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GENERALIZED U-H STABILITY OF CUBIC MAPPINGS

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Abstract. In this work, we investigate the generalized Ulam-Hyers stability of the ω-dimensional cubic functional equation

$$\wp(\sum_{\kappa=1}^{\omega}\kappa\varkappa_i) = \sum_{\kappa=1;\kappa\neq\lambda\neq\eta}^{\omega}\wp(\kappa\varkappa_\kappa+\lambda\varkappa_\lambda+\eta\varkappa_\eta)+(3-n)\sum_{\kappa=1;\kappa\neq\lambda\neq\eta}^{\omega}\wp(\kappa\varkappa_\kappa+\lambda\varkappa_\lambda)$$
$$+\left(\frac{\omega^2-5\omega+6}{2}\right)\sum_{\kappa=0}^{\omega-1}(\kappa+1)^3\wp(\varkappa_\kappa+1)$$

where $\omega \geq 4$, in Banach spaces using direct and fixed point methods.

1. Introduction

The stability problem for functional equations starts from the famous talk of Ulam and the partial solutions of Hyers [9] to the Ulams problem (see [16, 17]). Since then a great deal of work has been done by a number of authors and the problems concerned with the generalizations and the applications of the stability to a number of functional equations have been developed as well [1, 2, 4, 5, 6, 7, 8]. Jun and Kim [12] introduced the following cubic functional equation

$$\wp(2\varkappa+\mu)+\wp(2\varkappa-\mu) = 2\wp(\varkappa+\mu)+2\wp(\varkappa-\mu)+12\wp(\varkappa) \tag{1.1}$$

and they established the general solution and the generalized Hyers-Ulam stability for the functional equation 1.1. The function satisfies the functional equation 1.1, which is thus called a cubic functional equation. Every solution of the cubic functional equation is said to be cubic function. One of the most important problems in fuzzy topology is too obtain an appropriate concept. The applications of the stability to a number of functional equations have been developed as well. Recently, Choonkil Park and Dong Yun Shin [3] prove the Hyers-Ulam stability of Cauchy additive functional inequality, the Cauchy additive functional equation and quadratic functional equation in matrix paranormed spaces. Furthermore, some of the other few functional equations paper are very useful to develop this paper such as [10, 11, 13, 14, 15, 18].

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In this work, we investigate the generalized Ulam-Hyers stability of the ω-dimensional cubic functional equation

$$\wp(\sum_{\kappa=1}^{\omega}\kappa\varkappa_i) = \sum_{\kappa=1;\kappa\neq\lambda\neq\eta}^{\omega}\wp(\kappa\varkappa_\kappa+\lambda\varkappa_\lambda+\eta\varkappa_\eta)+(3-n)\sum_{\kappa=1;\kappa\neq\lambda\neq\eta}^{\omega}\wp(\kappa\varkappa_\kappa+\lambda\varkappa_\lambda)$$
$$+\left(\frac{\omega^2-5\omega+6}{2}\right)\sum_{\kappa=0}^{\omega-1}(\kappa+1)^3\wp(\varkappa_\kappa+1) \tag{1.2}$$

where $\omega \geq 4$, in Banach spaces using direct and fixed point methods.

Theorem 1.1. ***Banach Contraction Principle****: Let $(\chi; d)$ be a complete metric space and consider a mapping $\Phi : \chi \to \chi$ which is strictly contractive mapping, that is*

(B1) $d(\Phi\varkappa;\Phi\mu) \leq Ld(\varkappa,\mu)$ for some (Lipschitz constant) $L < 1$, then
  (1) The mapping Φ has one and only fixed point $\varkappa^\star = \Phi(\varkappa^\star)$;
  (2) The fixed point for each given element $\varkappa^\star$ is globally attractive that is
(B2) $\lim_{\omega\to\infty}\Phi^\omega\varkappa = \omega^\star$, for any starting point $\varkappa \in \chi$;
  (1) One has the following estimation inequalities:
(B3) $d(\Phi^\omega\varkappa,\varkappa^\star) \leq \frac{1}{1-L}d(\Phi^\omega\varkappa,\Phi^{\omega+1}\varkappa)$ for all $\omega \geq 0$, $\varkappa \in \chi$.
(B4) $d(\varkappa,\varkappa^\star) \leq \frac{1}{1-L}d(\varkappa,\varkappa^\star)\ \forall\varkappa \in \chi$.

We take χ be a normed space and Υ be a Banach space. For notational handiness, we define a function $F : \chi \to \Upsilon$ by

$$\Delta\wp(\varkappa_1,\varkappa_2,\ldots,\varkappa_n) = \wp(\sum_{\kappa=1}^{\omega}\kappa\varkappa_\kappa) - \sum_{\kappa=1,\kappa\neq\lambda\neq\eta}\wp(\kappa\varkappa_\kappa+\lambda\varkappa_\lambda+\eta\varkappa_\eta)$$
$$-(3-n)\sum_{\kappa=1;\kappa\neq\lambda\neq\eta}^{\omega}\wp(\kappa\varkappa_\kappa+\lambda\varkappa_\lambda)$$
$$+\left(\frac{\omega^2-5\omega+6}{2}\right)\sum_{\kappa=0}^{\omega-1}(\kappa+1)^3\wp(\varkappa_\kappa+1)$$

2. U - H STABILITY OF AN CUBIC FUNCTIONAL EQUATION 1.2 USING DIRECT METHOD

In this section, the authors discussed the generalized Ulam-Hyers stability of ω-dimensional functional equation 1.2 in Banach space using Direct Method.

Theorem 2.1. *Let $\gamma \in \{-1, 1\}$. Let $\varphi : \chi^\omega \to [0,\infty)$ be a function such that $\sum_{\xi=0}^{\infty}\frac{\varphi(2^{\xi\gamma}\varkappa_1,2^{\xi\gamma}\varkappa_2,\ldots,2^{\xi\gamma}\varkappa_n)}{2^{3\xi\gamma}}$ converges in $\mathbb{R}$ and*

$$\lim_{\xi\to\infty}\frac{\varphi(2^{\xi\gamma}\varkappa_1,2^{\xi\gamma}\varkappa_2,\ldots,2^{\xi\gamma}\varkappa_\omega)}{2^{3\xi\gamma}} = 0 \tag{2.1}$$

for all $\varkappa_1,\varkappa_2,\ldots,\varkappa_\omega \in \chi$. If a function $\wp : \chi \to \Upsilon$ be a function satisfies

$$\|\Delta\wp(\varkappa_1,\varkappa_2,\ldots,\varkappa_n)\| \leq \varphi(\varkappa_1,\varkappa_2,\ldots,\varkappa_n) \tag{2.2}$$

for all $\varkappa_1,\varkappa_2,\ldots,\varkappa_\omega \in \chi$, then there exist a unique cubic function $C : \chi \to \Upsilon$ which satisfies the functional equation 1.2 and

$$\|\wp(\varkappa) - C(\varkappa)\| \leq \frac{1}{8(\omega^2-5\omega+6)}\sum_{\xi=\frac{1-\gamma}{2}}^{\infty}\frac{\psi(2^{\xi\gamma}}{2^{3\xi\gamma}} \tag{2.3}$$

where

$$\psi(\varkappa) = \varphi(\varkappa, \varkappa, 0, \dots, 0) \tag{2.4}$$

for all $\varkappa \in \chi$: The function C is given by

$$C(\varkappa) = \lim_{\xi \to \infty} \frac{\wp(2^{\xi\gamma})\varkappa}{2^{3\xi\gamma}} \tag{2.5}$$

for all $\varkappa \in \chi$.

Proof. Assume that $\gamma = 1$. Replacing $\varkappa_1, \varkappa_2, \dots, \varkappa_\omega$ by $(\varkappa, \varkappa, 0, \dots 0)$ in 1.2, we get

$$\|8(\omega^2 - 5\omega + 6)\wp(\varkappa) - (\omega^2 - 5\omega + 6)\wp(2\varkappa)\| \leq \varphi(\varkappa, \varkappa, 0, \dots, 0) \tag{2.6}$$

for all $\varkappa \in \chi$. It follows from 2.6 that

$$\|\frac{\wp(2\varkappa}{2^3} - \wp(\varkappa)\| \leq \frac{1}{8(\omega^2 - 5\omega + 6)}\varphi(\varkappa, \varkappa, 0, \dots, 0) \tag{2.7}$$

for all $\varkappa \in \chi$. Now replacing $\varkappa$ by $2\varkappa$ and dividing by 2^3 in 2.7, we arrive

$$\|\frac{\wp(2^2\varkappa}{2^6} - \frac{\wp(2\varkappa}{2^3}\| \leq \frac{1}{2^6(\omega^2 - 5\omega + 6)}\varphi(2\varkappa, 2\varkappa, 0, \dots, 0) \tag{2.8}$$

for all $\varkappa \in \chi$. Adding 2.7 and 2.8, we have

$$\|\frac{\wp(2^2\varkappa}{2^3} - \wp(\varkappa)\| \leq \frac{1}{2^3(\omega^2 - 5\omega + 6)}\left(\varphi(\varkappa, \varkappa, 0, \dots, 0) + \frac{\varphi(2\varkappa, 2\varkappa, 0, \dots, 0)}{2^3}\right)$$

for all $\varkappa \in \chi$. In general for any positive integer δ, one can easy to verify that

$$\|\frac{\wp(2^\delta\varkappa}{2^{3\delta}} - \wp(\varkappa)\| \leq \frac{1}{2^3}\sum_{\xi=0}^{\infty}\frac{\psi(2^{\xi\varkappa})}{2^{3\xi\gamma}} \tag{2.9}$$

for all $\varkappa \in \chi$. In order to prove the convergence of the sequence $\{\frac{\wp(2^\delta\varkappa}{2^{3\delta}}\}$, replacing $\varkappa$ by $2^\tau\varkappa$ and dividing $2^{3\tau}$ in 2.9, for $\delta, \nu > 0$, we get

$$\|\frac{\wp(2^{\delta+\tau}\varkappa}{2^{3\delta+\tau}} - \frac{\wp(2^\tau\varkappa}{2^{3\tau}}\| \leq \frac{1}{2^3}\sum_{\xi=0}^{\delta-1}\frac{\psi(2^{(\xi+\tau)}\varkappa)}{2^{3(\xi+\tau)}} \to 0 \; as \; \tau \to \infty \tag{2.10}$$

for all $\varkappa \in \chi$. Hence the sequence $\{\frac{\wp(2^\delta\varkappa}{2^{3\delta}}\}$ is a Cauchy sequence. Since Υ is complete, there exists a mapping $C : \chi \to \Upsilon$ such that

$$C(\varkappa) = \lim_{\delta \to \infty} \frac{\wp(2^\delta\varkappa)}{2^{3\delta}}$$

for all $\varkappa \in \chi$. Letting $\delta \to \infty$ in 2.9 we see that 2.4 holds for $\varkappa \in \chi$. To prove that C satisfies 1.2, replacing $(\varkappa_1, \varkappa_2, \dots, \varkappa_\omega)$ by $(2^\tau\varkappa, 2^\tau\varkappa, \dots, 2^\tau\varkappa$ and dividing $2^{3\tau}$ in 2.2, we arrive

$$\frac{1}{2^{3\tau}}\|\Delta\wp((2^\tau\varkappa, 2^\tau\varkappa, \dots, 2^\tau\varkappa)\| \leq \frac{1}{2^{3\tau}}\varphi((2^\tau\varkappa, 2^\tau\varkappa, \dots, 2^\tau\varkappa)$$

for all $\varkappa_1, \varkappa_2, \dots, \varkappa_\omega \in \chi$. Letting $\tau \to \infty$ in above inequality and using the definition of $C(\varkappa)$, we see that $C(\varkappa_1, \varkappa_2, \dots, \varkappa_\omega) = 0$. Hence C satisfies 1.2 for all $\varkappa \in \chi$. To show that C is unique. Let Δ be the another cubic mapping satisfying 1.2 and 2.4, then

$$\begin{aligned}\|C(\varkappa)-\Delta(\varkappa)\| &\leq \frac{1}{2^{3\tau}}\{\|C(2^{\tau}\varkappa)-\wp(2^{\tau})\|+\|\Delta(2^{\tau}\varkappa\|\} \\ &\leq \frac{1}{2^{3}}\sum_{\xi=0}^{\infty}\frac{\psi(2^{\xi+\tau}\varkappa}{2^{3(\xi+\tau)}}\to 0\ as\ \tau\to\infty\end{aligned}$$

for all $\varkappa\in\chi$. Hence C is unique. For $\gamma=-1$, we can prove a similar stability result. The proof is completed. □

The following corollary is an immediate consequence of Theorem 1.1, concerning the stability of 1.2.

Corollary 2.2. *Let ϕ and δ be a positive real numbers. Let $\wp:\varkappa\in\chi$ be a function satisfying the inequality*

$$\|\Delta\wp(\varkappa_1,\varkappa_2,\dots,\varkappa_\omega)\|\leq\begin{cases}\phi\\ \phi\left(\sum_{\kappa=1}^{\omega}\|\varkappa_\kappa\|^{\delta}\right)\\ \phi\left(\prod_{\kappa=1}^{\omega}\|\varkappa_\kappa\|^{\delta}\right)+\left(\sum_{\kappa=1}^{\omega}\|\varkappa_\kappa\|^{\omega\delta}\right)\end{cases}$$

for all $arkappa_1,\varkappa_2,\dots,\varkappa_\omega\in\chi$. Then there exists a unique cubic function $C:\chi\to\Upsilon$ such that

$$\|\wp(\varkappa)-C(\varkappa)\|\leq\begin{cases}\frac{\phi}{(\omega^2-5\omega+6)|7|}\\ \frac{2\phi\|\varkappa\|}{(\omega^2-5\omega+6)2^3-2^{\delta}}; & \delta\neq 3\\ \frac{2\phi\|\varkappa\|}{(\omega^2-5\omega+6)2^3-2^{\omega\delta}}; & \delta\neq\frac{3}{\omega}\end{cases}$$

3. U - H Stability of an cubic functional equation 1.2 using Fixed Point Method

In this section, we found the generalized Ulam-Hyers stability of the ω-dimensional functional equation (1.2) in Banach space with the help of fixed point method.

Theorem 3.1. *Let $\wp:\chi\to\Upsilon$ be a mapping for which there exists a function $\Phi:\chi^{\omega}\to[0,\infty)$ with the condition*

$$\lim_{\xi\to\infty}\frac{\Phi(\rho_\kappa^{\varkappa 1},\rho_\kappa^{\varkappa 2}),\dots\rho_\kappa^{\varkappa\omega}}{\rho_\kappa^{3\xi}}=0 \tag{3.1}$$

where

$$\rho_\kappa=\begin{cases}2, & if\ \kappa=0\\ \frac{1}{2}, & if\ \kappa=1\end{cases}$$

such that the functional inequality

$$\|\Delta\wp(\varkappa_1,\varkappa_2,\dots\varkappa_\omega)\|\leq\Phi(\varkappa_1,\varkappa_2,\dots\varkappa_\omega) \tag{3.2}$$

for all $\varkappa_1,\varkappa_2,\dots\varkappa_\omega\in\chi$. If there exist $L=L(\kappa)$ such that the function

$$\varkappa\to\psi(\varkappa)=\frac{1}{\omega^2-5\omega+6)}\Phi\left(\frac{\varkappa}{2},\frac{\varkappa}{2},0,\dots,0\right)$$

has the property

$$\frac{1}{\rho_\kappa^4}\nu(\rho_\kappa\varkappa)=L\nu(\varkappa) \tag{3.3}$$

for all $\varkappa \in \chi$. Then there exists a unique cubic function $C : \chi \to \Upsilon$ satisfying the functional equation 1.2 and

$$\|\wp(\varkappa) - C(\varkappa)\| \leq \frac{L^{1-\kappa}}{1-L}\nu(\varkappa) \tag{3.4}$$

holds for all $\varkappa \in \chi$.

Corollary 3.2. *Let ϕ and δ be a non-negative real numbers. Let $\wp : \varkappa \in \chi$ be a function satisfying the inequality*

$$\|\Delta\wp(\varkappa_1, \varkappa_2, \dots, \varkappa_\omega)\| \leq \begin{cases} \phi \\ \phi\left(\sum_{\kappa=1}^{\omega} \|\varkappa_\kappa\|^\delta\right) \\ \phi\left(\prod_{\kappa=1}^{\omega} \|\varkappa_\kappa\|^\delta\right) + \left(\sum_{\kappa=1}^{\omega} \|\varkappa_\kappa\|^{\omega\delta}\right) \end{cases}$$

for all $arkappa_1, \varkappa_2, \dots, \varkappa_\omega \in \chi$. Then there exists a unique cubic function $C : \chi \to \Upsilon$ such that

$$\|\wp(\varkappa) - C(\varkappa)\| \leq \begin{cases} \frac{\phi}{(\omega^2-5\omega+6)|7|} \\ \frac{2\phi\|\varkappa\|}{(\omega^2-5\omega+6)2^3-2^\delta}; & \delta \neq 3 \\ \frac{2\phi\|\varkappa\|}{(\omega^2-5\omega+6)2^3-2^{\omega\delta}}; & \delta \neq \frac{3}{\omega} \end{cases}$$

4. CONCLUSION

We have introduced the following functional equation

$$\wp(\sum_{\kappa=1}^{\omega} \kappa\varkappa_i) = \sum_{\kappa=1;\kappa\neq\lambda\neq\eta}^{\omega} \wp(\kappa\varkappa_\kappa + \lambda\varkappa_\lambda + \eta\varkappa_\eta) + (3-n) \sum_{\kappa=1;\kappa\neq\lambda\neq\eta}^{\omega} \wp(\kappa\varkappa_\kappa + \lambda\varkappa_\lambda)$$
$$+ \left(\frac{\omega^2 - 5\omega + 6}{2}\right) \sum_{\kappa=0}^{\omega-1} (\kappa+1)^3 \wp(\varkappa_\kappa + 1).$$

We have solved the n-dimensional non-quadratic functional equation and also investigated the Hyers-Ulam stability of the new type of the functional equation using direct and fixed point methods.

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REFERENCES

[1] Baker, J.A. (2005). A general functional equation and its stability, Proc. Amer. Math. Soc., 133, pp. 1657-1664.
[2] Borelli, C. (1994). On the Hyers Ulam stability of Hosszus functional equation, Results Math., 26, pp. 221-224.
[3] Choonkil Park and Dong Yun Shin Functional equations in paranormed spaces Park and Shin Advances in Difference Equations 2012, 2012:123 Advances in difference equations.
[4] Chung, S.Y., Kim, D. and Lee, E.G. (2000). Periodic hyperfunctions and Fourier series, Proc. Amer. Math. Soc., 128, pp. 2421-2430.
[5] Czerwik, S. (1992). On the stability of the quadratic mapping in normed spaces, Abh. Math. Sem. Univ. Hamburg, 62, pp. 59-64.
[6] Gordji, M.E. and Savadkouhi, M.B. (2011). Stability of a mixed type additive, quadratic and cubic functional equation in random normed spaces, Filomat, 25(3), pp. 43-54.
[7] Govindan, V., Murthy, S. and Saravanan, M. (2018). Solution and stability of a cubic type functional equation: using direct and fixed point methods, Kragujevac J. Math., 44(1), pp. 7-26.

[8] Govindan, V., Tamilvanan, K. (2018). Stability of functional equation in banach space: using two different methods, Int. J. Math. and its Appl., 6(1-C) , pp. 527-536.
[9] Hyers, D.H. (1941). On the stability of the linear functional equation, Proc. Natl. Acad. Sci., USA, 27, pp. 222-224.
[10] Lee, J.R., Shin, D.Y. and Park, C. (2013). Hyers-Ulam stability of functional equations in matrix normed spaces, J. Inequal Appl., 2013(22), 11 pp. DOI:10.1186/1029-242X-2013-22.
[11] Moslehian, M.S. and Sadeghi, G. (2008). Stability of two types of cubic functional equations in non-Archimedean spaces, Real Anal. Exchange, 33(2), pp. 375-384.
[12] K. Jun and H. Kim, The generalized Hyers-Ulam-Rassias stability of a cubic functional equation, J. Math. Anal. Appl. 274 (2002), 867878.
[13] Park, C. and Rassias, Th.M. (2009). Fixed points and stability of the Cauchy functional equation, Aust. J. Math. Anal. Appl., 6(1), Art. ID 14, 9 pp.
[14] Rassias, J.M. (1994). On the stability of the non-linear Euler-Lagrange functional equation in real normed linear spaces, J. Math. Phys. Sci., 28, pp. 231-235.
[15] Sandra Pinelas, Govindan, V., Tamilvanan, K. (2018). Stability of a quartic functional equation, J. Fixed Point Theory and Appl., 20(4), 10 pages.
[16] S. M. Ulam, A collection of mathematical problems, Interscience Publishers, New York, U.S.A., (1960).
[17] S. M. Ulam, Problems in Modern Mathematics, Science Editions, Wiley, New York, 1964.
[18] Zhao, X., Yang, X. and Pang, C.-T. (2013). Solution and stability of a general mixed type cubic and quartic functional equation, J. Funct. Spaces Appl., 2013, Art.ID 673810, 8 pp.

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