. If (X,τ) is r-fuzzy almost $\Re s$-compact, then so is (Y,σ).*

Proof. Let $\{\eta_i|i\in J\}$ be an r-fuzzy regular closed cover of Y. By Theorem 2.4, $\{f^{-1}(\eta_i)|i\in J\}$ is an r-fuzzy regular closed cover of X. Since (X,τ) is r-fuzzy almost $\Re s$-compact, by Theorem 2.4, there exists a finite subset J_0 of J such that $\bigvee_{i\in J_0}f^{-1}(\eta_i)=\overline{1}$. From the surjectivity of f we have

$\overline{1}=f(\bigvee_{i\in J_0}f^{-1}(\eta_i))=\bigvee_{i\in J_0}f(f^{-1}(\eta_i))=\bigvee_{i\in J_0}\eta_i$.

Hence $\bigvee_{i\in J_0}\eta_i=\overline{1}$. Thus (Y,σ) is r-fuzzy almost $\Re s$-compact. □

Definition 3.3. A fts (X,τ) is called r-fuzzy extremally disconnected if $\tau(C_\tau(\lambda,r))\geq r$ for every $\lambda\in I^X$ with $\tau(\lambda)\geq r$.

Theorem 3.10. *Let (X, τ) and (Y, σ) be two fts, and let $f : (X, \tau) \to (Y, \sigma)$ be a surjective, fuzzy weakly open and fuzzy weakly continuous function. If (X, τ) is r-fuzzy extremally disconnected, then so is (Y, σ).*

Proof. Let $\lambda \in I^Y$ with $\sigma(\lambda) \geq r$. Then $\lambda = I_\sigma(\lambda, r)$. Hence $C_\tau(\lambda, r)$ is r-frc set. By Theorem 2.4, $f^{-1}(C_\sigma(\lambda, r))$ is r-frc, i.e., $f^{-1}(C_\sigma(\lambda, r)) = C_\tau(I_\tau(f^{-1}(C_\sigma(\lambda, r)), r), r)$. Since (X, τ) is r-fuzzy extremally disconnected and $\tau(I_\tau(f^{-1}(C_\sigma(\lambda, r)), r)) \geq r$ and $\tau(C_\tau(I_\tau(f^{-1}(C_\sigma(\lambda, r)), r), r)) \geq r$. From the surjectivity and fuzzy weakly openness of f we have

$C_\sigma(\lambda, r) = f(f^{-1}(C_\sigma(\lambda, r)))$
$= f(C_\tau(I_\tau(f^{-1}(C_\sigma(\lambda, r)), r), r))$
$\leq I_\sigma(f(C_\tau(I_\tau(f^{-1}(C_\sigma(\lambda, r)), r), r)), r)$
$= I_\sigma(f(C_\tau(f^{-1}(C_\sigma(\lambda, r)), r)), r)$
$= I_\sigma(f(f^{-1}(C_\sigma(\lambda, r))), r)$
$= I_\sigma(C_\sigma(\lambda, r), r)$.

Hence $C_\sigma(\lambda, r) = I_\sigma(C_\sigma(\lambda, r), r)$ and so $\sigma(C_\sigma(\lambda, r)) \geq r$. Thus (Y, σ) is r-fuzzy extremally disconnected. □

Theorem 3.11. *Let a fts (X, τ) be r-fuzzy extremally disconnected. If $\lambda \in I^X$ is r-frso, then $I_\tau(\lambda, r) = \lambda = C_\tau(\lambda, r)$.*

Proof. Let λ be an r-frso set. Then there exists an r-fro μ such that $\mu \leq \lambda \leq C_\tau(\mu, r)$. Since X is r-fuzzy extremally disconnected, $\mu = C_\tau(\mu, r)$. And we get $\mu = I_\tau(\mu, r)$, since μ is an r-fro set. Thus we have the following, $\mu = I_\tau(\mu, r) \leq I_\tau(\lambda, r) \leq \lambda \leq C_\tau(\lambda, r) \leq C_\tau(\mu, r) = \mu$. Hence $I_\tau(\lambda, r) = \lambda = C_\tau(\lambda, r)$. □

From the above theorem, we get the following:

Theorem 3.12. *Let a fts (X, τ) be r-fuzzy extremally disconnected. Then the following are equivalent:*

(1) (X, τ) is r-fuzzy weakly $\Re s$-compact.
(2) (X, τ) is r-fuzzy $\Re s$-compact.
(3) (X, τ) is r-fuzzy almost $\Re s$-compact.

Theorem 3.13. *For an r-fuzzy extremally disconnected fts (X, τ), the following are true:*

(1) r-fuzzy compactness implies r-fuzzy weakly $\Re s$-compactness.
(2) r-fuzzy nearly compactness implies r-fuzzy $\Re s$-compactness.
(3) r-fuzzy almost compactness implies r-fuzzy almost $\Re s$-compactness.

Proof. (2) Let (X, τ) be an r-fuzzy extremally disconnected and r-fuzzy nearly compact space, let $\{\lambda_i | i \in J\}$ be an r-fuzzy regular semiopen cover of X. Then there exists an r-fro set μ_i such that $\mu_i \leq \lambda_i \leq C_\tau(\mu_i, r)$ for each $i \in J$. Since (X, τ) is r-fuzzy extremally disconnected and $\mu_i = I_\tau(C_\tau(\mu_i, r))$ for each $i \in J$, $\lambda_i = I_\tau(\lambda_i, r)$ for each $i \in J$. Thus we get $\lambda_i = I_\tau(C_\tau(\lambda_i, r), r)$ for each $i \in J$ from Theorem 2.3. Hence (X, τ) is r-fuzzy $\Re s$-compact since X is r-fuzzy nearly compact.

(1) and (3) are similar to (2). □

Corollary 3.14. *If a fts (X, τ) is r-fuzzy extremally disconnected, then the following are equivalent:*

(1) r-fuzzy nearly compactness.
(2) r-fuzzy almost compactness.
(3) r-fuzzy S-closeness.

Proof. We get the results from Theorems 3.7, 3.12 and 3.13. □

4. r-FUZZY $\Re s$-CONNECTEDNESS

Definition 4.1. Let (X, τ) be a fts and $\lambda,\ \mu \in I^X$, $r \in I_0$. A r-fuzzy $\Re s$-separation on $\overline{1}$ is a pair of non null proper r-frso sets λ and μ such that $\lambda \wedge \mu = \overline{0}$ and $\lambda \vee \mu = \overline{1}$.

Definition 4.2. A fts (X, τ) is said to be r-fuzzy $\Re s$-connected if and only if there is no r-fuzzy $\Re s$-separation of $\overline{1}$. Otherwise, (X, τ) is said to be r-fuzzy $\Re s$-disconnected space.

Example 4.3. Let $X = \{a, b, c\}$, $\lambda, \mu, \delta \in I^X$, $r \in I_0$ are defined as $\lambda(a) = 0.2, \lambda(b) = 0.3, \lambda(c) = 0.4;\ \mu(a) = 0.6, \mu(b) = 0.3, \mu(c) = 0.4;\ \delta(a) = 0.7, \delta(b) = 0.4, \delta(c) = 0.5$. We define fuzzy topology $\tau : I^X \to I$ as follows:

$$\tau(\lambda) = \begin{cases} 1 & \text{if } \lambda \in \{\overline{0}, \overline{1}\}, \\ \frac{1}{3} & \text{if } \lambda = \lambda, \\ 0 & \text{otherwise.} \end{cases}$$

For $r = \frac{1}{3}$, μ and δ are r-frso sets in (X, τ), $\mu \neq \overline{0}$, $\delta \neq \overline{0}$, $\mu \vee \delta \neq \overline{1}$ and $\mu \wedge \delta \neq \overline{0}$. Hence (X, τ) is r-fuzzy $\Re s$-connected.

Proposition 4.1. *A fts (X, τ) is a r-fuzzy $\Re s$-connected if and only if there exists no non-null r-frso sets $\lambda,\ \mu \in I^X$, $r \in I_0$ such that $\lambda = \overline{1} - \mu$.*

Proof. Necessity: Let λ and μ be two r-frso sets in (X, τ) such that $\lambda \neq \overline{0},\ \overline{1} - \mu \neq \overline{0}$ and $\lambda = \overline{1} - \mu$. Therefore $\overline{1} - \mu$ is a r-frsc set. Since $\lambda \neq \overline{0},\ \mu \neq \overline{1}$. This implies that μ is a proper fuzzy set which is both r-frso and r-frsc in (X, τ). Hence (X, τ) is not a r-fuzzy $\Re s$-connected space. But this is a contradiction to our hypothesis. Thus there exists no non-null r-frso sets λ and μ in (X, τ) such that $\lambda = \overline{1} - \mu$.

Sufficiency: Let λ be both r-frso and r-frsc in (X, τ) such that $\lambda \neq \overline{0}$, $\lambda \neq \overline{1}$. Let $\overline{1} - \lambda = \mu$. Then μ is a r-frso set and $\overline{1} - \mu \neq \overline{1}$. This implies that $\mu = \overline{1} - \lambda \neq \overline{0}$, which is a contradiction to our hypothesis. Hence (X, τ) is a r-fuzzy $\Re s$-connected space. □

Proposition 4.2. *A fts (X, τ) is a r-fuzzy $\Re s$-connected space if and only if there exists no non-null r-frso sets $\lambda,\ \mu \in I^X$ with $r \in I_0$ such that $\lambda = \overline{1} - \mu$, $\mu = \overline{1} - RSC_\tau(\lambda)$ and $\lambda = \overline{1} - RSC_\tau(\mu)$.*

Proof. Necessity: Assume that there exists a fuzzy sets λ and μ such that $\lambda \neq \overline{0},\ \overline{1} - \mu \neq \overline{0}$, $\lambda = \overline{1} - \mu$, $\mu = \overline{1} - RSC_\tau(\lambda)$ and $\lambda = \overline{1} - RSC_\tau(\mu)$. Since $\overline{1} - RSC_\tau(\lambda)$ and $\overline{1} - RSC_\tau(\mu)$ are r-frso sets in (X, τ), λ and μ are r-frso sets in (X, τ). This implies (X, τ) is not a r-fuzzy $\Re s$-connected space, which is a contradiction. Thus there exists no non-null r-frso sets λ and μ in (X, τ) such that $\lambda = \overline{1} - \mu$, $\mu = \overline{1} - RSC_\tau(\lambda)$ and $\lambda = \overline{1} - RSC_\tau(\mu)$.

Sufficiency: Let λ be both r-frso and r-frsc in (X, τ) such that $\lambda \neq \overline{0}$, $\lambda \neq \overline{1}$. Now by taking $\overline{1} - \lambda = \mu$, we obtain a contradiction to our hypothesis. Hence (X, τ) is a r-fuzzy $\Re s$-connected space. □

Definition 4.4. A fts (X, τ) is said to be r-fuzzy C_5-disconnected if there exists fuzzy set $\lambda \in I^X$, $r \in I_0$, which is both r-fo and r-fc set such that $\lambda \neq \overline{0}$ and $\lambda \neq \overline{1}$. If (X, τ) is not r-fuzzy C_5-disconnected then it is said to be r-fuzzy C_5-connected.

Proposition 4.3. *Let (X, τ) and (Y, σ) be two fts's. Let $f : (X, \tau) \to (Y, \sigma)$ is a fuzzy regular semi continuous and surjective function. If (X, τ) is r-fuzzy $\Re s$-connected, then (Y, σ) is a r-fuzzy C_5-connected.*

Proof. Let (X, τ) is r-fuzzy $\Re s$-connected. Suppose (Y, σ) is not a r-fuzzy C_5-connected space, then there exists a proper fuzzy set $\lambda \in I^Y$, $r \in I_0$ which is both r-fo and r-fc set. Since f is a fuzzy regular semi continuous function, $f^{-1}(\lambda)$ is both r-frso and r-frsc in

(X, τ). But this is a contradiction to hypothesis. Hence (Y, σ) is a r-fuzzy C_5-connected space. □

Definition 4.5. A fuzzy set in a fts (X, τ) is said to be r-frsco set, which is both r-frso and r-frsc set.

Definition 4.6. A fts (X, τ) is said to be r-fuzzy $\Re s$-C_5-disconnected if there exists r-frsco set $\lambda \in I^X$, $r \in I_0$ such that $\lambda \neq \overline{0}$ and $\lambda \neq \overline{1}$. If (X, τ) is not r-fuzzy $\Re s$-C_5-disconnected then it is said to be r-fuzzy $\Re s$-C_5-connected.

Proposition 4.4. *A fts (X, τ) is r-fuzzy $\Re s$-C_5 connected, then it is r-fuzzy $\Re s$-connected.*

Proof. Suppose that there exists non-null r-frso sets λ and μ such that $\lambda \vee \mu = \overline{1}$ and $\lambda \wedge \mu = \overline{0}$ (r-fuzzy $\Re s$-disconnected), then $\lambda = \lambda \vee \mu$ and $\lambda = \lambda \wedge \mu$. In other words, $\lambda = \overline{1} - \mu$. Hence λ is a r-frsco set which implies that (X, τ) is r-fuzzy $\Re s$-C_5-disconnected. □

Remark. *The converse of the above Proposition need not be true as shown by the following example.*

Example 4.7. Let $X = \{a, b, c\}$, $\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \mu, \delta \in I^X$ are defined as $\lambda_1(a) = 0.4, \lambda_1(b) = 0.5, \lambda_1(c) = 0.6$; $\lambda_2(a) = 0.4, \lambda_2(b) = 0.5, \lambda_2(c) = 0.4$; $\lambda_3(a) = 0.5, \lambda_3(b) = 0.5, \lambda_3(c) = 0.5$; $\lambda_4(a) = 0.5, \lambda_4(b) = 0.5, \lambda_4(c) = 0.6$; $\lambda_5(a) = 0.4, \lambda_5(b) = 0.5, \lambda_5(c) = 0.5$; $\mu(a) = 0.5, \mu(b) = 0.5, \mu(c) = 0.4$; $\delta(a) = 0.6, \delta(b) = 0.5, \delta(c) = 0.6$. We define fuzzy topology $\tau : I^X \to I$ as follows:

$$\tau(\lambda) = \begin{cases} 1 & \text{if } \lambda \in \{\overline{0}, \overline{1}\}, \\ \frac{1}{3} & \text{if } \lambda = \lambda_1, \\ \frac{1}{3} & \text{if } \lambda = \lambda_2, \\ \frac{1}{3} & \text{if } \lambda = \lambda_3, \\ \frac{1}{3} & \text{if } \lambda = \lambda_4, \\ \frac{1}{3} & \text{if } \lambda = \lambda_5, \\ 0 & \text{otherwise.} \end{cases}$$

For $r = \frac{1}{3}$, The fuzzy sets μ and δ are r-frso sets over $\overline{1}$ (since there exist r-fro set λ_1 such that $\lambda_1 \leq \mu \leq C_\tau(\lambda_1) = \overline{1} - \lambda_4$ and there exist r-fro set λ_4 such that $\lambda_4 \leq \delta \leq C_\tau(\lambda_4) = \overline{1} - \lambda_2$). Also, $\mu \wedge \delta = \mu \neq \overline{0}$, $\mu \vee \delta = \delta \neq \overline{1}$, hence (X, τ) is r-fuzzy $\Re s$-connected. But (X, τ) is r-fuzzy $\Re s$-C_5-disconnected, since λ_3 is both r-frso set and r-frsc set.

Proposition 4.5. *Let (X, τ) and (Y, σ) be fts's. Let $f : (X, \tau) \to (Y, \sigma)$ be a fuzzy regular semi irresolute and surjective function. If (X, τ) is r-fuzzy $\Re s$-connected, then (Y, σ) is r-fuzzy $\Re s$-connected.*

Proof. Assume that (Y, σ) is not r-fuzzy $\Re s$-connected. Thus there exists non-null r-frso sets $\lambda, \mu \in I^Y$, $r \in I_0$ such that $\lambda \vee \mu = \overline{1}$ and $\lambda \wedge \mu = \overline{0}$. Since f is fuzzy regular semi irresolute function, $\nu = f^{-1}(\lambda)$, $\eta = f^{-1}(\mu)$ are r-frso sets in (X, τ). From $\lambda \neq \overline{0}$, we get $\nu = f^{-1}(\lambda) \neq \overline{0}$. (If $f^{-1}(\lambda) \neq \overline{0}$, then $\lambda = f(f^{-1}(\lambda)) = f(\overline{0}) = \overline{0}$, which is a contradiction.) Similarly we obtain $\eta = \overline{0}$. Now, $\lambda \vee \mu = \overline{1}$ $f^{-1}(\lambda) \vee f^{-1}(\mu) = f^{-1}(\overline{1})$, $\nu \vee \eta = \overline{1}$, $\lambda \wedge \mu = \overline{0}$ $f^{-1}(\lambda) \wedge f^{-1}(\mu) = f^{-1}(\overline{0})$ $\nu \wedge \eta = \overline{0}$. This implies that $\nu \vee \eta = \overline{1}$ and $\nu \wedge \eta = \overline{0}$. Thus (X, τ) is r-fuzzy $\Re s$-connected, which is a contradiction to our hypothesis. Hence (Y, σ) is r-fuzzy $\Re s$-connected. □

Proposition 4.6. *A fts (X, τ) is r-fuzzy $\Re s$-C_5-connected if and only if there exists no non-null r-frso sets $\lambda, \mu \in I^X$, $r \in I_0$ such that $\lambda = \overline{1} - \mu$.*

Proof. Suppose that λ and μ are r-frso sets in X such that $\lambda \neq \overline{0},\ \mu \neq \overline{0},\ \lambda = \overline{1} - \mu$. Since $\lambda = \overline{1} - \mu$, $\overline{1} - \mu$ is a r-frso set and μ is a r-frsc set. And $\lambda \neq \overline{0}$ implies $\mu \neq \overline{1}$. But this is a contradiction to the fact that (X, τ) is r-fuzzy $\Re s$-C_5-connected.

Conversely, let λ be both r-frso and r-frsc in X such that $\lambda \neq \overline{0}$, $\lambda \neq \overline{1}$. Now take $\mu = \overline{1} - \lambda$. In this case μ is a r-frso set and $\lambda \neq \overline{1}$. Which implies that $\mu = \overline{1} - \lambda = \overline{0}$, which is a contradiction. □

Proposition 4.7. *A fts (X, τ) is r-fuzzy $\Re s$-C_5-connected if and only if there exists no non-null fuzzy sets $\lambda,\ \mu \in I^X,\ r \in I_0$ such that $\overline{1} - \lambda = \mu$, $\mu = \overline{1} - RSC_\tau(\lambda)$, $\lambda = \overline{1} - RSC_\tau(\mu)$.*

Proof. Assume that there exists a fuzzy sets λ and μ such that $\lambda \neq \overline{0},\ \mu \neq \overline{0}, \overline{1} - \lambda = \mu$, $\mu = \overline{1} - RSC_\tau(\lambda)$ and $\lambda = \overline{1} - RSC_\tau(\mu)$. Since $\overline{1} - RSC_\tau(\lambda)$ and $\overline{1} - RSC_\tau(\mu)$ are r-frso sets over X, λ and μ are r-frso sets in X, which is a contradiction.

Conversely, let λ be both r-frso and r-frsc in X such that $\lambda \neq \overline{0}$, $\lambda \neq \overline{1}$. Taking $\mu = \overline{1} - \lambda$, we obtain a contradiction. □

Definition 4.8. A fts (X, τ) is said to be r-fuzzy $\Re s$-strongly connected if there exists no non-null r-frsc sets $\lambda,\ \mu \in I^X,\ r \in I_0$ such that $\lambda + \mu \leq \overline{1}$.

In otherwords, a fts (X, τ) is said to be r-fuzzy $\Re s$-strongly connected if there exists no non-null r-frsc sets $\lambda,\ \mu \in I^X,\ r \in I_0$ such that $\lambda \wedge \mu = \overline{1}$.

Proposition 4.8. *A fts (X, τ) is r-fuzzy $\Re s$-strongly connected if and only if there exists no non-null r-frso sets $\lambda,\ \mu \in I_X$ with $r \in I_0$ such that $\lambda \neq \overline{1},\ \mu \neq \overline{1}$ and $\lambda + \mu \geq \overline{1}$.*

Proof. Necessity: Let λ and μ are r-frso sets in (X, τ) such that $\lambda \neq \overline{1},\ \mu \neq \overline{1}$ and $\lambda + \mu \geq \overline{1}$. If we take $\nu = \overline{1} - \lambda$ and $\eta = \overline{1} - \mu$, then ν and η become r-frsc sets in X and $\nu \neq \overline{0},\ \eta \neq \overline{0}$ and $\nu + \eta \leq \overline{1}$. Which is a contradiction. Hence (X, τ) is r-fuzzy $\Re s$-strongly connected.

Sufficiency: Let λ and μ be non-null r-frsc sets in (X, τ) such that $\lambda + \mu \leq \overline{1}$. If $\nu = \overline{1} - \lambda$ and $\eta = \overline{1} - \mu$, then ν and η become r-frso sets in (X, τ) and $\nu \neq \overline{1}, \eta \neq \overline{1}$ and $\nu + \eta \geq \overline{1}$. Which is a contradiction. Thus there exists no non-null r-frso sets λ and μ in (X, τ) such that $\lambda \neq \overline{1}, \mu \neq \overline{1}$ and $\lambda + \mu \geq \overline{1}$. □

Proposition 4.9. *Let (X, τ) and (Y, σ) be fts's. Let $f : (X, \tau) \to (Y, \sigma)$ be a fuzzy regular semi irresolute and surjective function. If (X, τ) is r-fuzzy $\Re s$-strongly connected, then (Y, σ) is r-fuzzy $\Re s$-strongly connected.*

Proof. Suppose that (Y, σ) is not r-fuzzy $\Re s$-strongly connected. Then there exists non-null r-frsc sets ν_1 and ν_2 in (Y, σ) such that $\nu_1 \neq \overline{0}$, $\nu_2 \neq \overline{0}$, $\nu_1 + \nu_2 \leq \overline{0}$. Since f is fuzzy regular semi irresolute function, $f^{-1}(\nu_1)$, $f^{-1}(\nu_2)$ are r-frsc sets in (X, τ) and $f^{-1}(\nu_1) \wedge f^{-1}(\nu_2) = \overline{0}$, $f^{-1}(\nu_1) \neq \overline{0}$, $f^{-1}(\nu_2) \neq \overline{0}$. (If $f^{-1}(\nu_1) = \overline{0}$, then $f(f^{-1}(\nu_1)) = \nu_1$ which implies $f(\overline{0}) = \nu_1$. So $\overline{0} = \nu_1$ a contradiction.) Hence (X, τ) is r-fuzzy $\Re s$-strongly connected, a contradiction to our hypothesis. Thus (Y, σ) is r-fuzzy $\Re s$-strongly connected. □

Remark. *r-fuzzy $\Re s$-strongly connected does not imply r-fuzzy $\Re s$-C_5-connected.*

Example 4.9. In Example 4.7, (X, τ) is r-fuzzy $\Re s$-strongly connected, since there is no r-frsc sets $\lambda_1, \lambda_2, \lambda_1 + \lambda_2 \leq \overline{1}$. But (X, τ) is r-fuzzy $\Re s$-C_5-disconnected.

Remark. *r-fuzzy $\Re s$-C_5-connected does not imply r-fuzzy $\Re s$-strongly connected.*

Example 4.10. In Example 4.3, (X,τ) is r-fuzzy $\Re s$-C_5-strongly connected, since there is no fuzzy set λ is both r-frso and r-frsc set. But (X,τ) is not r-fuzzy $\Re s$-strongly connected, since there is the r-frsc sets λ and μ, $\lambda+\mu \leq \overline{1}$.

Definition 4.11. Let (X,τ) be fts, $\lambda,\ \mu \in I^X,\ r \in I_0$. The non-null fuzzy sets λ and μ are said to be

(1) r-fuzzy $\Re s$-weakly separated if $RSC_\tau(\lambda) \leq \overline{1}-\mu$ and $RSC_\tau(\mu) \leq \overline{1}-\lambda$.
(2) r-fuzzy $\Re s$-q-separated if $RSC_\tau(\lambda) \wedge \mu = \overline{0} = \lambda \wedge RSC_\tau(\mu)$.

Definition 4.12. A fts (X,τ) is said to be r-fuzzy $\Re s$-C_W-disconnected if there exists r-fuzzy $\Re s$-weakly separated non-null fuzzy sets λ and μ in X such that $\lambda \vee \mu = \overline{1}$.

Example 4.13. Let $X=\{a,b,c\}$, $\lambda,\ \mu \in I^X$, $r \in I_0$ are defined as $\lambda(a)=0, \lambda(b)=1, \lambda(c)=0;\ \mu(a)=1, \mu(b)=0, \mu(c)=1$. We define fuzzy topology $\tau: I^X \to I$ as follows:

$$\tau(\lambda)=\begin{cases}1 & \text{if } \lambda \in \{\overline{0},\overline{1}\},\\ \frac{1}{3} & \text{if } \lambda=\lambda,\\ \frac{1}{3} & \text{if } \lambda=\mu,\\ 0 & \text{otherwise.}\end{cases}$$

For $r=\frac{1}{3}$, the fuzzy sets λ and μ are r-frso sets in (X,τ), $RSC_\tau(\lambda) \leq \overline{1}-\mu$, $RSC_\tau(\mu) \leq \overline{1}-\lambda$. Hence λ and μ are r-fuzzy $\Re s$-weakly separated and $\lambda \vee \mu = \overline{1}$. Hence (X,τ) is r-fuzzy $\Re s$-C_W-disconnected.

Definition 4.14. A fts (X,τ) is said to be r-fuzzy $\Re s$-C_Q-disconnected if there exists r-fuzzy $\Re s$-q-separated non-null fuzzy sets λ and μ in X such that $\lambda \vee \mu = \overline{1}$.

Example 4.15. In Example 4.13, the fuzzy sets λ and μ are r-frso sets, $RSC_\tau(\lambda) = \overline{1}-\mu \wedge \mu = \overline{0}$ and $RSC_\tau(\mu) = \overline{1}-\lambda \wedge \lambda = \overline{0}$. Hence λ and μ are r-fuzzy $\Re s$-q-separated and $\lambda \vee \mu = \overline{1}$. Thus (X,τ) is r-fuzzy $\Re s$-C_Q-disconnected.

Remark. *A fts (X,τ) is said to be r-fuzzy $\Re s$-C_W-connected if and only if (X,τ) is r-fuzzy $\Re s$-C_Q-connected.*

Definition 4.16. Let (X,τ) be a fts and $Y \leq X$. Let λ^Y is defined as follows $\lambda^Y(x) = \begin{cases}1 & \text{if } x \in Y\\ 0 & \text{if } x \notin Y\end{cases}$. Let $\tau_Y = \{\lambda^Y \wedge \mu : \tau(\mu) \geq r\}$, then the fuzzy topology τ_Y on Y is called fuzzy subspace topology and (Y,τ_Y) is called fuzzy subspace of (X,τ).

Definition 4.17. A fuzzy subspace (Y,τ_Y) of fts (X,τ) is said to be r-fuzzy $\Re s$-open (resp. r- fuzzy $\Re s$-closed, r-fuzzy $\Re s$-connected) subspace if $\lambda^Y \in FRSO(X)$ (resp. $\lambda^Y \in FRSC(X)$, λ^Y is r-fuzzy $\Re s$-connected).

Theorem 4.10. *Let (Y,τ_Y) be a r-fuzzy $\Re s$-connected subspace of fts (X,τ) such that $\gamma^Y \wedge \mu \in FRSO(X)$. If $\overline{1}$ has a r-fuzzy $\Re s$-separations λ and μ, then either $\gamma^Y \leq \lambda$ or $\gamma^Y \leq \mu$.*

Proof. Let $\lambda,\ \mu$ be r-fuzzy $\Re s$-separation on $\overline{1}$. By hypothesis, $\lambda \wedge \gamma^Y \in FRSO(X)$, $\mu \wedge \gamma^Y \in FRSO(X)$ and $[\lambda \wedge \gamma^Y] \vee [\mu \wedge \gamma^Y] = \gamma^Y$. Since γ^Y is r-fuzzy $\Re s$-connected. Then either $\lambda \wedge \gamma^Y = \overline{0}$ or $\mu \wedge \gamma^Y = \overline{0}$. Therefore, either $\gamma^Y \leq \lambda$ or $\gamma^Y \leq \mu$. □

Theorem 4.11. *If (X,τ_2) is a r-fuzzy $\Re s$-connected space and τ_1 is fuzzy coarser than τ_2, then (X,τ_1) is also a r-fuzzy $\Re s$-connected.*

Proof. Let $\lambda,\ \mu \in I^X$, $r \in I_0$ be r-fuzzy $\Re s$-separation on (X, τ_1). Then λ, μ are r-frso sets. Since $\tau_1 \leq \tau_2$. Then $\lambda,\ \mu$ in (X, τ_2) such that $\lambda,\ \mu$ is r-fuzzy $\Re s$-separation on (X, τ_2), which is a contradiction with the r-fuzzy $\Re s$-connectedness of (X, τ_2). Hence, (X, τ_1) is r-fuzzy $\Re s$-connected. □

Remark. *The converse of Theorem 4.11 is not true in general, as shown in the following example.*

Example 4.18. Let $X = \{a, b, c, d, e, f\}$, $\lambda_1,\ \lambda_2,\ \lambda_3,\ \lambda_4 \in I^X$, $r \in I_0$ are defined as $\lambda_1(a) = 1, \lambda_1(b) = 1, \lambda_1(c) = 1, \lambda_1(d) = 0, \lambda_1(e) = 0, \lambda_1(f) = 0$; $\lambda_2(a) = 0.2, \lambda_2(b) = 0.3, \lambda_2(c) = 0.4, \lambda_2(d) = 0, \lambda_2(e) = 0, \lambda_2(f) = 0$; $\lambda_3(a) = 0, \lambda_3(b) = 0, \lambda_3(c) = 0, \lambda_3(d) = 1, \lambda_3(e) = 1, \lambda_3(f) = 1$; $\lambda_4(a) = 0.2, \lambda_4(b) = 0.3, \lambda_4(c) = 0.4, \lambda_4(d) = 1, \lambda_4(e) = 1, \lambda_4(f) = 1$. We define fuzzy topology $\tau_1, \tau_2 : I^X \to I$ as follows:

$$\tau_1(\lambda) = \begin{cases} 1 & \text{if } \lambda \in \{\overline{0}, \overline{1}\}, \\ 0 & \text{otherwise.} \end{cases} \quad \tau_2(\lambda) = \begin{cases} 1 & \text{if } \lambda \in \{\overline{0}, \overline{1}\}, \\ \frac{1}{3} & \text{if } \lambda = \lambda_1, \\ \frac{1}{3} & \text{if } \lambda = \lambda_2, \\ \frac{1}{3} & \text{if } \lambda = \lambda_3, \\ \frac{1}{3} & \text{if } \lambda = \lambda_4, \\ 0 & \text{otherwise.} \end{cases}$$

Let τ_1 be the indiscrete fuzzy $\Re s$-topology, then τ_1 is r-fuzzy $\Re s$-connected, on the other hand, Clearly, τ_2 defines a fuzzy topology on X such that $\tau_1 \leq \tau_2$. For $r = \frac{1}{3}$, λ_1 and λ_3 are r-frso sets in which form a r-fuzzy $\Re s$-separation of (X, τ_2) where $\lambda_1 \wedge \lambda_3 = \overline{0}$ and $\lambda_1 \vee \lambda_3 = \overline{1}$. Hence (X, τ_2) is r-fuzzy $\Re s$-disconnected.

Theorem 4.12. *A fuzzy subspace (Y, τ_Y) of a r-fuzzy $\Re s$-disconnected space (X, τ) is r-fuzzy $\Re s$-disconnected if $\gamma^Y \wedge \mu \in FRSO(X)$, $\forall \mu \in FRSO(X)$.*

Proof. Let (Y, τ_Y) be r-fuzzy $\Re s$-connected. Since (X, τ) is r-fuzzy $\Re s$-disconnected. Then there exists r-fuzzy $\Re s$-separation $\lambda,\ \mu$ on (X, τ). By hypothesis, $\lambda \wedge \gamma^Y \in FRSO(X)$, $\mu \wedge \gamma^Y \in FRSO(X)$ and $[\lambda \wedge \gamma^Y] \vee [\mu \wedge \gamma^Y] = \gamma^Y$, which is a contradiction with the r-fuzzy $\Re s$-connectedness of (Y, τ_Y). Therefore (Y, τ_Y) is r-fuzzy $\Re s$-disconnected. □

Remark. *A r-fuzzy $\Re s$-disconnectedness property is not hereditary property in general, as in the following example.*

Example 4.19. In Example 4.18, let $Y = \{a, b\} \leq X$. We consider the fuzzy set λ^Y of Y defined as follows, $\lambda^Y(a) = 1, \lambda^Y(b) = 1$. Then we define fuzzy subspace topology $\tau_Y : I^Y \to I$ as follows:

$$\tau_Y(\lambda) = \begin{cases} 1 & \text{if } \lambda \in \{\overline{0}, \overline{1}\}, \\ \frac{1}{3} & \text{if } \lambda = \lambda^Y \wedge \lambda_1, \\ \frac{1}{3} & \text{if } \lambda = \lambda^Y \wedge \lambda_2, \\ \frac{1}{3} & \text{if } \lambda = \lambda^Y \wedge \lambda_3, \\ \frac{1}{3} & \text{if } \lambda = \lambda^Y \wedge \lambda_4, \\ 0 & \text{otherwise.} \end{cases}$$

Thus, the collection $\tau_Y = \{\lambda^Y \wedge \mu : \tau(\mu) \geq r\}$ is a fuzzy subspace topology on Y in which there is no r-fuzzy $\Re s$-separation on (Y, τ_Y). Therefore, (Y, τ_Y) is r-fuzzy $\Re s$-connected at the time that (X, τ) is r-fuzzy $\Re s$-disconnected as shown in Example 4.18.

5. Conclusion

Sostak's fuzzy topology has been recently of major interest among fuzzy topologies. In this paper, we have introduced r-fuzzy regular semi compactness and gave basic definition and theorems of the concept. Also, we introduce r-fuzzy regular semi connectedness, r-fuzzy regular semi strongly connectedness and r-fuzzy regular semi-C_5-connectedness. Some interesting properties of these notions are studied.

6. Acknowledgements

The author would like to thank from the anonymous reviewers for carefully reading of the manuscript and giving useful comments, which will help us to improve the paper.

References

[1] B. Amudhambigai, M. K. Uma, and E. Roja. On rarely $\widetilde{g}$-continuous functions in smooth fuzzy topological spaces, The Journal of Fuzzy Mathematics, 20(2)(2012), 433–442.
[2] C. L. Chang. Fuzzy topological spaces, J. Math. Anal. Appl., 24(1968), 182–190.
[3] K. C. Chattopadhyay, and S. K. Samanta. Fuzzy topology, Fuzzy Sets and Systems, 54(1993), 207–212.
[4] K. C. Chattopadhyay, R. N. Hazra, and S. K. Samanta. Gradation of openness, Fuzzy Sets and Systems, 49(2) (1992), 237–242.
[5] C. K. Park, and W. K. Min. R-Semi-generalized fuzzy compactness, Korean J. Math., 16(3) 291–300.
[6] E. E. Kerre, and M. A. Fath Alla. On fuzzy regular semi open sets and αS^*-closed fuzzy topological spaces, The Journal of Fuzzy Mathematics, 11(1)(2003), 225–235.
[7] R. N. Hazra, S. K. Samanta, and K. C. Chattopadhyay. Fuzzy topology redefined, Fuzzy Sets and Systems, 45(1992), 79–82.
[8] A. S. Mashhour, M. H. Ghanim, and M. A. Fath Alla. α-separation axioms and α-compactness in fuzzy topological spaces, Rocky Moutain Journal of Mathematics, 16(3)(1986), 591–600.
[9] A. A. Ramadan, S. E. Abbas, and Y. C. Kim. Fuzzy irresolute mappings in smooth fuzzy topological spaces, The Journal of Fuzzy Mathematics, 9(4)(2001), 865–877.
[10] Seok Jong Lee, and Eun Pyo Lee. Fuzzy r-regular open sets and fuzzy almost r-continuous maps, Bull. Korean Math. Soc., 39(3)(2002), 441–453.
[11] A. P. $\check{S}$ostak. On a fuzzy topological structure, Rend. Circ. Matem. Palermo Ser II, 11(1986), 89–103.
[12] A. Vadivel, and E. Elavarasan. r-fuzzy regular semi open sets in smooth topological spaces, Sahand Communications i Mathematical Analysis, 6(1)(2017), 1–17.
[13] A. M. Zahran. Fuzzy regular semi open sets and αS-closed spaces, The Journal of Fuzzy Mathematics, 2(4)(1997), 579–586.

E. Elavarasan
Department of Mathematics, Shree Raghavendra Arts and Science College (Affiliated to Thiruvalluvar University), Keezhamoongiladi, Chidambaram-608102, Tamil Nadu, India.
Email address: maths.aras@gmail.com