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FUZZY STABILITY RESULTS OF ADDITIVE FUNCTIONAL EQUATION IN DIFFERENT APPROCHES

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ABSTRACT. In this paper, we investigate some stability results of the following finite dimensional additive functional equation

$$f\left(\sum_{i=1}^{n} kx_i\right) + \sum_{j=1}^{n} f\left(-kx_j + \sum_{i=1;i\neq j}^{n} kx_i\right) = (n-1)\left[\sum_{i=1}^{n}(2i-1)f(x_i)\right]$$

where n is the positive integer with $N - \{0, 1, 2\}$ and k is the only odd positive integers, in Fuzzy Normed space using direct and fixed point approaches.

1. INTRODUCTION

Hyers [7] gave a positive answer to the question of Ulam for Banach spaces. In 1950, T. Aoki [3] was the second author to treat this problem for additive mappings. One of the most famous functional equations is the additive functional equation

$$f(x+y) = f(x) + f(y) \tag{1.1}$$

In 1821, it was first solved by A. L. Cauchy in the class of continuous real-valued function. It is often called Cauchy additive functional equation in honor of A. L. Cauchy. The theory of additive functional equations is frequently applied to the development of theories of other functional equations. Moreover, the properties of additive functional equations are tools in almost every field of natural and social sciences. Every solution of the additive functional equation (1.1) is called an additive function. The solution and stability of the following additive functional equations

$$f(x+ay) + af(x-y) = f(x-ay) + af(x+y), \quad a \neq -1, 0, 1 \tag{1.2}$$

$$f(2x-y) + f(x-2y) = 3f(x) - 3f(y) \tag{1.3}$$

were studied by K. W. Jun and H. M. Kim [8] and D. O. Lee [16]. Few functional equation papers which were discussed additive and non-additive functional equations properties in fuzzy normed spaces [see ([1, 2, 6, 15, 17, 18, 22]. Mishra et al., and Tamilvanan et al.,

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were used to develop this work as more clear (see. Ref.[4, 9, 10, 11, 12, 13, 14, 19, 20, 21]). In this paper, the authors investigate the stability of the n-dimensional additive functional equation

$$f\left(\sum_{i=1}^{n} kx_i\right)+\sum_{j=1}^{n} f\left(-kx_j+\sum_{i=1;i\neq j}^{n} kx_i\right)=(n-1)\left[\sum_{i=1}^{n}(2i-1)f(x_i)\right] \quad (1.4)$$

where n is the positive integer with $N-\{0,1,2\}$ and k is the only odd positive integers in Fuzzy Normed Space is discussed.

2. Fuzzy Stability Results

In this section, the authors present basic definition in fuzzy normed space and investigate the fuzzy stability of the finite dimensional functional equation (1.4).

Definition 2.1. Let X be a real linear space. A function $N : X\times\mathbb{R}\longrightarrow[0,1]$ is said to be fuzzy norm on X if for all $x,y\in X$ and $a,b\in\mathbb{R}$.
(N_1) $N(x,a)=0$ for $a\leq 0$;
(N_2) $x=0$ iff $N(x,a)=1$ for all $a>0$;
(N_3) $N(ax,b)=N\left(x,\frac{b}{|a|}\right)$ if $a\neq 0$
(N_4) $N(x+y,a+b)\geq min\{N(x,a),N(y,b)\}$;
(N_5) $N(x,.)$ is a non-decreasing function on $\mathbb{R}$ and $lim_{a\longrightarrow\infty}N(x,a)=1$.
(N_6) For $x\neq 0, N(x,.)$ is continuous on $\mathbb{R}$.

The pair (X,N) is called a fuzzy normed linear space. One may regard $N(x,a)$ as the truth value of the statement the norm of x is less than or equal to the real number a.

Definition 2.2. Let (X,N) be a fuzzy normed linear space. Let x_n be a sequence in X. Then x_n is said to be convergent if there exists $x\in X$ such that $lim_{n\to\infty}N(x_n-x,t)=1$ for all $t>0$. In that case, x is called the limit of the sequence x_n and we denote it by $N-lim_{n\to\infty}x_n=x$.

Definition 2.3. A sequence x_n in X is called Cauchy if for each $\epsilon>0$ and each $t>0$ there exists n_0 such that for all $n\geq n_0$ and all $p>0$, we have $N(x_{n+p}-x_n,t)>1-\epsilon$.

Definition 2.4. Every convergent sequence in a fuzzy normed space is Cauchy. If each Cauchy sequence is convergent, then the fuzzy norm is said to be complete and the fuzzy normed space is called a fuzzy Banach space.

Definition 2.5. A mapping $f : X\to Y$ between fuzzy normed spaces X and Y is continuous at a point x_0 if for each sequence $\{x_n\}$ converging to x_0 in X, the sequence $f\{x_n\}$ converges to $f\{x_0\}$. If f is continuous at each point of $x_0\in X$, then f is said to be continuous on X.

In section 3 and 4, assume that X, (Z,F') and (Y,F) are linear space, Fuzzy Normed space and Fuzzy Banach space respectively. We define a function $f : X\to Y$ by

$$Df(x_1,x_2,\cdots,x_n)=f\left(\sum_{i=1}^{n} kx_i\right)+\sum_{j=1}^{n} f\left(-kx_j+\sum_{i=1;i\neq j}^{n} kx_i\right)-(n-1)\left[\sum_{i=1}^{n}(2i-1)f(x_i)\right]$$

for all $x_1,x_2,\cdots,x_n\in X$.

3. Stability of the functional equation (1.4)- Direct Method

Theorem 3.1. *Let $\beta \in \{1, -1\}$ be fixed and let $\psi : X^n \longrightarrow Z$ be a mapping such that for some $d > 0$ with $0 < \left(\frac{d}{3}\right)^{\beta} < 1$.*

$$N'\left(\psi(0, 2^{\beta}x, 0, \cdots, 0), r\right) \geq N'(d^{\beta}\psi(0, x, 0, \cdots, 0), r) \tag{3.1}$$

for all $x \in X$ and all $r > 0, d > 0$, and

$$lim_{k\longrightarrow\infty} N'\left(\psi(3^{\beta k}x_1, 3^{\beta k}x_2, ..., 3^{\beta k}x_n), 3^{\beta k}r\right) = 1 \tag{3.2}$$

for all $x_1, x_2, ..., x_n \in X$ and all $r > 0$. Suppose that a function $f : X \longrightarrow Y$ satisfies the inequality

$$N(Df(x_1, x_2, ..., x_n), r) \geq N'(\psi(x_1, x_2, ..., x_n), r) \tag{3.3}$$

for all $r > 0$ and $x_1, x_2, ..., x_n \in X$. Then the limit

$$A(x) = N - lim_{k\longrightarrow\infty} \frac{f(3^{\beta k}x)}{3^{\beta k}} \tag{3.4}$$

exists for all $x \in X$ and the mapping $A : X \longrightarrow Y$ is the unique additive mapping such that

$$N(f(x) - A(x), r) \geq N'(\psi(0, x, 0, \cdots, 0), (n-1)r|3-d|) \tag{3.5}$$

for all $x \in X$ and for all $r > 0$.

Proof. First assume that $\beta = 1$. Letting $(x_1, x_2, x_3, ..., x_n)$ by $(0, x, 0, ..., 0)$ in (3.3), we get

$$N\left(3(n-1)f(x) - (n-1)f(3x), r\right) \geq N'(\psi(0, x, 0, ..., 0), r) \tag{3.6}$$

for all $x \in X$ and all $r > 0$. Replacing x by $3^k x$ in (3.6), we obtain

$$N\left(\frac{f(3^{k+1}x)}{3} - f(3^k x), \frac{r}{3(n-1)}\right) \geq N'(\psi(0, 3^k x, 0, ..., 0), r) \tag{3.7}$$

for all $x \in X$ and for all $r > 0$. Using (3.1), we get

$$N\left(\frac{f(3^{k+1}x)}{3} - f(3^k x), \frac{r}{3(n-1)}\right) \geq N'(\psi(0, x, 0, ..., 0), \frac{r}{d^k}) \tag{3.8}$$

for all $x \in X$ and for all $r > 0$. It is easy to verify from (3.8) that

$$N\left(\frac{f(3^{k+1}x)}{3^{k+1}} - \frac{f(3^k x)}{3^k}, \frac{r}{3(n-1)3^k}\right) \geq N'(\psi(0, x, 0, ..., 0), \frac{r}{d^k}) \tag{3.9}$$

holds for all $x \in X$ and for all $r > 0$. Replacing r by $d^k r$ in (3.9),we get

$$N\left(\frac{f(3^{k+1}x)}{3^{k+1}} - \frac{f(3^k x)}{3^k}, \frac{d^k r}{3(n-1)3^k}\right) \geq N'(\psi(x, x, 0, ..., 0), r) \tag{3.10}$$

for all $x \in X$ and for all $r > 0$. It is follows from

$$\frac{f(3^k x)}{3^k} - f(x) = \sum_{i=0}^{k-1} \left[\frac{f(3^{i+1}x)}{3^{i+1}} - \frac{f(3^i x)}{3^i}\right] \tag{3.11}$$

for all $x \in X$. From the equations (3.10) and (3.11), we arrive

$$N\left(\frac{f(3^k x)}{3^k} - f(x), \sum_{i=0}^{k-1} \frac{d^i r}{3(n-1)3^i}\right)$$

$$\geq min \bigcup_{i=1}^{k-1} \left\{N\left(\frac{f(3^{i+1}x)}{3^{i+1}} - \frac{f(3^i x)}{3^i}, \frac{d^i r}{3(n-1)3^i}\right)\right\}$$

$$\geq min \bigcup_{i=1}^{k-1} N'(\psi(0,x,0,...,0),r)$$
$$\geq N'(\psi(0,x,0,...,0),r) \tag{3.12}$$

for all $x \in X$ and for all $r > 0$. Replacing x by $3^m x$ in (3.12), we obtain

$$N\left(\frac{f(3^{k+m}x)}{3^{k+m}} - \frac{f(3^m x)}{3^m}, \sum_{i=m}^{m+k-1} \frac{d^i r}{3(n-1)3^i}\right) \geq N'(\psi(0,x,0,...,0), \frac{r}{d^m}) \tag{3.13}$$

for all $x \in X$ and for all $r > 0$ and all $m, k \geq 0$. Replacing r by $d^m r$ in (3.13), we get

$$N\left(\frac{f(3^{k+m}x)}{3^{k+m}} - \frac{f(3^m x)}{3^m}, \sum_{i=0}^{k-1} \frac{d^i r}{3(n-1)3^i}\right) \geq N'(\psi(0,x,0,...,0), r) \tag{3.14}$$

for all $x \in X$ and for all $r > 0$ and all $m, k \geq 0$. Using (N_3) in (3.13), we obtain

$$N\left(\frac{f(3^{k+m}x)}{3^{k+m}} - \frac{f(3^m x)}{3^m}, r\right) \geq N'\left(\psi(0,x,0,...,0), \frac{r}{\sum_{i=m}^{m+k-1} \frac{d^i}{3(n-1)3^i}}\right) \tag{3.15}$$

for all $x \in X$, $r > 0$ and all $m, k \geq 0$. Since $0 < d < 3$ and $\sum_{i=0}^{k} \left(\frac{d}{3}\right)^i < \infty$, the Cauchy criterion for convergence and (N_5) implies that $\left\{\frac{f(3^k x)}{3^k}\right\}$ is a Cauchy sequence in (Y, N). Since $(Y, N)$is a fuzzy Banach Space, this sequence converges to some point $A(x) \in Y$. So one can define the mapping $A : X \longrightarrow Y$ by

$$A(x) := N - lim_{k \longrightarrow \infty} \frac{f(3^k x)}{3^k}$$

for all $x \in X$. Letting $m = 0$ in (3.15), we get

$$N\left(\frac{f(3^k x)}{3^k} - f(x), r\right) \geq N'\left(\psi(0,x,0,...,0), \frac{r}{\sum_{i=0}^{k-1} \frac{d^i}{3(n-1)3^i}}\right) \tag{3.16}$$

for all $x \in X$. Taking the limits as $k \longrightarrow \infty$ and using (N_6), we arrive

$$N(f(x) - A(x), r) \geq N'(\psi(0,x,0,...,0), 3(n-1)r(3-d))$$

for all $x \in X$ and for all $r > 0$. Now, we claim that A is additive. Replacing $(x_1, x_2, x_3, ..., x_n)$ by $(3^k x_1, 3^k x_2, ..., 3^k x_n)$ in (3.3), we obtain

$$N\left(\frac{1}{3^k} Df(3^k x_1, 3^k x_2, ..., 3^k x_n), r\right) \geq N'(\psi(3^k x_1, 3^k x_2, ..., 3^k x_n), 3^k r) \tag{3.17}$$

for all $r > 0$ and for all $x_1, x_2, ..., x_n \in X$. Since

$$lim_{k \longrightarrow \infty} N'\left(\psi(3^{\beta k} x_1, 3^{\beta k} x_2, ..., 3^{\beta k} x_n), 3^{\beta k} r\right) = 1$$

Hence A satisfies the additive functional equation (1.4). In order to prove $A(x)$ is unique, we let $A'(x)$ be another additive functional equation satisfying (1.4) and (3.5). Hence

$$N(A(x) - A'(x), r) = N\left(\frac{A(3^k x)}{3^k} - \frac{A'(3^k x)}{3^k}, r\right)$$
$$\geq min\left\{N\left(\frac{A(3^k x)}{3^k} - \frac{f(3^k x)}{3^k}, \frac{r}{2}\right), N\left(\frac{f(3^k x)}{3^k} - \frac{A'(3^k x)}{3^k}, \frac{r}{2}\right)\right\}$$
$$\geq N'\left(\psi(0, 3^k x, 0, ..., 0), \frac{3r(n-1)3^k(3-d)}{2}\right)$$

$$\geq N'\left(\psi(0, 3^k x, 0, ..., 0), \frac{3r(n-1)3^k(3-d)}{2d^k}\right)$$

for all $x \in X$ and $r > 0$. Since $lim_{k\longrightarrow\infty}\frac{3r(n-1)3^k(3-d)}{2d^k} = \infty$, we obtain

$$lim_{k\to\infty}N'\left(\psi(0, x, 0, ..., 0), \frac{3r(n-1)3^k(3-d)}{2d^k}\right) = 1$$

Thus $N(A(x) - A'(x), r) = 1$ for all $x \in X$ and $r > 0$. Hence $A(x) = A'(x)$. Therefore $A(x)$ is unique. For $\beta = -1$, we can prove the result by a similar method. This completes the proof of the theorem. □

The following corollary is an immediate consequence of Theorem 3.1, concerning the stability for the functional equation (1.4).

Corollary 3.2. *Suppose that the function $f : X \longrightarrow Y$ satisfies the inequality*

$$N(Df(x_1.x_2, ..., x_n), r) \geq \begin{cases} N'(\theta, r) \\ N'(\theta\sum_{i=1}^n ||x_i||^s, r) \\ N'(\theta(\sum_{i=1}^n ||x_i||^{ns} + \Pi_{i=1}^n||x_i||^s), r) \end{cases}$$

for all $x_1, x_2, ..., x_n \in X$ and all $r > 0$, where θ, s are constants then there exists a unique additive mapping $A : X \to Y$ such that

$$N(f(x) - A(x), r) \geq \begin{cases} N'(\theta, |2|r(n-1) \\ N'\left(\theta||x||^s, r(n-1) \mid 3 - 3^s \mid\right) & ; s \neq 1 \\ N'\left(\theta||x||^{ns}, r(n-1) \mid 3 - 3^{ns} \mid\right) & ; s \neq \frac{1}{n} \end{cases}$$

4. Stability of the Functional Equation (1.4) Fixed Point Method

In this section, the authors investigate the generalized Ulam-Hyers stability of the functional equation (1.4) in fuzzy normed space using fixed point method.

For to prove the stability result, we define the following μ_i is a constant such that

$$\eta_i = \begin{cases} 3 & if \quad i = 0 \\ \frac{1}{3} & if \quad i = 1 \end{cases}$$

and Ω is the set such that $\Omega = \{t/t : X \longrightarrow Y, t(0) = 0\}$.

Theorem 4.1. *Let $f : X \longrightarrow Y$ be a mapping for which there exists a function $\psi : X^n \longrightarrow Z$ with condition*

$$lim_{k\longrightarrow\infty}N'\left(\psi(\eta^k x_1, \eta^k x_2, ..., \eta^k x_n), \eta^k r\right) = 1 \tag{4.1}$$

for all $x_1, x_2, ..., x_n \in X$ and all $r > 0$ and satisfying the inequality

$$N(Df(x_1, x_2, ..., x_n), r) \geq N'(\psi(x_1, x_2, ..., x_n), r) \tag{4.2}$$

for all $x \in X$ and $r > 0$. If there exist $L = L[i]$ such that the function $x \longrightarrow \beta(x) = \frac{1}{(n-1)}\psi\left(0, \frac{x}{3}, 0, ..., 0\right)$ has the property

$$N'\left(L\frac{1}{\eta_i}\beta(\eta_i x), r\right) = N'\left(\beta(x), r\right) \tag{4.3}$$

for all $x \in X$ and $r > 0$, then there exist unique additive function $A : X \longrightarrow Y$ satisfying the functional equation (1.4) and

$$N(f(x) - A(x), r) \geq N'\left(\frac{L^{1-i}}{1-L}\beta(x), r\right)$$

for all $x \in X$ and $r > 0$.

Proof. Let d be a general metric on Ω such that

$$d(t,u) = inf\left\{k \in (0,\infty) | N(t(x) - u(x), r) \geq N'(\beta(x), kr), x \in X, r > 0\right\}$$

It is easy to see that (Ω, d) is complete. Define $T : \Omega \longrightarrow \Omega$ by $Tt(x) = \frac{1}{\eta_i}t(\eta_i x)$ for all $x \in X$,for $t, u \in \Omega$, we have

$$\begin{aligned} d(t,u) = k &\Rightarrow N\left(t(x) - u(x), r\right) \geq N'(\beta(x), kr) \\ &\Rightarrow N\left(\frac{t(\eta_i x)}{\eta_i} - \frac{u(\eta_i x)}{\eta_i}, r\right) \geq N'(\beta(\eta_i x), k\eta_i r) \qquad (4.4) \\ &\Rightarrow N(Tt(x) - Tu(x), r) \geq N'(\beta(\eta_i x), k\eta_i r) \\ &\Rightarrow N(Tt(x) - Tu(x), r) \geq N'(\beta(x), kLr) \\ &\Rightarrow d(Tt(x) - Tu(x)) \geq kL \\ &\Rightarrow d(Tt - Tu, r) \geq Ld(t,u) \end{aligned}$$

for all $t, u \in \Omega$.Therefore T is strictly contractive mapping on Ω with Lipschitz constant L, replacing $(x_1, x_2, x_3, ..., x_n)$ by $(0, x, 0, ..., 0)$ in (4.2), we get

$$N\left((n-1)f(3x) - 3(n-1)f(x), r\right) \geq N'\left(\psi(0, x, 0, ..., 0), r\right) \qquad (4.5)$$

for all $x \in X$ and $r > 0$. Using (N_3) in (4.5), we arrive

$$N\left(\frac{f(3x)}{3} - f(x), r\right) \geq N'\left(\frac{\psi(0, x, 0, ..., 0)}{(n-1)}, r\right) \qquad (4.6)$$

for all $x \in X$ and $r > 0$ with the help of (4.3) when $i = 0$, it follows from (4.6) that

$$\begin{aligned} &\Rightarrow N\left(\frac{f(3x)}{3} - f(x), r\right) \geq N'(L\beta(x), r) \\ &\Rightarrow d(Tf, f) \geq L = L^1 = L^{1-i}. \qquad (4.7) \end{aligned}$$

Replacing x by $\frac{x}{3}$ in (4.5), we obtain

$$N\left(f(x) - 3f\left(\frac{x}{3}\right), r\right) \geq N'\left(\frac{3}{(n-1)}\psi\left(0, \frac{x}{3}, 0, ..., 0\right), r\right) \qquad (4.8)$$

for all $x \in X$ and $r > 0$, when $i = 1$, it follows from (4.8), we get

$$\begin{aligned} &\Rightarrow N\left(f(x) - 3f\left(\frac{x}{3}\right), r\right) \geq N'(\beta(x), r) \\ &\Rightarrow T(f, Tf) \leq 1 = L^0 = L^{1-i}. \qquad (4.9) \end{aligned}$$

Then from (4.7) and (4.9), we can conclude

$$\Rightarrow T(f - Tf) \leq L^{1-i} < \infty.$$

Now from the fixed point alternative in both cases, it follows that there exists a fixed point A of T in Ω such that

$$A(x) = N - lim_{k \longrightarrow \infty}\frac{f(\eta^k x)}{\eta^k}$$

for all $x \in X$ and $r > 0$. Replacing $(x_1, x_2, ..., x_n)$ by $(\eta_i^k x_1, \eta_i^k x_2, ..., \eta_i^k x_n)$ in (4.2), we arrive

$$N\left(\frac{1}{\eta_i^k} Df(\eta_i^k x_1, \eta_i^k x_2, , \eta_i^k x_n), r\right) \geq N'(\psi(\eta_i^k x_1, \eta_i^k x_2, ..., \eta_i^k x_n), \eta_i^k r)$$

for all $r > 0$ and all $x_1, x_2, ..., x_n \in X$. By proceeding the same procedure of the Theorem 3.1 , we can prove the function $A : X \longrightarrow Y$ is additive and its satisfies the functional equation (1.4). By a fixed point alternative, since A is unique fixed point of T in the set

$$\Delta = \{f \in \Omega / d(f, A) < \infty\}.$$

Therefore A is a unique function such that

$$N(f(x) - A(x), r) \geq N'(\beta(x), kr)$$

for all $x \in X$ and $r > 0$. Again using the fixed point alternative, we obtain

$$d(f, A) \leq \frac{1}{1-L} d(f, Tf)$$
$$\Rightarrow d(f, A) \leq \frac{L^{1-i}}{1-L}$$
$$\Rightarrow N(f(x) - A(x), r) \geq N'\left(\beta(x) \frac{L^{1-i}}{1-L}, r\right)$$

for all $x \in X$ and $r > 0$. This completes the proof of the Theorem. $\square$

The following Corollary is an immediate consequence of Theorem 4.1 concerning the stability of (1.4).

Corollary 4.2. *Suppose a function $f : X \longrightarrow Y$ satisfies the inequality*

$$N(Df(x_1.x_2, ..., x_n), r) \geq \begin{cases} N'(\theta, r) \\ N'(\theta \sum_{i=1}^n ||x_i||^s, r) \\ N'(\theta(\sum_{i=1}^n ||x_i||^{ns} + \Pi_{i=1}^n ||x_i||^s), r) \end{cases}$$

for all $x_1, x_2, ..., x_n \in X$ and $r > 0$, where θ, s are constants with $\theta > 0$. Then there exists a unique additive mapping $A : X \longrightarrow Y$ such that

$$N(f(x) - A(x), r) \geq \begin{cases} N'(\theta, r(n-1) \mid 2 \mid \\ N'\left(\theta ||x||^s, r(n-1) \mid 3 - 3^s \mid\right) & ; s \neq 1 \\ N'\left(\theta ||x||^{ns}, r(n-1) \mid 3 - 3^{ns} \mid\right) & ; s \neq \frac{1}{n} \end{cases}$$

for all $x \in X$ and $r > 0$.

Proof. Setting

$$\psi(x_1, x_2, x_3, ..., x_n) \leq \begin{cases} \theta \\ \theta(\sum_{i=1}^n ||x_i||^s) \\ \theta(\prod_{i=1}^n ||x_i||^s + \sum_{i=1}^n ||x_i||^{ns}) \end{cases}$$

for all $x_1, x_2, ..., x_n \in X$. Then

$$N'\left(\psi\left(\eta_i^k x_1, \eta_i^k x_2, ..., \eta_i^k x_n\right), \eta_i^k r\right) = \begin{cases} N'(\theta, \eta_i^k r) \\ N'\left(\theta \sum_{i=1}^n ||x_i||^s, \eta_i^{(1-s)k} r\right) \\ N'\left(\theta(\sum_{i=1}^n ||x_i||^{ns} + \Pi_{i=1}^n ||x_i||^s), \eta_i^{(1-ns)k} r\right) \end{cases}$$

$$= \begin{cases} \longrightarrow & 1 \quad as \quad k \longrightarrow \infty, \\ \longrightarrow & 1 \quad as \quad k \longrightarrow \infty, \\ \longrightarrow & 1 \quad as \quad k \longrightarrow \infty. \end{cases}$$

Thus, (4.1) is holds. But we have

$$\beta(x) = \frac{1}{(n-1)}\psi\left(0, \frac{x}{3}, 0, ..., 0\right)$$

has the property

$$N'\left(L\frac{1}{\eta_i}\beta(\eta_i x), r\right) \geq N'(\beta(x), r)$$

for all $x \in X$ and $r > 0$. Hence

$$N'(\beta(x), r) = N'\left(\psi\left(0, \frac{x}{3}, 0, ..., 0\right), (n-1)r\right)$$

$$= \begin{cases} N'(\theta, r(n-1)) \\ N'\left(\frac{1}{3^s}\theta||x||^s, r(n-1)\right) \\ N'\left(\frac{1}{3^{ns}}\theta||x||^{ns}, r(n-1)\right). \end{cases}$$

Now,

$$N'\left(\frac{1}{\eta_i}\beta(\eta_i x), r\right) = \begin{cases} N'\left(\frac{\theta}{\eta_i}, r(n-1)\right) \\ N'\left(\frac{\theta}{\eta_i}\left(\frac{1}{3^s}\right)||\eta_i x||^s, r(n-1)\right) \\ N'\left(\frac{\theta}{\eta_i}\left(\frac{1}{3^{ns}}\right)||\eta_i x||^{ns}, r(n-1)\right) \end{cases}$$

$$= \begin{cases} N'(\eta_i^{-1}\beta(x), r) \\ N'(\eta_i^{s-1}\beta(x), r) \\ N'(\eta_i^{ns-1}\beta(x), r) \end{cases}$$

Now from the following cases for the conditions (i) and (vi).

Case(i):$L = 3^{-1} \quad for \quad s = 0 \quad if \quad i = 0.$

$$N(f(x) - A(x), r) \geq N'\left(\frac{L^{1-i}}{1-L}\beta(x), r\right) \geq N'\left(\frac{3^{-1}}{1-3^{-1}}\frac{\theta}{(n-1)}, r\right) \geq N'\left(\theta, 2r(n-1)\right)$$

Case(ii):$L = \left(\frac{1}{3}\right)^{-1} \quad for \quad s = 0 \quad if \quad i = 1.$

$$N(f(x)-A(x), r) \geq N'\left(\frac{L^{1-i}}{1-L}\beta(x), r\right) \geq N'\left(\frac{1}{1-\left(\frac{1}{3}\right)^{-1}}\frac{\theta}{(n-1)}, r\right) \geq N'\left(\theta, -2r(n-1)\right)$$

Case(iii):$L = (3)^{s-1} \quad for \quad s < 1 \quad if \quad i = 0.$

$$N(f(x) - A(x), r) \geq N'\left(\frac{L^{1-i}}{1-L}\beta(x), r\right) \geq N'\left(\frac{3^{s-1}}{1-3^{s-1}}\frac{\theta||x||^s}{(n-1)3^s}, r\right)$$
$$\geq N'\left(\theta||x||^s, r(n-1)(3-3^s)\right)$$

Case(iv):$L = (3)^{1-s} \quad for \quad s > 1 \quad if \quad i = 1.$

$$N(f(x) - A(x), r) \geq N'\left(\frac{L^{1-i}}{1-L}\beta(x), r\right) \geq N'\left(\frac{3^{1-s}}{1-3^{1-s}}\frac{\theta||x||^s}{(n-1)3^s}, r\right)$$
$$\geq N'\left(\theta||x||^s, r(n-1)(3^s-3)\right)$$

Case(v): $L = (3)^{ns-1} \quad for \quad s < \frac{1}{n} \quad if \quad i = 0.$

$$\begin{aligned} N(f(x) - A(x), r) &\geq N'\left(\frac{L^{1-i}}{1-L}\beta(x), r\right) \geq N'\left(\frac{3^{ns-1}}{1-3^{ns-1}}\frac{\theta||x||^{ns}}{(n-1)3^{ns}}, r\right) \\ &\geq N'\left(\theta||x||^{ns}, r(n-1)(3-3^{ns})\right) \end{aligned}$$

Case(vi): $L = (3)^{1-ns} \quad for \quad s < \frac{1}{n} \quad if \quad i = 1.$

$$\begin{aligned} N(f(x) - A(x), r) &\geq N'\left(\frac{L^{1-i}}{1-L}\beta(x), r\right) \geq N'\left(\frac{3^{1-ns}}{1-3^{1-ns}}\frac{\theta||x||^{ns}}{(n-1)2^{ns}}, r\right) \\ &\geq N'\left(\theta||x||^{ns}, r(n-1)(3^{ns}-3)\right) \end{aligned}$$

Hence the proof is completed. □

5. Conclusion

In this work, in section 3, we investigated Hyers-Ulam stability results in Fuzzy normed spaces by means of direct method, in section 4, we examined the Hyers-Ulam stability results in fuzzy normed spaces by means of fixed point method.

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