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GENERALIZED FUZZY Γ-IDEALS OF ORDERED Γ-SEMIGROUPS

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ABSTRACT. In this paper, the notions of $(\in, \in \vee(k^*, q_k))$-fuzzy left Γ-ideals, $(\in, \in \vee(k^*, q_k))$-fuzzy right Γ-ideals and $(\in, \in \vee(k^*, q_k))$-fuzzy Γ-ideals in ordered Γ-semigroups are introduced and their related properties are investigated. Furthermore, (k^*, k)-lower parts of $(\in, \in \vee(k^*, q_k))$-fuzzy left Γ-ideals, $(\in, \in \vee(k^*, q_k))$-fuzzy right Γ-ideals and $(\in, \in \vee(k^*, q_k))$-fuzzy Γ-ideals are also defined. Finally, left regular, right regular and regular ordered Γ-semigroups in terms of $(\in, \in \vee(k^*, q_k))$-fuzzy left Γ-ideals and $(\in, \in \vee(k^*, q_k))$-fuzzy right Γ-ideals are characterized.

1. INTRODUCTION

In 1986, Sen and Saha [26] introduced the notion of a Γ-semigroup as follows: Let S and Γ be two nonempty sets. Then S is called a Γ-semigroup if there exists a mapping from $S \times \Gamma \times S$ to S which maps $(a, \alpha, b) \to a\alpha b$ satisfying $(a\gamma b)\mu c = a\gamma(b\mu c)$ for all $a, b, c \in S$ and $\gamma, \mu \in \Gamma$. Later on in 1993, the notion of an ordered Γ-semigroup was introduced by Sen and Seth [27]. Many classical notions such as ideals, bi-ideals and quasi-ideals in ordered semigroups and regular ordered semigroups have been generalized to ordered Γ-semigroups, and these classical notions of ordered Γ-semigroups have been studied by [6, 7, 3, 10, 14]

Zadeh [29], in 1965, introduced the concept of a fuzzy set. The concept of a fuzzy subgroup introduced by Rosenfeld [23]. In 1979, Kuroki [20] introduced fuzzy sets in semigroup theory. Fuzzy sets in ordered semigroups were first studied by Kehayopulu and Tsingelis in [11]. In [28], Tang characterized ordered semigroups by $(\in, \in \vee q)$-fuzzy ideals. Later on, the concept of $(\in, \in \vee q_k)$- fuzzy subalgebras in BCK/BCI-algebras is introduced by Jun [1]. In [24] Shabir et al. characterized the regular semigroups by $(\in, \in \vee q_k)$-fuzzy ideals. In [16, 17, 18] Khan et al characterized ordered semigroups in terms of fuzzy bi-ideals and intuitionistics fuzzy bi-ideals. The concepts $(\in, \in \vee(k^*, q_k))$-fuzzy left ideals, $(\in, \in \vee(k^*, q_k))$-fuzzy right ideals and $(\in, \in \vee(k^*, q_k))$-fuzzy generalized bi-ideals of ordered semigroups are introduced by Khan et al. [13]. Recently, Muhiuddin et al. [21] introduced the concept of $(\in, \in \vee(k^*, q_k))$-fuzzy semiprime ideals using the concept of $(k*, k)$ qusi-coincidence.

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The $(\in, \in \vee q_k)$-fuzzy left Γ-ideals, $(\in, \in \vee q_k)$-fuzzy right Γ-ideals and $(\in, \in \vee q_k)$-fuzzy ideals of an ordered Γ-semigroup are introduced and characterized by Khan et al. [19]. In [4], Gambo et al. characterized left regular, right regular, regular and completely regular ordered Γ-semigroups in terms of $(\in, \in \vee q_k)$-fuzzy left Γ-ideals, $(\in, \in \vee q_k)$-fuzzy right Γ-ideals and $(\in, \in \vee q_k)$-fuzzy ideals. By generalizing the concept of fuzzy generalized bi Γ-ideals, the concept of $(\in, \in \vee q_k)$-fuzzy bi Γ-ideals in ordered Γ-semigroups is introduced by Gambo et al. [5].

Motivated by the work of Khan et al. [19] and Gambo et al. [4, 5], in the present paper, we have given some new types of fuzzy left (resp. right) Γ-ideals and Γ-ideals in ordered Γ-semigroups. As a follow up, we introduce the concept of an $(\in, \in \vee(k^*, q_k))$-fuzzy left Γ-ideal, $(\in, \in \vee(k^*, q_k))$-fuzzy right Γ-ideal and $(\in, \in \vee(k^*, q_k))$-fuzzy ideal in an ordered Γ-semigroup, and investigate some vital properties of these new types of fuzzy Γ-ideals. Moreover, we characterize left (resp. right) regular ordered Γ-semigroups in terms (k^*, k)-lower part of $(\in, \in \vee(k^*, q_k))$-fuzzy left (resp. right) Γ-ideals and also regular ordered Γ-semigroups in terms of $(\in, \in \vee(k^*, q_k))$-fuzzy left Γ-ideals and $(\in, \in \vee(k^*, q_k))$-fuzzy right Γ-ideals.

2. Preliminaries

Let S and Γ be the nonempty sets. Then the triplet $(S, \Gamma, \leq)$ is called an ordered Γ-semigroup if S is a Γ-semigroup and $(S, \leq)$ is a partially ordered set such that

$$a \leq b \Rightarrow a\gamma c \leq b\gamma c \text{ and } c\gamma a \leq c\gamma b$$

for all $a, b, c \in S$ and $\gamma \in \Gamma$.

For a subset A of an ordered Γ-semigroup S, we denote $(A] = \{t \in S \mid t \leq a \text{ for some } a \in A\}$. For any nonempty subsets A and B of S, the following properties hold: (1) $A \subseteq (A]$; (2) $((A]] = (A]$; (3) If $A \subseteq B$, then $(A] \subseteq (B]$; (4) $(A]\Gamma(B] \subseteq (A\Gamma B]$ and (5) $((A]\Gamma(B]] = (A\Gamma B]$.

A nonempty subset T of S is said to be a Γ-subsemigroup of S if for all $x, y \in T$ and $\gamma \in \Gamma$, $x\gamma y \in T$. A nonempty subset A of S is called left (right) Γ-ideal of S if $S\Gamma A \subseteq A$ $(A\Gamma S \subseteq A)$ and $(A] \subseteq A$. If A is both left and right Γ-ideal of S, then A is called a Γ-ideal of S.

Let S be an ordered Γ-semigroup and let A be any nonempty subset of S. Then by $L(A), R(A)$ and $J(A)$, we denote the left Γ-ideal, the right Γ-ideal and the Γ-ideal of S generated by A respectively. It is easy to verify that $L(A) = (A \cup S\Gamma A]$, $R(A) = (A \cup A\Gamma S]$ and $J(A) = (A \cup S\Gamma A \cup A\Gamma S \cup S\Gamma A\Gamma S]$.

Let $(S, \Gamma, \leq)$ be an ordered Γ-semigroup. A mapping f from S to real closed interval $[0, 1]$ is called the fuzzy subset of S (or fuzzy set of S). We denote by f_A the characteristic function of a subset A of S, which is defined as:

$$f_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A. \end{cases}$$

For any fuzzy subsets f and g of S, the fuzzy subsets $f \cap g$, $f \cup g$ and $f \circ g$ are defined as follows:

$$\begin{aligned} (f \cap g)(x) &= \min\{f(x), g(x)\} = f(x) \wedge g(x) \\ (f \cup g)(x) &= \max\{f(x), g(x)\} = f(x) \vee g(x) \end{aligned}$$

and

$$(f \circ g)(x) = \begin{cases} \bigvee_{(y,z)\in A_x} \{f(y) \wedge g(z)\} & \text{if } A_x \neq \phi \\ 0 & \text{if } A_x = \phi, \end{cases}$$

where $A_x = \{(y,z) \in S \times S \mid x \leq y\alpha z$ for some $\alpha \in \Gamma\}$. Define an order relation $\preceq$ on the set of all fuzzy subsets of S by

$$f \preceq g \Leftrightarrow f(x) \leq g(x) \text{ for all } x \in S.$$

If f, g are fuzzy subsets of S such that $f \preceq g$, then for each fuzzy subset h of S, $f \circ h \preceq g \circ h$ and $h \circ f \preceq h \circ g$.

A fuzzy subset f of S is called a fuzzy Γ-subsemigroup of S if $f(x\alpha y) \geq \min\{f(x), f(y)\}$ for all $x, y \in S$ and $\alpha \in \Gamma$. A fuzzy subset f of S is called a fuzzy left (resp. right) Γ-ideal of S if (1) $x \leq y \Rightarrow f(x) \geq f(y)$ and (2) $f(x\alpha y) \geq f(y)$ (resp. $f(x\alpha y) \geq f(x)$) for all $x, y \in S$ and $\alpha \in \Gamma$. A fuzzy subset f of S is called a fuzzy Γ-ideal of S if it is both a fuzzy left and right Γ-ideal of S.

Lemma 2.1. [12] *Let S be an ordered Γ-semigroyps. Then the following are equivalent:*

(1) *S is regular;*
(2) *$A \cap B = (AB]$ for every right ideal A and left ideal B of S.*

3. $(\in, \in \vee(k^*, q_k))$-FUZZY Γ-IDEALS OF ORDERED Γ-SEMIGROUPS

Let S be an ordered Γ-semigroup, $a \in S$ and $u \in (0,1]$. An ordered fuzzy point a_u is a mapping from S into [0,1] which is defined as follows:

$$a_u(x) = \begin{cases} u, & \text{if } x \in (a], \\ 0, & \text{if } x \notin (a]. \end{cases}$$

For any fuzzy subset f of S, we shall also denote $a_u \subseteq f$ by $a_u \in f$ in the sequel. Then $a_u \in f$ if and only if $f(a) \geq u$.

An ordered fuzzy point a_u of an ordered Γ-semigroup S is said to be quasi-coincident with a fuzzy subset f of S, written as $a_u q f$, if $f(a) + u > 1$.

Definition 3.1. An ordered fuzzy point x_u of ordered Γ-semigroup S, for any $k^* \in (0,1]$, is said to be (k^*, q)-quasi-coincident with a fuzzy subset f of S, written as $x_u(k^*, q)f$, if

$$f(x) + u > k^*.$$

Let S be an ordered Γ-semigroup and $0 \leq k < k^* \leq 1$. For an ordered fuzzy point x_u, we say that

(1) $x_u(k^*, q_k)f$ if $f(x) + u + k > k^*$;
(2) $x_u \in \vee(k^*, q_k)f$ if $x_u \in f$ or $x_u(k^*, q_k)f$;
(3) $x_u \overline{\alpha} f$ if $x_u \alpha f$ does not hold for $\alpha \in \{(k^*, q_k), \in \vee(k^*, q_k)\}$.

Definition 3.2. A fuzzy subset f of an ordered Γ-semigroup S is called an $(\in, \in \vee(k^*, q_k))$-fuzzy Γ-subsemigroup of S if $x_u \in f$ and $y_v \in f$ imply $(x\gamma y)_{\min\{u,v\}} \in \vee(k^*, q_k)f$ for all $x, y \in S, \gamma \in \Gamma$ and $u, v \in (0,1]$.

Definition 3.3. A fuzzy subset f of an ordered Γ-semigroup S is called an $(\in, \in \vee(k^*, q_k))$-fuzzy left (resp. right) Γ-ideal of S if:

(1) $x \leq y, y_u \in f \Rightarrow x_u \in \vee(k^*, q_k)f$ and
(2) $x \in S, y_u \in f$ implies $(x\gamma y)_u \in \vee(k^*, q_k)f$ (resp. $(y\gamma x)_u \in \vee(k^*, q_k)f$)

for all $x, y \in S$ and $\gamma \in \Gamma$.

A fuzzy subset f of an ordered Γ-semigroup S is called an $(\in, \in \vee(k^*, q_k))$-fuzzy Γ-ideal if it is both $(\in, \in \vee(k^*, q_k))$-fuzzy left Γ-ideal and $(\in, \in \vee(k^*, q_k))$-fuzzy right Γ-ideal of S.

Example 3.4. Let $S = \{0, w, b, y\}$ and $\Gamma = \{\alpha, \beta\}$ be the nonempty sets. Define binary operations as:

α	0	w	x	y
0	0	0	0	0
w	0	x	0	w
x	0	x	0	y
y	0	0	0	x

β	0	w	x	y
0	0	0	0	0
w	w	w	w	w
x	0	0	0	0
y	w	w	w	y

Define order relation on S as, $\leq := \{(0,0),(w,w),(x,x),(y,y),(0,w),(0,x),(0,y)\}$. Clearly S is an ordered Γ-semigroup. The fuzzy set $\mu : S \rightarrow [0,1]$ is defined by

$$\mu(a) = \begin{cases} 0.2 & \text{if } a \in \{0, w, x\} \\ 0 & \text{if } a = y. \end{cases}$$

Take $k^* = 0.5$ and $k = 0.1$. It is easy to verify that μ is an $(\in, \in \vee(k^*, q_k))$-fuzzy left Γ-ideal of S.

Theorem 3.1. *Let S be an ordered Γ-semigroup and A be a nonempty subset of S. Then the fuzzy subset f_A of A defined as*

$$f_A(x) = \begin{cases} \frac{k^*-k}{2}, & \text{if } x \in A; \\ 0, & \text{if } x \notin A, \end{cases}$$

is an $(\in, \in \vee(k^*, q_k))$-fuzzy left (resp. right) Γ-ideal of S if and only if A is a left (resp. right) Γ-ideal of S.

Proof. Suppose that A is left Γ-ideal of S. Let $x, y \in S$ with $x \leq y$ and $u \in (0,1]$ such that $y_u \in f_A$. Then $f_A(y) \geq u$. As L is a left Γ-ideal of S, $x \in A$. Thus $f_A(x) = \frac{k^*-k}{2}$. If $u \leq \frac{k^*-k}{2}$, then $f_A(x) \geq u$, so $x_u \in f_A$. If $u > \frac{k^*-k}{2}$, then $f_A(x) + u > \frac{k^*-k}{2} + \frac{k^*-k}{2} = k^* - k$. Thus $x_u(k^*, q_k)f_A$. Therefore $x_u \in \vee(k^*, q_k)f_A$.

Let $x, y \in S$ and $u \in (0,1]$ such that $y_u \in f$. Then $y \in A$, $f(y) \geq u$. As A is a left Γ-ideal of S, we have $x\gamma y \in A$ for each $\gamma \in \Gamma$. Thus $f(x\gamma y) \geq \frac{k^*-k}{2}$. If $u \leq \frac{k^*-k}{2}$, then $f(x\gamma y) \geq u$. Therefore $(x\gamma y)_u \in f$. Again, if $u > \frac{k^*-k}{2}$, then $f(x\gamma y) + u > \frac{k^*-k}{2} + \frac{k^*-k}{2} = k^* - k$. So $(x\gamma y)_u(k^*, q_k)f$. Therefore $(x\gamma y)_u \in \vee(k^*, q_k)f$.

Conversely, assume that f_A is an $(\in, \in \vee(k^*, q_k))$-fuzzy left Γ-ideal of S. Let $x, y \in S$ such that $x \leq y$. If $y \in A$, then $f_A(y) = \frac{k^*-k}{2}$. As f_A is an $(\in, \in \vee(k^*, q_k))$-fuzzy left Γ-ideal of S and $x \leq y$, we have $f_A(x) \geq \min\{f_A(y), \frac{k^*-k}{2}\} = \frac{k^*-k}{2}$. It follows that $f_A(x) = \frac{k^*-k}{2}$ and so $x \in A$. Let $x \in S$ and $y \in A$. Then $f_A(y) = \frac{k^*-k}{2}$. Now we have

$$f_A(x\gamma y) \geq \min\left\{f_A(y), \frac{k^*-k}{2}\right\} = \frac{k^*-k}{2}.$$

Thus $f_A(x\gamma y) = \frac{k^*-k}{2}$ and so $x\gamma y \in A$. Hence A is a left Γ-ideal of S. □

Corollary 3.2. *Let S be an ordered Γ-semigroup and A be a nonempty subset of S. Then the fuzzy subset f_A of A is defined as*

$$f_A(x) = \begin{cases} \frac{k^*-k}{2}, & \text{if } x \in A; \\ 0, & \text{if } x \notin A, \end{cases}$$

is an $(\in, \in \vee(k^*, q_k))$-fuzzy Γ-ideal of S if and only if A is a Γ-ideal of S.

Theorem 3.3. *A fuzzy subset f of S is an $(\in, \in \vee(k^*, q_k))$-fuzzy left (resp. right) Γ-ideal of S if and only if*

(1) $x \leq y \Rightarrow f(x) \geq \min\{f(y), \frac{k^*-k}{2}\}$;

(2) $f(x\gamma y) \geq \min\{f(y), \frac{k^*-k}{2}\}$ *(resp.* $f(x\gamma y) \geq \min\{f(x), \frac{k^*-k}{2}\}$*);*

for all $x, y \in S$ and $\gamma \in \Gamma$.

Proof. Let f be an $(\in, \in \vee(k^*, q_k))$-fuzzy left Γ-ideal of S and $x, y \in S$. If $f(x) < \min\{f(y), \frac{k^*-k}{2}\}$ for some $x, y \in S$. Choose $u \in (0, 1]$ such that $f(x) < u \leq \min\{f(y), \frac{k^*-k}{2}\}$. Then $y_u \in f$, but $(x)_u \overline{\in \vee(k^*, q_k)} f$, a contradiction. Thus $f(x) \geq \min\{f(y), \frac{k^*-k}{2}\}$. Again, if $f(x\gamma y) < \min\{f(y), \frac{k^*-k}{2}\}$ for some $x, y \in S$ and $\gamma \in \Gamma$. Choose $u \in (0, 1]$ such that $f(x\gamma y) < u \leq \min\{f(y), \frac{k^*-k}{2}\}$. Then $y_u \in f$ but $(x\gamma y)_u \overline{\in \vee(k^*, q_k)} f$, which is a contradiction. Hence $f(x\gamma y) \geq \min\{f(y), \frac{k^*-k}{2}\}$.

Conversely assume that $f(x) \geq \min\{f(y), \frac{k^*-k}{2}\}$ for all $x, y \in S$. Let $y_u \in f$ $(u \in (0, 1])$. Then $f(y) \geq u$. So $f(x) \geq \min\{f(y), \frac{k^*-k}{2}\} \geq \min\{u, \frac{k^*-k}{2}\}$. If $u \leq \frac{k^*-k}{2}$, then $f(x) \geq u$ implies $x_u \in f$. Again, if $u > \frac{k^*-k}{2}$, then $f(x) \geq \frac{k^*-k}{2}$. So $f(x) + u > \frac{k^*-k}{2} + \frac{k^*-k}{2} = k^* - k$, which implies that $x_u(k^*, q_k)f$. Thus $x_u \in \vee(k^*, q_k)f$. Let $f(x\gamma y) \geq \min\{f(y), \frac{k^*-k}{2}\}$ for all $x, y \in S$ and $\gamma \in \Gamma$. Let $y_u \in f$ $(u \in (0, 1])$. Then $f(y) \geq u$. So $f(x\gamma y) \geq \min\{f(y), \frac{k^*-k}{2}\} \geq \min\{u, \frac{k^*-k}{2}\}$. If $u \leq \frac{k^*-k}{2}$, then $f(x\gamma y) \geq u$ implies $(x\gamma y)_u \in f$. If $u > \frac{k^*-k}{2}$, then $f(x\gamma y) \geq \frac{k^*-k}{2}$. So $f(x\gamma y) + u > \frac{k^*-k}{2} + \frac{k^*-k}{2} = k^* - k$, it follows that $(x\gamma y)_u(k^*, q_k)f$. Thus $(x\gamma y)_u \in \vee(k^*, q_k)f$. Hence f is an $(\in, \in \vee(k^*, q_k))$-fuzzy left Γ-ideal of S. □

Corollary 3.4. *A fuzzy subset f of S is an $(\in, \in \vee(k^*, q_k))$-fuzzy Γ-ideal of S if and only if*

(1) $x \leq y \Rightarrow f(x) \geq \min\{f(y), \frac{k^*-k}{2}\}$;

(2) $f(x\gamma y) \geq \min\{f(y), \frac{k^*-k}{2}\}$ *and* $f(x\gamma y) \geq \min\{f(x), \frac{k^*-k}{2}\}$;

for all $x, y \in S$ and $\gamma \in \Gamma$.

Definition 3.5. Let f be any fuzzy subset of an ordered Γ-semigroup S. For any $u \in (0, 1]$, the set

$$U(f; u) = \{x \in S \mid f(x) \geq u\},$$

is called a level subset of f.

Theorem 3.5. *Let f be a fuzzy subset of S. Then f is an $(\in, \in \vee(k^*, q_k))$-fuzzy left (resp. right) Γ-ideal of S if and only if $U(f; u)(\neq \emptyset)$ $(u \in (0, \frac{k^*-k}{2}])$ is a left (resp. right) Γ-ideal of S.*

Proof. Suppose that f is an $(\in, \in \vee(k^*, q_k))$-fuzzy left Γ-ideal of S. Let $x, y \in S$ be such that $x \leq y \in U(f; u)$, where $u \in (0, \frac{k^*-k}{2}]$. Then $f(y) \geq u$. By Theorem 3.3, $f(x) \geq \min\{f(y), \frac{k^*-k}{2}\} \geq \min\{u, \frac{k^*-k}{2}\} = u$. Therefore $x \in L$ and $y \in U(f; u)$. Let $x \in S$ and $y \in U(f; u)$. Then $f(x) \geq u$. So, by Theorem 3.3, $f(x\gamma y) \geq \min\{f(y), \frac{k^*-k}{2}\} \geq \min\{u, \frac{k^*-k}{2}\} = u$. Thus $f(x\gamma y) \geq u$. Therefore $x\gamma y \in U(f; u)$. Hence $U(f; u)$ is a left Γ-ideal.

Conversely assume that $U(f; u)(\neq \emptyset)$ is a left Γ-ideal of S for all $u \in (0, \frac{k^*-k}{2}]$. Take any $x, y \in S$ with $x \leq y$. If $f(x) < \min\{f(y), \frac{k^*-k}{2}\}$. Then $f(x) < u \leq \min\{f(y), \frac{k^*-k}{2}\}$, for some $u \in (0, \frac{k^*-k}{2}]$. It follows that $y \in U(f; u)$ but $x \notin U(f; u)$, which is a contradiction. Thus $f(x) \geq \min\{f(y), \frac{k^*-k}{2}\}$ for all $x, y \in S$ with $x \leq y$.

Again, if $f(x\gamma y) < \min\{f(y), \frac{k^*-k}{2}\}$ for some $x, y \in S$. Therefore there exist $u \in (0, \frac{k^*-k}{2}]$ such that $f(x\gamma y) < u \leq \min\{f(y), \frac{k^*-k}{2}\}$ implies $y_u \in U(f;u)$ but $(x\gamma y)_u \notin U(f;u)$, which is again a contradiction. Thus $f(x\gamma y) \geq \min\{f(y), \frac{k^*-k}{2}\}$ for all $x, y \in S$. Hence by Theorem 3.3, f is an $(\in, \in \vee(k^*, q_k))$-fuzzy left Γ-ideal of S. □

Corollary 3.6. *Let f be a fuzzy subset of S. Then f is an $(\in, \in \vee(k^*, q_k))$-fuzzy Γ-ideal of S if and only if $U(f;u)(\neq \emptyset)$ $(u \in (0, \frac{k^*-k}{2}])$ is a Γ-ideal of S.*

Lemma 3.7. *If f is a nonzero $(\in, \in \vee(k^*, q_k))$-fuzzy left (resp. right) Γ-ideal of S, then the set $f_0 = \{x \in S \mid f(x) > 0\}$ is a left (resp. right) Γ-ideal of S.*

Proof. Let f be an $(\in, \in \vee(k^*, q_k))$-fuzzy left Γ-ideal of S. Let $x, y \in S$, $x \leq y$ and $y \in f_0$. Then $f(y) > 0$. As f is an $(\in, \in \vee(k^*, q_k))$-fuzzy left Γ-ideal of S, $f(x) \geq \min\{f(y), \frac{k^*-k}{2}\} > 0$. Since $f(y) > 0$, we have $f(x) > 0$ and so $x \in f_0$. Let $x \in S$ and $y \in f_0$. Then $f(y) > 0$. Therefore $f(x\gamma y) \geq \min\{f(y), \frac{k^*-k}{2}\} > 0$. Thus $x\gamma y \in f_0$, as required. □

Corollary 3.8. *If f is a nonzero $(\in, \in \vee(k^*, q_k))$-fuzzy Γ-ideal of S, then the set $f_0 = \{x \in S \mid f(x) > 0\}$ is a Γ-ideal of S.*

4. (k^*, k)-LOWER PARTS OF $(\in, \in \vee(k^*, q_k))$-FUZZY Γ-IDEALS

Let f be a fuzzy subset of an ordered Γ-semigroup S. The (k^*, k)-lower part $\underline{f_k^{k^*}}$ of f is defined as follows:

$$\underline{f_k^{k^*}}(x) = \min\left\{f(x), \frac{k^*-k}{2}\right\}$$

for all $x \in S$ and $0 \leq k < k^* \leq 1$.

For any nonempty subset A of S and fuzzy subset f of S, $(\underline{f_A})_k^{k^*}$, the (k^*, k)-lower part of the characteristic function f_A, will be denoted by $(\underline{f_k^{k^*}})_A$ in the sequel.

Let f and g be any fuzzy subsets of S. Define $f(\cap)_k^{k^*} g$, $f(\cup)_k^{k^*} g$ and $f(\circ)_k^{k^*} g$ as follows:

$$(f(\cap)_k^{k^*} g)(x) = \min\left\{(f \cap g)(x), \frac{k^*-k}{2}\right\}$$

$$(f(\cup)_k^{k^*} g)(x) = \min\left\{(f \cup g)(x), \frac{k^*-k}{2}\right\}$$

$$(f(\circ)_k^{k^*} g)(x) = \min\left\{(f \circ g)(x), \frac{k^*-k}{2}\right\}$$

for all $x \in S$ and $0 \leq k < k^* \leq 1$.

Let f and g be any fuzzy subsets of S. Then we have: (1) $(\underline{f_k^{k^*}})_k^{k^*} = \underline{f_k^{k^*}}$ and $\underline{f_k^{k^*}} \subseteq f$; (2) If $f \subseteq g$, and $h \in F(S)$, then $f(\circ)_k^{k^*} h \subseteq g(\circ)_k^{k^*} h$ and $h(\circ)_k^{k^*} f \subseteq h(\circ)_k^{k^*} g$; (3) $f(\cap)_k^{k^*} g = \underline{f_k^{k^*}} \cap \underline{g_k^{k^*}}$; (4) $f(\cup)_k^{k^*} g = \underline{f_k^{k^*}} \cup \underline{g_k^{k^*}}$; (5) $f(\circ)_k^{k^*} g = \underline{f_k^{k^*}} \circ \underline{g_k^{k^*}}$.

Lemma 4.1. *Let A and B be any nonempty subsets of an ordered Γ-semigroup S. Then*

(1) $f_A(\cap)_k^{k^*} f_B = (\underline{f_k^{k^*}})_{A\cap B}$;
(2) $f_A(\cup)_k^{k^*} f_B = (\underline{f_k^{k^*}})_{A\cup B}$;
(3) $f_A(\circ)_k^{k^*} f_B = (\underline{f_k^{k^*}})_{(A\Gamma B]}$.

Proof. Straightforward. □

Theorem 4.2. *The (k^*, k)-lower part $(\underline{f_k^{k^*}})_A$ of the characteristic function f_A of a nonempty subset A is an $(\in, \in \vee(k^*, q_k))$-fuzzy left (resp. right) Γ-ideal of S if and only if A is a left (resp. right) Γ-ideal of S.*

Proof. Let A be a left Γ-ideal of S. Let $x, y \in S$, $x \leq y$ and $u \in (0,1]$ be such that $y_u \in (\underline{f_k^{k^*}})_A$. Then $y \in A$, $(\underline{f_k^{k^*}})_A(y) \geq u$. As A is a left Γ-ideal of S and $x \leq y \in A$, $x \in A$. Thus $f_A(x) \geq \frac{k^*-k}{2}$. If $u \leq \frac{k^*-k}{2}$, then $f_A(x) \geq u$, so we have $x_u \in (\underline{f_k^{k^*}})_A$. If $u > \frac{k^*-k}{2}$, then $f_A(x) + u > \frac{k^*-k}{2} + \frac{k^*-k}{2} = k^* - k$. So $x_u(k^*, q_k)(\underline{f_k^{k^*}})_A$. Therefore $x_u \in \vee(k^*, q_k)(\underline{f_k^{k^*}})_A$.

Next suppose that $x \in S, y_u \in (\underline{f_k^{k^*}})_A$ and $\gamma \in \Gamma$. Then $y \in A$, $(\underline{\eta_k^{k^*}})_A(y) \geq u$. As A is a left Γ-ideal of S, $x\gamma y \in A$. Thus $f_A(x\gamma y) \geq \frac{k^*-k}{2}$. If $u \leq \frac{k^*-k}{2}$, then $f_A(x\gamma y) \geq u$, so $(x\gamma y)_u \in (\underline{f_k^{k^*}})_A$. If $u > \frac{k^*-k}{2}$, then $f_A(x\gamma y) + u > \frac{k^*-k}{2} + \frac{k^*-k}{2} = k^* - k$. So $(x\gamma y)_u(k^*, q_k)(\underline{f_k^{k^*}})_A$. Therefore $(x\gamma y)_u \in \vee(k^*, q_k)(\underline{f_k^{k^*}})_A$.

Conversely assume that $(\underline{f_k^{k^*}})_A$ is an $(\in, \in \vee(k^*, q_k))$-fuzzy left Γ-ideal of S. Let $x, y \in S$ such that $x \leq y$. If $y \in A$, then $(\underline{f_k^{k^*}})_A(y) = \frac{k^*-k}{2}$. Since $(\underline{f_k^{k^*}})_A$ is an $(\in, \in \vee(k^*, q_k))$-fuzzy left Γ-ideal of S, and $x \leq y$, we have $(\underline{f_k^{k^*}})_A(x) \geq \min\{(\underline{f_k^{k^*}})_A(y), \frac{k^*-k}{2}\} = \frac{k^*-k}{2}$. It follows that $(\underline{f_k^{k^*}})_A(x) = \frac{k^*-k}{2}$ and so $x \in A$. Let $x \in S$ and $y \in A$. $(\underline{f_k^{k^*}})_A(y) = \frac{k^*-k}{2}$. Now, $(\underline{f_k^{k^*}})_A(x\gamma y) \geq \min\{(\underline{f_k^{k^*}})_A(y), \frac{k^*-k}{2}\} = \frac{k^*-k}{2}$. Hence $(\underline{f_k^{k^*}})_A(x\gamma y) = \frac{k^*-k}{2}$ and so $x\gamma y \in A$. Therefore A is a left Γ-ideal of S. □

Corollary 4.3. *The (k^*, k)-lower part $(\underline{f_k^{k^*}})_A$ of the characteristic function f_A of a nonempty subset A is an $(\in, \in \vee(k^*, q_k))$-fuzzy Γ-ideal of S if and only if A is a Γ-ideal of S.*

Lemma 4.4. *If f is an $(\in, \in \vee(k^*, q_k))$-fuzzy left (resp. right) Γ-ideal of S, then $\underline{f_k^{k^*}}$ is a fuzzy left (resp. right) Γ-ideal of S.*

Proof. Let $x, y \in S$ such that $x \leq y$. As f is an $(\in, \in \vee(k^*, q_k))$-fuzzy left Γ-ideal of S, $f(x) \geq \min\{f(y), \frac{k^*-k}{2}\}$. It follows that $\min\{f(x), \frac{k^*-k}{2}\} \geq \min\{f(y), \frac{k^*-k}{2}\}$ and so, $(\underline{f_k^{k^*}})(x) \geq (\underline{f_k^{k^*}})(y)$. Let $x, y \in S$ and $\gamma \in \Gamma$, then we have $f(x\gamma y) \geq \min\{f(y), \frac{k^*-k}{2}\}$. Then $\min\{f(x\gamma y), \frac{k^*-k}{2}\} \geq \min\{f(y), \frac{k^*-k}{2}\}$, and so $(\underline{f_k^{k^*}})(x\gamma y) \geq \min\{(\underline{f_k^{k^*}})(y)\}$. Therefore $\underline{f_k^{k^*}}$ is a fuzzy left $\Gamma-$ideal of S. □

Corollary 4.5. *If f is an $(\in, \in \vee(k^*, q_k))$-fuzzy Γ-ideal of S, then $\underline{f_k^{k^*}}$ is a fuzzy Γ-ideal of S.*

Theorem 4.6. *An ordered Γ-semigroup S is left (resp. right) regular if and only if $(\underline{f_k^{k^*}})(a) = (\underline{f_k^{k^*}})(a\alpha a)$, for each $(\in, \in \vee(k^*, q_k))$-fuzzy left (resp. right) Γ-ideal f of S and for each $a \in S, \alpha \in \Gamma$.*

Proof. Let $a \in S$. As S is left regular, $a \in (S\Gamma a\Gamma a]$. Then there exist $x \in S$ and $\gamma, \beta \in \Gamma$ such that $a \leq x\gamma a\alpha a$. Since f is an $(\in, \in \vee(k^*, q_k))$-fuzzy left Γ-ideal of S, we have

$$
\begin{aligned}
f(a) &\geq \min\left\{f(x\gamma a\alpha a), \frac{k^*-k}{2}\right\} \\
&\geq \min\left\{f(a\alpha a), \frac{k^*-k}{2}\right\}
\end{aligned}
$$

$$\geq \min\left\{f(a), \frac{k^*-k}{2}\right\}$$

Therefore $(\underline{f_k^{k^*}})(a) = (\underline{f_k^{k^*}})(a\alpha a)$.

Conversely take any $a \in S, \alpha \in \Gamma$. Consider the left Γ-ideal $L(a\alpha a) = (a\alpha a \cup S\Gamma a\alpha a]$ of S generated by $a\alpha a (a \in S)$. Then, by Theorem 4.2, $(\underline{f_k^{k^*}})_{L(a\alpha a)}$ is an $(\in, \in \vee(k^*, q_k))$-fuzzy left Γ-ideal of S. Therefore by hypothesis, $(\underline{f_k^{k^*}})_{L(a\alpha a)}(a) = (\underline{f_k^{k^*}})_{L(a\alpha a)}(a\alpha a)$. Since $a^2 \in L(a\alpha a)$, we have $(\underline{f_k^{k^*}})_{L(a\alpha a)}(a\alpha a) = \frac{k^*-k}{2}$, and so, $(\underline{f_k^{k^*}})_{L(a\alpha a)}(a) = \frac{k^*-k}{2}$. Thus, $a \in L(a\alpha a)$. Thus $a \leq a\alpha a$ or $a \leq x\gamma a\alpha a$. If $a \leq a\alpha a$, then $a \leq a\alpha a = a\alpha a \leq a\alpha a\alpha a = a\alpha a\alpha a \in S\Gamma a\Gamma a$ and $a \in (S\Gamma a\Gamma a]$. Again, if $a \leq x\gamma a\alpha a$, then $a \in (S\Gamma a\Gamma a]$. So $a \in (S\Gamma a\Gamma a]$. Therefore S is left regular. □

Theorem 4.7. *The following assertions are equivalent in S:*

(1) *S is regular.*
(2) *$f(\circ)_k^{k^*} g = f(\cap)_k^{k^*} g$ for each $(\in, \in \vee(k^*, q_k))$-fuzzy right Γ-ideal f and $(\in, \in \vee(k^*, q_k))$-fuzzy left Γ-ideal g of S.*

Proof. $(1) \Rightarrow (2)$. Suppose that f and g are $(\in, \in \vee(k^*, q_k))$-fuzzy right Γ-ideal and an $(\in, \in \vee(k^*, q_k))$-fuzzy left Γ-ideal of S, and $a \in S$. Then there exists $r \in S$ and α, β such that $a \leq ara$, it follows that $(a\alpha r, a) \in A_a$. Then we have

$$\begin{aligned}
(f(\circ)_k^{k^*} g)(a) &= \bigvee_{(y,z)\in A_a} \min\left\{\underline{f_k^{k^*}}(y), \underline{g_k^{k^*}}(z)\right\} \\
&\geq \min\left\{\underline{f_k^{k^*}}(a\alpha r), \underline{g_k^{k^*}}(a)\right\} \\
&= \min\left\{f(a\alpha r), \underline{g_k^{k^*}}(a), \frac{k^*-k}{2}\right\} \\
&= \min\left\{f(a), \underline{g_k^{k^*}}(a), \frac{k^*-k}{2}\right\} \\
&= \min\left\{\underline{f_k^{k^*}}(a), \underline{g_k^{k^*}}(a)\right\} \\
&= (f(\cap)_k^{k^*} g)(a).
\end{aligned}$$

Thus $f(\cap)_k^{k^*} g \subseteq f(\circ)_k^{k^*} g$. Inverse inclusion is obvious. Therefore $f(\cap)_k^{k^*} g = f(\circ)_k^{k^*} g$.

$(2) \Rightarrow (1)$. Suppose that L and R are left and right Γ-ideal of S. Take any $x \in L \cap R$. Then $x \in L$ and $x \in R$. As L is a left Γ-ideal and R is a right Γ-ideal of S, by Theorem 4.2, $(\underline{f_k^{k^*}})_L$ is an $(\in, \in \vee(k^*, q_k))$-fuzzy left Γ-ideal and $(\underline{f_k^{k^*}})_R$ is an $(\in, \in \vee(k^*, q_k))$-fuzzy right Γ-ideal of S. So, we have $(f_R(\circ)_k^{k^*} f_L)(x) \geq (f_R(\cap)_k^{k^*} f_L)(x) = \min\{(\underline{f_k^{k^*}})_R(x), (\underline{f_k^{k^*}})_L(x)\}$. Since $x \in L$ and $x \in R$, $(\underline{f_k^{k^*}})_L = \frac{k^*-k}{2}$ and $(\underline{f_k^{k^*}})_R = \frac{k^*-k}{2}$. Thus $(f_R(\cap)_k^{k^*} f_A)(x) = \frac{k^*-k}{2}$. Therefore $(f_R(\circ)_k^{k^*} f_A)(x) \geq \frac{k^*-k}{2}$. Since $(f_R(\circ)_k^{k^*} f_L)(x) \leq \frac{k^*-k}{2}$. Therefore $(f_R(\circ)_k^{k^*} f_L)(x) = \frac{k^*-k}{2}$. By Lemma 4.1(3), we have

$$(f_R(\circ)_k^{k^*} f_L)(x) = (\underline{f_k^{k^*}})_{(R\Gamma L]}(x).$$

Therefore, $(\underline{f_k^{k^*}})_{(R\Gamma L]}(x) = \frac{k^*-k}{2}$ and so, $x \in (R\Gamma L]$. It follows that $L \cap R \subseteq (R\Gamma L]$. Also $(R\Gamma L] \subseteq L \cap R$. Thus $L \cap R = (R\Gamma L]$. Hence by Lemma 2.1, S is regular. □

5. Conclusion

The main purpose of the present paper is to introduce the concept of an $(\in, \in \vee(k^*, q_k))$-fuzzy Γ-ideal, $(\in, \in \vee(k^*, q_k))$-fuzzy left Γ-ideal and $(\in, \in \vee(k^*, q_k))$-fuzzy right Γ-ideal in ordered semigroups by generalizing the concept of $(\in, \in \vee q_k)$-fuzzy Γ-ideal, $(\in, \in \vee q_k)$-fuzzy left Γ-ideal and $(\in, \in \vee q_k)$-fuzzy right Γ-ideal. Also, we enhance the understanding of regular ordered Γ-semigroups by considering the structural influence of the $(\in, \in \vee(k^*, q_k))$-fuzzy Γ-ideals, $(\in, \in \vee(k^*, q_k))$-fuzzy left Γ-ideals and $(\in, \in \vee(k^*, q_k))$-fuzzy right Γ-ideals. In our future work, by using the concept of (k^*, q)-quasi-coincident with a fuzzy subset of ordered Γ-semigroups, the concept of $(\in, \in \vee(k^*, q_k))$-fuzzy interior Γ-ideals and $(\in, \in \vee(k^*, q_k))$-fuzzy quasi Γ-ideals of ordered Γ-semigroups will be introduced.

Following are the particular cases of the present paper:

(1). If we take $k^* = 1$, then the definition of an $(\in, \in \vee(k^*, q_k))$-fuzzy Γ-ideal, $(\in, \in \vee(k^*, q_k))$-fuzzy left Γ-ideal and $(\in, \in \vee(k^*, q_k))$-fuzzy right Γ-ideal reduce to a concept and we call as an $(\in, \in \vee q_k)$-fuzzy Γ-ideal, $(\in, \in \vee q_k)$-fuzzy left Γ-ideal and $(\in, \in \vee q_k)$-fuzzy right Γ-ideal. Thus we can apply all the results of this paper in the setting of $(\in, \in \vee q_k)$-fuzzy Γ-ideals.

(2). If we take $k^* = 1$ and $k = 0$, then the definition of an $(\in, \in \vee(k^*, q_k))$-fuzzy Γ-ideal, $(\in, \in \vee(k^*, q_k))$-fuzzy left Γ-ideal and $(\in, \in \vee(k^*, q_k))$-fuzzy right Γ-ideal reduce to a concept and we call as an $(\in, \in \vee q)$-fuzzy Γ-ideal, $(\in, \in \vee q)$-fuzzy left Γ-ideal and $(\in, \in \vee q)$-fuzzy right Γ-ideal. Thus all the results of this paper may be applied in the setting of $(\in, \in \vee q)$-fuzzy (m, n)-ideals.

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