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ON GENERALIZED DIRECT PRODUCT OF FUZZY MULTIGROUPS

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ABSTRACT. Fuzzy multigroup is a structure that generalizes the idea of fuzzy group. In fact, the concept of fuzzy multigroups is the application of fuzzy multisets to group theory. The idea of direct product of fuzzy multigroups has been established. This paper extends the notion of direct product between two fuzzy multigroups to the case of finitely many fuzzy multigroups. Some properties of generalized direct product of fuzzy multigroups are elucidated. It is shown that generalized direct product of fuzzy multigroups is a fuzzy multigroup. Finally, a number of results are obtained and duly verified with respect to alph-cuts and level sets.

1. INTRODUCTION

Fuzzy set theory was proposed by Zadeh [26] as a technique for representing imprecision in real-world situations. Fuzzy set theory has achieved a great success in several fields due to its ability to cope uncertainty. Fuzzy set is characterized by a membership function, μ which takes value from a crisp set to a unit interval, $I = [0, 1]$. The concept of fuzzy sets has grown amazingly with several applications in groups, like fuzzy groups [19], metric spaces [10], etc. Elaborate researches have been conducted on fuzzy group theory [1, 16, 18, 20]. Just as fuzzy groups were drawn from fuzzy sets, in like manner, the notion of multigroups had been studied [11, 12, 17] via multisets.

By a way of generalization, Yager [25] proposed the idea of fuzzy multisets which is a generalization of fuzzy sets in multisets framework [22, 23]. For some details on fuzzy multisets, see [4, 13, 24]. Shinoj *et al.* [21] followed the footsteps of Rosenfeld [19] to introduced a non-classical group called fuzzy multigroup. The concept of fuzzy multigroups constitutes an application of fuzzy multisets to the notion of group. The ideas of abelian fuzzy multigroup and order of fuzzy multigroups have been studied in [2, 6]. Ejegwa [5] introduced the concept of fuzzy multigroupoids and presented the idea of fuzzy submultigroups with a number of results. The notions of centre and centralizer in fuzzy multigroup setting were established with some relevant results [6]. In [7], the idea of homomorphism in the setting of fuzzy multigroups was defined and some homomorphic properties of fuzzy multigroups were elaborated. Recently, the idea of direct product of fuzzy multigroups was proposed and a number of results were established [9].

The objectives of this paper are to generalize direct product of fuzzy multigroups and deduce some related results therein, motivated by the results in [9]. In fact, generalized

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direct product of fuzzy multigroups is the extension of the work in [9]. In recap, this paper proposes the notion of generalized direct product of fuzzy multigroups and explicates some of its properties in details. By organization, the paper is thus presented: Section 2 provides some preliminaries while Section 3 proposes the idea of generalized direct product of fuzzy multigroups and discusses some of its properties. Also, a number of results are obtained. Finally, Section 4 concludes the paper and provides direction for future studies.

2. PRELIMINARIES

Definition 2.1. [25] Assume X is a set of elements. Then, a fuzzy bag/multiset A drawn from X can be characterized by a count membership function CM_A such that

$$CM_A : X \to Q,$$

where Q is the set of all crisp bags or multisets from the unit interval $I = [0, 1]$.

From [24], a fuzzy multiset can also be characterized by a high-order function. In particular, a fuzzy multiset A can be characterized by a function

$$CM_A : X \to N^I \text{ or } CM_A : X \to [0, 1] \to N,$$

where $I = [0, 1]$ and $N = \mathbb{N} \cup \{0\}$.

By [14], it implies that $CM_A(x)$ for $x \in X$ is given as

$$CM_A(x) = \{\mu_A^1(x), \mu_A^2(x), ..., \mu_A^n(x), ...\},$$

where $\mu_A^1(x), \mu_A^2(x), ..., \mu_A^n(x), ... \in [0, 1]$ such that $\mu_A^1(x) \geq \mu_A^2(x) \geq ... \geq \mu_A^n(x) \geq ...$, whereas in a finite case, we write

$$CM_A(x) = \{\mu_A^1(x), \mu_A^2(x), ..., \mu_A^n(x)\},$$

for $\mu_A^1(x) \geq \mu_A^2(x) \geq ... \geq \mu_A^n(x)$.

A fuzzy multiset A can be represented in the form

$$A = \{\langle \frac{CM_A(x)}{x} \rangle \mid x \in X\} \text{ or } A = \{\langle x, CM_A(x) \rangle \mid x \in X\}.$$

In a simple term, a fuzzy multiset A of X is characterized by the count membership function $CM_A(x)$ for $x \in X$, that takes the value of a multiset of a unit interval $I = [0, 1]$ [3, 15].

We denote the set of all fuzzy multisets by $FMS(X)$.

Definition 2.2. [13] Let $A, B \in FMS(X)$. Then, A is called a fuzzy submultiset of B written as $A \subseteq B$ if $CM_A(x) \leq CM_B(x)\ \forall x \in X$. Also, if $A \subseteq B$ and $A \neq B$, then A is called a proper fuzzy submultiset of B and denoted as $A \subset B$.

Definition 2.3. [24] Let $A, B \in FMS(X)$. Then, the intersection and union of A and B, denoted by $A \cap B$ and $A \cup B$, respectively, are defined by the rules that for any object $x \in X$,

(i) $CM_{A\cap B}(x) = CM_A(x) \wedge CM_B(x)$,
(ii) $CM_{A\cup B}(x) = CM_A(x) \vee CM_B(x)$,

where $\wedge$ and $\vee$ denote minimum and maximum respectively.

Definition 2.4. [13] Let $A, B \in FMS(X)$. Then, A and B are comparable to each other if $A \subseteq B$ or $B \subseteq A$, and $A = B$ if $CM_A(x) = CM_B(x) \forall x \in X$.

Definition 2.5. [21] Let X be a group. Then, A fuzzy multiset A of X is said to be a fuzzy multigroup of X if it satisfies the following two conditions:

(i) $CM_A(xy) \geq CM_A(x) \wedge CM_A(y)\ \forall x, y \in X$,

(ii) $CM_A(x^{-1}) \geq CM_A(x)\, \forall x \in X$.

It follows immediately that,

$$CM_A(x^{-1}) = CM_A(x)\, \forall x \in X$$

since

$$CM_A(x) = CM_A((x^{-1})^{-1}) \geq CM_A(x^{-1}).$$

Also,

$$CM_A(e) \geq CM_A(x)\, \forall x \in X$$

because

$$CM_A(e) = CM_A(xx^{-1}) \geq CM_A(x) \wedge CM_A(x) = CM_A(x).$$

In fact, every fuzzy multigroup is a fuzzy multiset but the converse is not true. The set of all fuzzy multigroups of X is denoted by $FMG(X)$.

Definition 2.6. [5] Let $A \in FMG(X)$. A fuzzy submultiset B of A is called a fuzzy submultigroup of A, i.e., $B \subseteq A$ if B form a fuzzy multigroup. A fuzzy submultigroup B of A is a proper fuzzy submultigroup, i.e., $B \subset A$, if $B \subseteq A$ and $A \neq B$.

Definition 2.7. [2] Let $A \in FMG(X)$. Then, A is said to be commutative if for all $x, y \in X, CM_A(xy) = CM_A(yx)$.

Definition 2.8. [8] Let $A, B \in FMG(X)$ such that $A \subseteq B$. Then, A is called a normal fuzzy submultigroup of B if for all $x, y \in X$, it satisfies

$$CM_A(xyx^{-1}) \geq CM_A(y).$$

Definition 2.9. [8] Two fuzzy multigroups A and B of X are conjugate to each other if for all $x, y \in X$,

$$CM_A(x) = CM_B(yxy^{-1}) \text{ and } CM_B(y) = CM_A(xyx^{-1}).$$

Proposition 2.1. [8] *Let $A, B \in FMG(X)$. Then, the following statements are equivalent.*

(i) A is a normal fuzzy submultigroup of B.
(ii) $CM_A(xyx^{-1}) = CM_A(y) \forall x, y \in X$.
(iii) $CM_A(xy) = CM_A(yx) \forall x, y \in X$.

Proposition 2.2. [5, 21] *Let $A \in FMG(X)$. Then, the sets A_* and A^* are defined as*

(i) $A_* = \{x \in X \mid CM_A(x) > 0\}$ and
(ii) $A^* = \{x \in X \mid CM_A(x) = CM_A(e)\}$, (where e is the identity element of X) are subgroups of X.

Definition 2.10. Let $A \in FMG(X)$. Then, the sets $A_{[\alpha]}$ and $A_{(\alpha)}$ defined as

(i) $A_{[\alpha]} = \{x \in X \mid CM_A(x) \geq \alpha\}$ and
(ii) $A_{(\alpha)} = \{x \in X \mid CM_A(x) > \alpha\}$

are called strong upper alpha-cut and weak upper alpha-cut of A, where $\alpha \in [0, 1]$.

Definition 2.11. Let $A \in FMG(X)$. Then, the sets $A^{[\alpha]}$ and $A^{(\alpha)}$ defined as

(i) $A^{[\alpha]} = \{x \in X \mid CM_A(x) \leq \alpha\}$ and
(ii) $A^{(\alpha)} = \{x \in X \mid CM_A(x) < \alpha\}$

are called strong lower alpha-cut and weak lower alpha-cut of A, where $\alpha \in [0, 1]$.

Theorem 2.3. *Let $A \in FMG(X)$. Then, $A_{[\alpha]}$ is a subgroup of X for all $\alpha \leq CM_A(e)$ and $A^{[\alpha]}$ is a subgroup of X for all $\alpha \geq CM_A(e)$, where e is the identity element of X and $\alpha \in [0,1]$.*

Definition 2.12. Let $A, B \in FMG(X)$ such that $A \subseteq B$. Then, A is called a characteristic (fully invariant) fuzzy submultigroup of B if

$$CM_{A^{\theta}}(x) = CM_A(x)\ \forall x \in X$$

for every automorphism, θ of X. That is, $\theta(A) \subseteq A$ for every $\theta \in Aut(X)$.

Proposition 2.4. *Let X be a group. Every characteristic fuzzy submultigroup of a fuzzy multigroup B of X is normal.*

Definition 2.13. [7] Let X and Y be groups and let $f : X \to Y$ be a homomorphism. Suppose A and B are fuzzy multigroups of X and Y, respectively. Then, f induces a homomorphism from A to B which satisfies

(i) $CM_A(f^{-1}(y_1y_2)) \geq CM_A(f^{-1}(y_1)) \wedge CM_A(f^{-1}(y_2))\ \forall y_1, y_2 \in Y$,
(ii) $CM_B(f(x_1x_2)) \geq CM_B(f(x_1)) \wedge CM_B(f(x_2))\ \forall x_1, x_2 \in X$,

where

(i) the image of A under f, denoted by $f(A)$, is a fuzzy multiset over Y defined by

$$CM_{f(A)}(y) = \begin{cases} \bigvee_{x \in f^{-1}(y)} CM_A(x), & f^{-1}(y) \neq \emptyset \\ 0, & \text{otherwise} \end{cases}$$

for each $y \in Y$.

(ii) the inverse image of B under f, denoted by $f^{-1}(B)$, is a fuzzy multiset over X defined by

$$CM_{f^{-1}(B)}(x) = CM_B(f(x)) \forall x \in X.$$

Definition 2.14. [9] Let X and Y be groups, $A \in FMG(X)$ and $B \in FMG(Y)$, respectively. Then, the direct product of A and B depicted by $A \times B$ is a function

$$CM_{A \times B} : X \times Y \to Q$$

defined by

$$CM_{A \times B}((x, y)) = CM_A(x) \wedge CM_B(y)\ \forall x \in X,\ \forall y \in Y,$$

where Q is the set of all multisets from the unit interval $I = [0, 1]$.

Theorem 2.5. [9] *Let A and B be fuzzy multigroups of X and Y, respectively. Then, $A \times B$ is a fuzzy multigroup of $X \times Y$.*

Theorem 2.6. [9] *Let $A \in FMG(X)$ and $B \in FMG(Y)$. Suppose C and D are two fuzzy submultisets of A and B, respectively. Then, $C \times D$ is a fuzzy submultigroup of $A \times B$ if and only if both C and D are fuzzy submultigroups of A and B, respectively.*

Theorem 2.7. [9] *Let $f : W \times X \to Y \times Z$ be an isomorphism, A, B, C and D be fuzzy multigroups of W, X, Y and Z, respectively. Then, the following statements hold.*

(i) $f(A \times B) \in FMG(Y \times Z)$.
(ii) $f^{-1}(C) \times f^{-1}(D) \in FMG(W \times X)$.

3. GENERALIZED DIRECT PRODUCT OF FUZZY MULTIGROUPS

In this section, we define direct product of k^{th} fuzzy multigroups and obtain some results.

Definition 3.1. Let $A_1, A_2, ..., A_k$ be fuzzy multigroups of groups $X_1, X_2, ..., X_k$, respectively. Then, the direct product of $A_1, A_2, ..., A_k$ is a function

$$CM_{A_1 \times A_2 \times ... \times A_k} : X_1 \times X_2 \times ... \times X_k \to Q$$

defined by

$$CM_{A_1 \times A_2 \times ... \times A_k}(x) = CM_{A_1}(x_1) \wedge CM_{A_2}(x_2) \wedge ... \wedge CM_{A_{k-1}}(x_{k-1}) \wedge CM_{A_k}(x_k),$$

where Q is the set of all multisets from the unit interval $I = [0, 1]$ and $x = (x_1, x_2, ..., x_{k-1}, x_k)$, $\forall x_1 \in X_1, \forall x_2 \in X_2, ..., \forall x_k \in X_k$. If we denote $A_1, A_2, ..., A_k$ by $A_i, (i \in I), X_1, X_2, ..., X_k$ by $X_i, (i \in I)$, $A_1 \times A_2 \times ... \times A_k$ by $\prod_{i=1}^k A_i$ and $X_1 \times X_2 \times ... \times X_k$ by $\prod_{i=1}^k X_i$. Then the direct product of A_i is a function

$$CM_{\prod_{i=1}^k A_i} : \prod_{i=1}^k X_i \to Q$$

defined by

$$CM_{\prod_{i=1}^k A_i}((x_i)_{i \in I}) = \wedge_{i \in I} CM_{A_i}((x_i)) \; \forall x_i \in X_i, \; I = 1, ..., k.$$

Unless otherwise specified, it is assumed that X_i is a group with identity e_i for all $i \in I$, $X = \prod_{i \in I}^k X_i$, and so $e = (e_i)_{i \in I}$.

Proposition 3.1. *Let $A_1, A_2, ..., A_k$ be fuzzy multisets of the sets $X_1, X_2, ..., X_k$, respectively. Then*

(i) $(A_1 \times A_2 \times ... \times A_k)_* = A_{1*} \times A_{2*} \times ... \times A_{k*}$.
(ii) $(A_1 \times A_2 \times ... \times A_k)^* = A_1^* \times A_2^* \times ... \times A_k^*$.

Proof. (i) Let $(x_1, x_2, ..., x_k) \in (A_1 \times A_2 \times ... \times A_k)_*$. Synthesizing Proposition 2.2, we have

$$CM_{A_1 \times A_2 \times ... \times A_k}((x_1, x_2, ..., x_k)) = (CM_{A_1}(x_1) \wedge CM_{A_2}(x_2) \wedge ... \wedge CM_{A_k}(x_k)) \geq 0.$$

This implies that $CM_{A_1}(x_1) \geq 0$, $CM_{A_2}(x_2) \geq 0, ..., C_{A_k}(x_k) \geq 0$ and $x_1 \in A_{1*}$, $x_2 \in A_{2*}, ..., x_k \in A_{k*}$. Thus, $(x_1, x_2, ..., x_k) \in A_{1*} \times A_{2*} \times ... \times A_{k*}$.

Also, let $(x_1, x_2, ..., x_k) \in A_{1*} \times A_{2*} \times ... \times A_{k*}$. Then $x_i \in A_{i*}$, for $i = 1, 2, ..., k$, $CM_{A_1}(x_1) \geq 0$, $CM_{A_2}(x_2) \geq 0, ..., CM_{A_k}(x_k) \geq 0$. Thus,

$$(CM_{A_1}(x_1) \wedge CM_{A_2}(x_2) \wedge ... \wedge CM_{A_k}(x_k)) \geq 0.$$

That is,

$$CM_{A_1 \times A_2 \times ... \times A_k}((x_1, x_2, ..., x_k)) \geq 0,$$

implies that

$$(x_1, x_2, ..., x_k) \in (A_1 \times A_2 \times ... \times A_k)_*.$$

Hence, $(A_1 \times A_2 \times ... \times A_k)_* = A_{1*} \times A_{2*} \times ... \times A_{k*}$.

(ii) Synthesizing Proposition 2.2 and following the logic in (i), the result follows. □

Theorem 3.2. *Let $A_1, A_2, ..., A_k$ be fuzzy multisets of the sets $X_1, X_2, ..., X_k$, respectively and let $\alpha \in [0, 1]$. Then*

(i) $(A_1 \times A_2 \times ... \times A_k)_{[\alpha]} = A_{1[\alpha]} \times A_{2[\alpha]} \times ... \times A_{k[\alpha]}$.
(ii) $(A_1 \times A_2 \times ... \times A_k)^{[\alpha]} = A_1^{[\alpha]} \times A_2^{[\alpha]} \times ... \times A_k^{[\alpha]}$.

Proof. (i) Let $(x_1, x_2, ..., x_k) \in (A_1 \times A_2 \times ... \times A_k)_{[\alpha]}$. From Definition 2.10, we have

$$CM_{A_1 \times A_2 \times ... \times A_k}((x_1, x_2, ..., x_k)) = (CM_{A_1}(x_1) \wedge CM_{A_2}(x_2) \wedge ... \wedge CM_{A_k}(x_k)) \geq \alpha.$$

This implies that

$$CM_{A_1}(x_1) \geq \alpha, CM_{A_2}(x_2) \geq \alpha, ..., C_{A_k}(x_k) \geq \alpha$$

and $x_1 \in A_{1[\alpha]}, x_2 \in A_{2[\alpha]}, ..., x_k \in A_{k[\alpha]}$. Thus,

$$(x_1, x_2, ..., x_k) \in A_{1[\alpha]} \times A_{2[\alpha]} \times ... \times A_{k[\alpha]}.$$

Again, let $(x_1, x_2, ..., x_k) \in A_{1[\alpha]} \times A_{2[\alpha]} \times ... \times A_{k[\alpha]}$. Then $x_i \in A_{i[\alpha]}$, for $i = 1, 2, ..., k$, $CM_{A_1}(x_1) \geq \alpha, CM_{A_2}(x_2) \geq \alpha, ..., CM_{A_k}(x_k) \geq \alpha$. Thus,

$$(CM_{A_1}(x_1) \wedge CM_{A_2}(x_2) \wedge ... \wedge CM_{A_k}(x_k)) \geq \alpha.$$

That is,

$$CM_{A_1 \times A_2 \times ... \times A_k}((x_1, x_2, ..., x_k)) \geq \alpha,$$

implies that

$$(x_1, x_2, ..., x_k) \in (A_1 \times A_2 \times ... \times A_k)_{[\alpha]}.$$

Hence, $(A_1 \times A_2 \times ... \times A_k)_{[\alpha]} = A_{1[\alpha]} \times A_{2[\alpha]} \times ... \times A_{k[\alpha]}$.

(ii) By Definition 2.11 and following the logic in (i), the result is complete. □

Theorem 3.3. *Let $A_1, A_2, ..., A_k$ be fuzzy multigroups of groups $X_1, X_2, ..., X_k$, respectively. Then, $A_1 \times A_2 \times ... \times A_k$ is a fuzzy multigroup of $X_1 \times X_2 \times ... \times X_k$.*

Proof. Let $(x_1, x_2, ..., x_k), (y_1, y_2, ..., y_k) \in X_1 \times X_2 \times ... \times X_k$. We get
$CM_{A_1 \times ... \times A_k}((x_1, ..., x_k)(y_1, ..., y_k)) = CM_{A_1 \times ... \times A_k}((x_1y_1, ..., x_ky_k))$
$= CM_{A_1}(x_1y_1) \wedge ... \wedge CM_{A_k}(x_ky_k) \geq (CM_{A_1}(x_1) \wedge CM_{A_1}(y_1)) \wedge ... \wedge (CM_{A_k}(x_k) \wedge CM_{A_k}(y_k)) = \wedge(\wedge(CM_{A_1}(x_1), CM_{A_1}(y_1)), ..., \wedge(CM_{A_k}(x_k), CM_{A_k}(y_k))$
$= \wedge(\wedge(CM_{A_1}(x_1), ..., CM_{A_k}(x_k)), \wedge(CM_{A_1}(y_1), ..., CM_{A_k}(y_k)))$
$= CM_{A_1 \times ... \times A_k}((x_1, ..., x_k)) \wedge CM_{A_1 \times ... \times A_k}((y_1, ..., y_k))$.
Also,

$$\begin{aligned} CM_{A_1 \times ... \times A_k}((x_1, ..., x_k)^{-1}) &= CM_{A_1 \times ... \times A_k}((x_1^{-1}, ..., x_k^{-1})) \\ &= CM_{A_1}(x_1^{-1}) \wedge ... \wedge CM_{A_k}(x_k^{-1}) \\ &= CM_{A_1}(x_1) \wedge ... \wedge CM_{A_k}(x_k) \\ &= CM_{A_1 \times ... \times A_k}((x_1, ..., x_k)). \end{aligned}$$

Hence, $A_1 \times A_2 \times ... \times A_k$ is a fuzzy multigroup of $X_1 \times X_2 \times ... \times X_k$. □

Theorem 3.4. *Let $A_1, A_2, ..., A_k$ and $B_1, B_2, ..., B_k$ be fuzzy multigroups of $X_1, X_2, ..., X_k$, respectively, such that $A_1, A_2, ..., A_k \subseteq B_1, B_2, ..., B_k$. If $A_1, A_2, ...,$ A_k are normal fuzzy submultigroups of $B_1, B_2, ..., B_k$, then $A_1 \times A_2 \times ... \times A_k$ is a normal fuzzy submultigroup of $B_1 \times B_2 \times ... \times B_k$.*

Proof. By Theorem 3.3, $A_1 \times A_2 \times ... \times A_k$ is a fuzzy multigroup of $X_1, X_2, ..., X_k$. Likewise, $B_1 \times B_2 \times ... \times B_k$ is a fuzzy multigroup of $X_1, X_2, ..., X_k$. Now, we let $(x_1, x_2, ..., x_k), (y_1, y_2, ..., y_k) \in X_1 \times X_2 \times ... \times X_k$. Then we get

$$\begin{aligned} CM_{A_1 \times ... \times A_k}((x_1, ..., x_k)(y_1, ..., y_k)) &= CM_{A_1 \times ... \times A_k}((x_1y_1, ..., x_ky_k)) \\ &= CM_{A_1}(x_1y_1) \wedge ... \wedge CM_{A_k}(x_ky_k) \\ &= CM_{A_1}(y_1x_1) \wedge ... \wedge CM_{A_k}(y_kx_k) \\ &= CM_{A_1 \times ... \times A_k}((y_1x_1, ..., y_kx_k)) \\ &= CM_{A_1 \times ... \times A_k}((y_1, ..., y_k)(x_1, ..., x_k)). \end{aligned}$$

Thus, $A_1 \times A_2 \times ... \times A_k$ is a normal fuzzy submultigroup of $B_1 \times B_2 \times ... \times B_k$ by Proposition 2.1. □

Theorem 3.5. *If $A_1, A_2, ..., A_k$ are fuzzy multigroups of $X_1, X_2, ..., X_k$, respectively, then*

(i) $(A_1 \times A_2 \times ... \times A_k)_*$ *is a subgroup of* $X_1 \times X_2 \times ... \times X_k$,
(ii) $(A_1 \times A_2 \times ... \times A_k)^*$ *is a subgroup of* $X_1 \times X_2 \times ... \times X_k$,
(iii) $(A_1 \times A_2 \times ... \times A_k)_{[\alpha]}$ *is a subgroup of* $X_1 \times X_2 \times ... \times X_k$, $\forall\, \alpha \leq [CM_{A_1}(e_1), CM_{A_2}(e_2), ..., CM_{A_k}(e_k)]$, $\alpha \in [0, 1]$.
(iv) $(A_1 \times A_2 \times ... \times A_k)^{[\alpha]}$ *is a subgroup of* $X_1 \times X_2 \times ... \times X_k$, $\forall\, \alpha \geq [CM_{A_1}(e_1), CM_{A_2}(e_2), ..., CM_{A_k}(e_k)]$, $\alpha \in [0, 1]$.

Proof. Combining Proposition 2.2, Theorems 2.3 and 3.3, the results follow. □

Corollary 3.6. *Let $A_1, A_2, ..., A_k$ and $B_1, B_2, ..., B_k$ be fuzzy multigroups of $X_1, X_2, ..., X_k$ such that $A_1, A_2, ..., A_k \subseteq B_1, B_2, ..., B_k$. If $A_1, A_2, ..., A_k$ are normal fuzzy submultigroups of $B_1, B_2, ..., B_k$, then*

(i) $(A_1 \times A_2 \times ... \times A_k)_*$ *is a normal subgroup of* $(B_1 \times B_2 \times ... \times B_k)_*$,
(ii) $(A_1 \times A_2 \times ... \times A_k)^*$ *is a normal subgroup of* $(B_1 \times B_2 \times ... \times B_k)^*$,
(iii) $(A_1 \times A_2 \times ... \times A_k)_{[\alpha]}$ *is a normal subgroup of* $(B_1 \times B_2 \times ... \times B_k)_{[\alpha]}$, $\forall\, \alpha \leq [CM_{A_1}(e_1), CM_{A_2}(e_2), ..., CM_{A_k}(e_k)]$, $\alpha \in [0, 1]$.
(iv) $(A_1 \times A_2 \times ... \times A_k)^{[\alpha]}$ *is a normal subgroup of* $(B_1 \times B_2 \times ... \times B_k)^{[\alpha]}$, $\forall\, \alpha \geq [CM_{A_1}(e_1), CM_{A_2}(e_2), ..., CM_{A_k}(e_k)]$, $\alpha \in [0, 1]$.

Proof. Combining Proposition 2.2, Theorems 2.3, 3.3 and 3.4, the results follow. □

Theorem 3.7. *Let $A_1, A_2, ..., A_k$ and $B_1, B_2, ..., B_k$ be fuzzy multigroups of groups $X_1, X_2, ..., X_k$, respectively. If $A_1, A_2, ..., A_k$ are conjugate to $B_1, B_2, ..., B_k$, then the fuzzy multigroup $A_1 \times A_2 \times ... \times A_k$ of $X_1 \times X_2 \times ... \times X_k$ is conjugate to the fuzzy multigroup $B_1 \times B_2 \times ... \times B_k$ of $X_1 \times X_2 \times ... \times X_k$.*

Proof. By Definition 2.9, if fuzzy multigroup A_i of X_i conjugates to fuzzy multigroup B_i of X_i, then exist $x_i \in X_i$ such that for all $y_i \in X_i$,

$$CM_{A_i}(y_i) = CM_{B_i}(x_i^{-1} y_i x_i), i = 1, 2, ..., k.$$

Then we have

$$\begin{aligned} CM_{A_1 \times ... \times A_k}((y_1, ..., y_k)) &= CM_{A_1}(y_1) \wedge ... \wedge CM_{A_k}(y_k) \\ &= CM_{B_1}(x_1^{-1} y_1 x_1) \wedge ... \wedge CM_{B_k}(x_k^{-1} y_k x_k) \\ &= CM_{B_1 \times ... \times B_k}((x_1^{-1} y_1 x_1, ..., x_k^{-1} y_k x_k)). \end{aligned}$$

This completes the proof. □

Theorem 3.8. *Let $A_1, A_2, ..., A_k$ be fuzzy multisets of the groups $X_1, X_2, ..., X_k$, respectively. Suppose that $e_1, e_2, ..., e_k$ are identities elements of $X_1, X_2, ..., X_k$, respectively. If $A_1 \times A_2 \times ... \times A_k$ is a fuzzy multigroup of $X_1 \times X_2 \times ... \times X_k$, then for at least one $i = 1, 2, ..., k$, the statement*

$$CM_{A_1 \times A_2 \times ... \times A_{i-1} \times A_{i+1} \times ... \times A_k}((e_1, e_2, ..., e_{i-1}, e_{i+1}, ..., e_k)) \geq CM_{A_i}((x_i))$$

$\forall x_i \in X_i$ holds.

Proof. Let $A_1 \times A_2 \times ... \times A_k$ be a fuzzy multigroup of $X_1 \times X_2 \times ... \times X_k$. By contraposition, suppose that for none of $i = 1, 2, ..., k$, the statement holds. Then we can find $(a_1, a_2, ..., a_k) \in X_1 \times X_2 \times ... \times X_k$, respectively, such that

$$CM_{A_i}((a_i)) > CM_{A_1 \times A_2 \times ... \times A_{i-1} \times A_{i+1} \times ... \times A_k}((e_1, e_2, ..., e_{i-1}, e_{i+1}, ..., e_k)).$$

Then we have $CM_{A_1 \times ... \times A_k}((a_1, ..., a_k)) = CM_{A_1}(a_1) \wedge ... \wedge CM_{A_k}(a_k) >$ $CM_{A_1 \times ... \times A_{i-1} \times A_{i+1} \times ... \times A_k}((e_1, ..., e_{i-1}, e_{i+1}, ..., e_k)) = CM_{A_1}(e_1) \wedge ...$

$\wedge CM_{A_{i-1}}(e_{i-1}) \wedge CM_{A_{i+1}}(e_{i+1}) \wedge ... \wedge CM_{A_k}(e_k) = CM_{A_1}(e_1) \wedge ... \wedge CM_{A_k}(e_k)$
$= CM_{A_1 \times ... \times A_k}((e_1, ..., e_k))$.

Thus, $A_1 \times A_2 \times ... \times A_k$ is not a fuzzy multigroup of $X_1 \times X_2 \times ... \times X_k$. Hence, for at least one $i = 1, 2, ..., k$, the inequality

$$CM_{A_1 \times ... \times A_{i-1} \times A_{i+1} \times ... \times A_k}((e_1, ..., e_{i-1}, e_{i+1}, ..., e_k)) \geq CM_{A_i}((x_i))$$

is satisfied for all $x_i \in X_i$. □

Theorem 3.9. *Let $A_1, A_2, ..., A_k$ be fuzzy multisets of the groups $X_1, X_2, ..., X_k$, respectively, such that*

$$CM_{A_i}((x_i)) \leq CM_{A_1 \times A_2 \times ... \times A_{i-1} \times A_{i+1} \times ... \times A_k}((e_1, e_2, ..., e_{i-1}, e_{i+1}, ..., e_k))$$

$\forall x_i \in X_i$, e_i being the identity element of X_i. If $A_1 \times ... \times A_{i-1} \times A_{i+1} \times ... \times A_k$ is a fuzzy multigroup of $X_1 \times ... \times X_{i-1} \times X_{i+1} \times ... \times X_k$, then A_i is a fuzzy multigroup of X_i.

Proof. Let $A_1 \times ... \times A_{i-1} \times A_{i+1} \times ... \times A_k$ be a fuzzy multigroup of $X_1 \times ... \times X_{i-1} \times X_{i+1} \times ... \times X_k$ and $x_i, y_i \in X_i$. Then

$$(e_1, ..., x_i, ..., e_k), (e_1, ..., y_i, ..., e_k) \in X_1 \times ... \times X_i \times ... \times X_k.$$

Now, using the given inequality, we have
$CM_{A_i}((x_i y_i)) = CM_{A_i}((x_i y_i)) \wedge CM_{A_1 \times ... \times A_{i-1} \times A_{i+1} \times ... \times A_k}((e_1, ..., e_{i-1}, e_{i+1}, ..., e_k)(e_1, ..., e_{i-1}, e_{i+1}, ..., e_k)) = CM_{A_1 \times ... \times A_i \times ... \times A_k}((e_1, ..., x_i, ..., e_k)(e_1, ..., y_i, ..., e_k)) \geq$
$CM_{A_1 \times ... \times A_i \times ... \times A_k}((e_1, ..., x_i, ..., e_k)) \wedge CM_{A_1 \times ... \times A_i \times ... \times A_k}((e_1, ..., y_i, ..., e_k)) =$
$\wedge(CM_{A_i}((x_i)) \wedge CM_{A_1 \times ... \times A_{i-1} \times A_{i+1} \times ... \times A_k}((e_1, ..., e_{i-1}, e_{i+1}, ..., e_k)), CM_{A_i}((y_i)) \wedge$
$CM_{A_1 \times ... \times A_{i-1} \times A_{i+1} \times ... \times A_k}((e_1, ..., e_{i-1}, e_{i+1}, ..., e_k)))$
$= CM_{A_i}((x_i)) \wedge CM_{A_i}((y_i))$.
Also, $CM_{A_i}((x_i^{-1})) = CM_{A_i}((x_i^{-1})) \wedge CM_{A_1 \times ... \times A_{i-1} \times A_{i+1} \times ... \times A_k}((e_1^{-1}, ...,$
$e_{i-1}^{-1}, e_{i+1}^{-1}, ..., e_k^{-1})) = CM_{A_1 \times ... \times A_i \times ... \times A_k}((e_1^{-1}, ..., x_i^{-1}, ..., e_k^{-1})) =$
$CM_{A_1 \times ... \times A_i \times ... \times A_k}((e_1, ..., x_i, ..., e_k)^{-1}) = CM_{A_1 \times ... \times A_i \times ... \times A_k}((e_1, ..., x_i, ...,$
$e_k)) = CM_{A_i}((x_i)) \wedge CM_{A_1 \times ... \times A_{i-1} \times A_{i+1} \times ... \times A_k}((e_1, ..., e_{i-1}, e_{i+1}, ..., e_k)) = CM_{A_i}((x_i))$.
Hence, $A_i \in FMG(X_i)$. □

Theorem 3.10. *Let $A_1, A_2, ..., A_k$ be fuzzy multisets of the groups $X_1, X_2, ..., X_k$, respectively, such that*

$$CM_{A_1 \times A_2 \times ... \times A_{i-1} \times A_{i+1} \times ... \times A_k}((x_1, x_2, ..., x_{i-1}, x_{i+1}, ..., x_k)) \leq CM_{A_i}((e_i))$$

$\forall (x_1, x_2, ..., x_{i-1}, x_{i+1}, ..., x_k) \in X_1 \times X_2 \times ... \times X_{i-1} \times X_{i+1} \times ... \times X_k$, e_i being the identity element of X_i. If $A_1 \times A_2 \times ... \times A_k$ is a fuzzy multigroup of $X_1 \times X_2 \times ... \times X_k$, then $A_1 \times A_2 \times ... \times A_{i-1} \times A_{i+1} \times ... \times A_k$ is a fuzzy multigroup of $X_1 \times X_2 \times ... \times X_{i-1} \times X_{i+1} \times ... \times X_k$.

Proof. Let $A_1 \times A_2 \times ... \times A_k$ be a fuzzy multigroup of $X_1 \times X_2 \times ... \times X_k$ and $(x_1, x_2, ..., x_{i-1}, x_{i+1}, ..., x_k), (y_1, y_2, ..., y_{i-1}, y_{i+1}, ..., y_k) \in X_1 \times X_2 \times ... \times X_{i-1} \times X_{i+1} \times ... \times X_k$. Using the given inequality, we arrive at
$CM_{A_1 \times ... \times A_{i-1} \times A_{i+1} \times ... \times A_k}((x_1, ..., x_{i-1}, x_{i+1}, ..., x_k)(y_1, ..., y_{i-1}, y_{i+1}, ..., y_k))$
$= CM_{A_1 \times ... \times A_{i-1} \times A_{i+1} \times ... \times A_k}((x_1, ..., x_{i-1}, x_{i+1}, ..., x_k)(y_1, ..., y_{i-1}, y_{i+1}, ..., y_k)) \wedge$
$CM_{A_i}((e_i)) = CM_{A_1 \times ... \times A_i \times ... \times A_k}((x_1, ..., e_i, ..., x_k)(y_1, ..., e_i, ..., y_k))$
$\geq CM_{A_1 \times ... \times A_i \times ... \times A_k}((x_1, ..., e_i, ..., x_k)) \wedge CM_{A_1 \times ... \times A_i \times ... \times A_k}((y_1, ..., e_i, ..., y_k))$
$= \wedge(CM_{A_i}((e_i)) \wedge CM_{A_1 \times ... \times A_{i-1} \times A_{i+1} \times ... \times A_k}((x_1, ..., x_{i-1}, x_{i+1}, ..., x_k)),$
$CM_{A_i}((e_i)) \wedge CM_{A_1 \times ... \times A_{i-1} \times A_{i+1} \times ... \times A_k}((y_1, ..., y_{i-1}, y_{i+1}, ..., y_k))) =$

$CM_{A_1\times...\times A_{i-1}\times A_{i+1}\times...\times A_k}((x_1,...,x_{i-1},x_{i+1},...,x_k))$
$\wedge\, CM_{A_1\times...\times A_{i-1}\times A_{i+1}\times...\times A_k}((y_1,y_2,...,y_{i-1},y_{i+1},...,y_k))$.

Again,
$CM_{A_1\times...\times A_{i-1}\times A_{i+1}\times...\times A_k}((x_1^{-1},...,x_{i-1}^{-1},x_{i+1}^{-1},...,x_k^{-1}))$
$= CM_{A_1\times...\times A_{i-1}\times A_{i+1}\times...\times A_k}((x_1^{-1},...,x_{i-1}^{-1},x_{i+1}^{-1},...,x_k^{-1})) \wedge CM_{A_i}((e_i^{-1}))$
$= CM_{A_1\times...\times A_i\times...\times A_k}((x_1^{-1},...,e_i^{-1},...,x_k^{-1}))$
$= CM_{A_1\times...\times A_i\times...\times A_k}((x_1,...,e_i,...,x_k)^{-1})$
$= CM_{A_1\times...\times A_i\times...\times A_k}((x_1,...,e_i,...,x_k))$
$= CM_{A_1\times...\times A_{i-1}\times A_{i+1}\times...\times A_k}((x_1,...,x_{i-1},x_{i+1},...,x_k)) \wedge CM_{A_i}((e_i))$
$= CM_{A_1\times...\times A_{i-1}\times A_{i+1}\times...\times A_k}((x_1,...,x_{i-1},x_{i+1},...,x_k))$.

Hence, $A_1 \times A_2 \times ... \times A_{i-1} \times A_{i+1} \times ... \times A_k$ is a fuzzy multigroup of $X_1 \times X_2 \times ... \times X_{i-1} \times X_{i+1} \times ... \times X_k$. □

Theorem 3.11. *Let $B_1, ..., B_k$ be fuzzy multigroups of groups $X_1, ..., X_k$. Suppose $A_1, ..., A_k$ are fuzzy submultisets of $B_1, ..., B_k$, respectively. Then $A_1 \times ... \times A_k$ is a fuzzy submultigroup of $B_1 \times ... \times B_k$ if and only if $A_1, ..., A_k$ are fuzzy submultigroups of $B_1, ..., B_k$.*

Proof. Similar to Theorem 2.6, so we omit the proof. □

Remark. Let $B_1, ..., B_k$ be fuzzy multigroups of groups $X_1, ..., X_k$. Suppose $A_1, ..., A_k$ are fuzzy submultigroups of $B_1, ..., B_k$, respectively. Then

(i) $A_1 \times ... \times A_k$ is a normal fuzzy submultigroup of $B_1 \times ... \times B_k$ if and only if $A_1, ..., A_k$ are normal fuzzy submultigroups of $B_1, ..., B_k$.
(ii) $A_1 \times ... \times A_k$ is a characteristic fuzzy submultigroup of $B_1 \times ... \times B_k$ if and only if $A_1, ..., A_k$ are characteristic fuzzy submultigroups of $B_1, ..., B_k$.
(iii) $A_1 \times ... \times A_k$ is a normal fuzzy submultigroup of $B_1 \times ... \times B_k$ if $A_1, ..., A_k$ are characteristic fuzzy submultigroups of $B_1, ..., B_k$.

Remark. Let $B_1, ..., B_k$ be fuzzy multigroups of groups $X_1, ..., X_k$. Suppose $A_1, ..., A_k$ are fuzzy submultigroups of $B_1, ..., B_k$, respectively. Then

(i) $A_1 \times ... \times A_k$ is a normal fuzzy submultigroup of $B_1 \times ... \times B_k$ if and only if $A_1, ..., A_k$ are normal fuzzy submultigroups of $B_1, ..., B_k$.
(ii) $A_1 \times ... \times A_k$ is a characteristic fuzzy submultigroup of $B_1 \times ... \times B_k$ if and only if $A_1, ..., A_k$ are characteristic fuzzy submultigroups of $B_1, ..., B_k$.
(iii) $A_1 \times ... \times A_k$ is a normal fuzzy submultigroup of $B_1 \times ... \times B_k$ if $A_1, ..., A_k$ are characteristic fuzzy submultigroups of $B_1, ..., B_k$.

Theorem 3.12. *Let $X_1, X_2, ..., X_k$ and $Y_1, Y_2, ..., Y_k$ be groups, and*

$$f : X_1 \times X_2 \times ... \times X_k \to Y_1 \times Y_2 \times ... \times Y_k$$

be homomorphism. If $A_1 \times A_2 \times ... \times A_k \in FMG(X_1 \times X_2 \times ... \times X_k)$ and $B_1 \times B_2 \times ... \times B_k \in FMG(Y_1 \times Y_2 \times ... \times Y_k)$, then

(i) $f(A_1 \times A_2 \times ... \times A_k) \in FMG(Y_1 \times Y_2 \times ... \times Y_k)$,
(ii) $f^{-1}(B_1 \times B_2 \times ... \times B_k) \in FMG(X_1 \times X_2 \times ... \times X_k)$.

Proof. Similar to Theorem 2.7, so we omit the proof. □

Theorem 3.13. *Let $A_1, ..., A_k$ be fuzzy multigroups of groups $X_1, ..., X_k$. Then $A_1, ..., A_k$ are commutative if and only if $A_1 \times ... \times A_k$ is a commutative fuzzy multigroup of $X_1 \times ... \times X_k$.*

Proof. Suppose $A_1, ..., A_k$ are commutative. We show that $A_1 \times ... \times A_k$ is a commutative fuzzy multigroup of $X_1 \times ... \times X_k$. It is a known fact that $A_1 \times ... \times A_k$ is a fuzzy multigroup of $X_1 \times ... \times X_k$ by Theorem 3.3. Let $(x_1, ..., x_k), (y_1, ..., y_k) \in X_1 \times ... \times X_k$. Then, we get

$$\begin{aligned} CM_{A_1 \times ... \times A_k}((x_1, ..., x_k)(y_1, ..., y_k)) &= CM_{A_1 \times ... \times A_k}((x_1 y_1), ..., (x_k y_k)) \\ &= CM_{A_1}(x_1 y_1) \wedge ... \wedge CM_{A_k}(x_k y_k) \\ &= CM_{A_1}(y_1 x_1) \wedge ... \wedge CM_{A_k}(y_k x_k) \\ &= CM_{A_1 \times ... \times A_k}((y_1 x_1), ..., (y_k x_k)) \\ &= CM_{A_1 \times ... \times A_k}((y_1, ..., y_k)(x_1, ..., x_k)). \end{aligned}$$

Hence, $A_1 \times ... \times A_k$ is a commutative fuzzy multigroup of $X_1 \times ... \times X_k$ by Definition 2.7.

Conversely, suppose $A_1 \times ... \times A_k$ is a commutative fuzzy multigroup of $X_1 \times ... \times X_k$. Then,

$$CM_{A_1 \times ... \times A_k}((x_1, ..., x_k)(y_1, ..., y_k)) = CM_{A_1 \times ... \times A_k}((y_1, ..., y_k)(x_1, ..., x_k)).$$

Consequently, $A_1, ..., A_k$ are commutative fuzzy multigroups of groups $X_1, ..., X_k$. □

4. CONCLUSIONS

The concept of fuzzy multigroups is an application of fuzzy multisets to group theory. In the continuation of the study of fuzzy multigroups, we have proposed the idea of generalized direct product of fuzzy multigroups as an extension of direct product of fuzzy multigroups and obtained some related results. More results on generalized direct product of fuzzy multigroups could be exploited for future research.

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REFERENCES

[1] H. Aktas and N. Cagman. Generalized product of fuzzy subgroups and $t-$ level subgroups. Mathematical Communications, 11 (2006), 121–128.
[2] A. Baby, T.K. Shinoj and J.J. Sunil. On abelian fuzzy multigroups and order of fuzzy multigroups. Journal of New Theory, 5(2) (2015), 80–93.
[3] R. Biswas. An application of Yager's bag theory in multicriteria based decision making problems. International Journal of Intelligent Systems, 14 (1999), 1231–1238.
[4] P.A. Ejegwa. Correspondence between fuzzy multisets and sequences. Global Journal of Science Frontier Research: Mathematics and Decision Science, 14(7) (2014), 61–66.
[5] P.A. Ejegwa. On fuzzy multigroups and fuzzy submultigroups. Journal of Fuzzy Mathematics, 26(3) (2018), 641–654.
[6] P.A. Ejegwa. On abelian fuzzy multigroups. Journal of Fuzzy Mathematics, 26(3) (2018), 655–668.
[7] P.A. Ejegwa. Homomorphism of fuzzy multigroups and some of its properties. Applications and Applied Mathematics, 13(1) (2018), 114–129.
[8] P.A. Ejegwa. On normal fuzzy submultigroups of a fuzzy multigroup. Theory and Applications of Mathematics and Computer Science, 8(1) (2018), 64–80.
[9] P.A. Ejegwa. Direct product of fuzzy multigroups. Journal of New Theory, 28 (2019), 62–73.
[10] P.A. Ejegwa and J.M. Agbetayo. Norm and metric defined over fuzzy sets. Annals of Pure and Applied Mathematics, 10(1) (2015), 59–64.
[11] P.A. Ejegwa and A.M. Ibrahim. Homomorphism of cuts of multigroups. Gulf Journal of Mathematics, 6(1) (2018), 61–73.

[12] P.A. and A.M. Ibrahim. Some properties of multigroups. Palestine Journal of Mathematics, 9(1) (2020), 31–47.
[13] S. Miyamoto. Basic operations of fuzzy multisets. Journal of Japan Society of Fuzzy Theory and Systems, 8(4) (1996), 639–645.
[14] S. Miyamoto and K. Mizutani. Fuzzy multiset model and methods for nonlinear document clustering for information retrieval. Springer-Verlag Berlin Heidelberg, (2004) pp. 273–283.
[15] K. Mizutani, R. Inokuchi and S. Miyamoto. Algorithms of nonlinear document clustering based on fuzzy multiset model. International Journal of Intelligent Systems, 23 (2008), 176–198.
[16] J.M. Mordeson, K.R. Bhutani and A. Rosenfeld. Fuzzy group theory. Springer-Verlag Berlin Heidelberg, 2005.
[17] Sk. Nazmul, P. Majumdar and S.K. Samanta. On multisets and multigroups. Annals of Fuzzy Mathematics and Informatics, 6(3) (2013), 643–656.
[18] A.K. Ray. On product of fuzzy subgroups. Fuzzy Sets and Systems, 105 (1999), 181–183.
[19] A. Rosenfeld. Fuzzy subgroups. Journal of Mathematical Analysis and Applications, 35 (1971), 512–517.
[20] B. Seselja and A. Tepavcevic. A note on fuzzy groups. Yugoslav Journal of Operation Research, 7(1) (1997), 49–54.
[21] T.K. Shinoj, A. Baby and J.J. Sunil. On some algebraic structures of fuzzy multisets. Annals of Fuzzy Mathematics and Informatics, 9(1) (2015), 77–90.
[22] D. Singh, A.M. Ibrahim, T. Yohanna, and J.N. Singh. An overview of the applications of multisets. Novi Sad Journal of Mathematics, 37(2) (2007), 73–92.
[23] A. Syropoulos. Mathematics of multisets. Springer-Verlag Berlin Heidelberg, (2001) pp. 347–358.
[24] A. Syropoulos. On generalized fuzzy multisets and their use in computation. Iranian Journal of Fuzzy Systems, 9(2) (2012), 113–125.
[25] R.R. Yager. On the theory of bags. International Journal of General Systems, 13 (1986), 23–37.
[26] L.A. Zadeh. Fuzzy sets. Information and Control, 8 (1965), 338–353.

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