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A NOTE ON SPACETIMES IN $f(R)$-GRAVITY

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ABSTRACT. In this work, the investigation of spacetimes admitting semiconformal curvature tensor in $f(R)$-gravity theory is the major goal. First, semiconformally flat spacetimes in the presence of $f(R)$-gravity are investigated, and the relationship between isotropic pressure and energy density is discovered. Some energy conditions are then taken into account. Finally, spacetimes with divergence-free semiconformal curvature tensors in $f(R)$-gravity are explored.

1. INTRODUCTION AND PRELIMINARIES

Ishii [17], introduced the set of conharmonic transformations as a subgroup of the conformal group of transformations as given by

$$\widetilde{g}_{lm} = e^{2\gamma} g_{lm}, \tag{1.1}$$

and it satisfying the condition

$$\nabla_l \gamma^l + \nabla\gamma\nabla\gamma^l = 0, \tag{1.2}$$

where g_{lm} and $\widetilde{g}^{lm}$ are the metric tensors for Riemannian spaces $\mathbb{V}$ and $\widetilde{\mathbb{V}}$, respectively, and γ is a real scalar function. A rank four tensor $\mathcal{H}^j_{lmi}$ which is invariant under conharmonic transformation, on Riemannian manifold $(\Theta^n, g), n \geq 4$ is defined as [1]:

$$\mathcal{H}^j_{lmi} = \mathcal{R}^j_{lmi} + \frac{1}{n-2}[\delta^j_m \mathcal{R}_{li} - \delta^j_i \mathcal{R}_{lm} + g_{li}\mathcal{R}^j_m - g_{lm}\mathcal{R}^j_i], \tag{1.3}$$

where $\mathcal{R}^j_{lmi}$, $\mathcal{R}_{li}$ are Riemann and Ricci curvature tensors respectively. The geometrical and physical significance of these tensor has been discussed by several authors (see, [2, 4, 5]). Recently, Kim (see, [18, 19]) introduced a curvature like tensor which is remains invariant under condition (1.2), this curvature-like tensor $\mathcal{S}^j_{lmi}$ of type $(1,3)$ on a (Θ^n, g) is called semiconformal curvature tensor and is given by:

$$\mathcal{S}^j_{lmi} = -(n-2)\mu\mathcal{W}^j_{lmi} + [\nu + (n-2)\mu]\mathcal{H}^j_{lmi}, \tag{1.4}$$

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where μ, ν are constants which are not simultaneously zero and the Weyl conformal curvature tensor $\mathcal{W}^j_{lmi}$ as noted [24]:

$$\mathcal{W}^j_{lmi} = \mathcal{R}^j_{lmi} + \frac{1}{n-2}[\delta^j_m \mathcal{R}_{li} - \delta^j_i \mathcal{R}_{lm} + g_{li}\mathcal{R}^j_m - g_{lm}\mathcal{R}^j_i] + \frac{\mathcal{R}}{(n-1)(n-2)}[\delta^j_i g_{lm} - \delta^j_m g_{li}], \tag{1.5}$$

For ν=1 and μ=-$\frac{1}{(n-2)}$, the semiconformal curvature tensor reduces to conformal curvature tensor, while for ν=1 and μ=0, it reduces to conharmonic curvature tensor.
In Einstein's theory of gravity, Einstein's field equations (EFE)

$$\mathcal{R}_{lm} - \frac{\mathcal{R}}{2} g_{lm} = \kappa \mathcal{T}_{lm}, \tag{1.6}$$

where κ being the Newtonian constant and $\mathcal{T}_{lm}$ is the EMT [20], imply that the EMT, $\mathcal{T}_{lm}$ is of vanishing divergence if, $\mathcal{T}_{lm}$ is covariantly constant.
The $f(R)$-gravity theory is the most popular of such modification of the standard theory of gravity. This important modification was first introduced in [6]. This modified theory can be obtained by replacing the scalar curvature $\mathcal{R}$ with a generic function $f(R)$ in the Einstein-Hilbert action. We recall the modified Einstein-Hilbert action term [12]:

$$\mathcal{G} = \frac{1}{\kappa^2} \int f(R)\sqrt{-g}dx^4x + \int \mathcal{L}_m \sqrt{-g}dx^4x, \tag{1.7}$$

where $f(R)$ is an arbitrary function of the Ricci scalar $\mathcal{R}$, $\mathcal{L}_m$ is the matter Lagrangian density, and we define the stress-energy tensor of matter as

$$\mathcal{T}_{lm} = -\frac{2}{\sqrt{-g}} \frac{\delta(\sqrt{-g}\mathcal{L}_m)}{\delta g^{lm}}. \tag{1.8}$$

By varying the action $\mathcal{G}$ of the gravitational field with respect to the metric tensor components g^{ij} and using the least action principle the field equations of $f(R)$ gravity are given as.

$$f^{'}(\mathcal{R})\mathcal{R}_{lm} - \frac{1}{2} f(R) g_{lm} + g_{lm}(\diamond - \nabla_l \nabla_m) f^{'}(\mathcal{R}) = \kappa^2 \mathcal{T}_{lm}, \tag{1.9}$$

where $\diamond$ denote the d'Alembertian operator,$f^{'}$=$\frac{\partial f(R)}{\partial R}$ which must be positive to ensure attractive gravity [9].The $f(R)$ gravity represents a higher order and well-studied theory of gravity. For example, an earlier investigation of quintessence and cosmic acceleration in $f(R)$ gravity theory as a higher order gravity theory are considered in [10]. Also, Capoziello et al. proved that, in a generalized Robertson-Walker spacetime with divergence free conformal curvature tensor, the higher order gravity tensor has the form of perfect fluid [11]. In a series of recent studies, weakly Ricci symmetric spacetimes $(WRS)_4$, almost pseudo-Ricci symmetric spacetimes$(APRS)_4$, conformally flat generalized Ricci recurrent spacetimes, energy conditions for a $(WRS)_4$ spacetime in $f(R)$-gravity and spacetime with concircular curvature tensor are investigated in $f(R)$ gravity theory, whereas $\mathcal{LP}$-Sasakian spacetimes is also studied (see, [13, 14, 15, 16, 26]). Motivated by these studies, the main aim of this paper is to study some conditions on semiconformal curvature tensor in $f(R)$ gravity. Also we deduce the energy conditions under the condition of the functional form of $f(R)$.

2. SEMICONFORMALLY FLAT SPACETIME IN $f(R)$ GRAVITY

In this section, we assume that the semiconformally flat spacetime in $f(R)$ gravity and find out some results.

With the help of (1.3) and (1.4), equation (1.4) reduces for n=4:

$$\mathcal{S}^j_{lmi} = \nu[\mathcal{R}^j_{lmi} + \frac{1}{2}(\delta^j_m \mathcal{R}_{li} - \delta^j_i \mathcal{R}_{lm} + g_{li}\mathcal{R}^j_m - g_{lm}\mathcal{R}^j_i] - \frac{\mu\mathcal{R}}{3}(\delta^j_i g_{lm} - \delta^j_m g_{li}), \quad (2.1)$$

If $\mathcal{S}^j_{lmi}$=0, then (2.1) leads to

$$\nu\mathcal{R}^j_{lmi} = -\frac{\nu}{2}(\delta^j_m \mathcal{R}_{li} - \delta^j_i \mathcal{R}_{lm} + g_{li}\mathcal{R}^j_m - g_{lm}\mathcal{R}^j_i) - \frac{\mu\mathcal{R}}{3}(\delta^j_i g_{lm} - \delta^j_m g_{li}). \quad (2.2)$$

After taking the transvection over j and m, we get

$$\mathcal{R}_{li} = -\left(\frac{\nu + 2\mu}{\nu}\right)\frac{\mathcal{R}}{4} g_{li}. \quad (2.3)$$

Also, let $\mathbb{V}_4$ be an Einstein spacetime, then (2.1) takes the form

$$\mathcal{S}^j_{lmi} = \nu\mathcal{R}^j_{lmi} + \frac{3\nu + 4\mu}{12}\mathcal{R}(\delta^j_m g_{li} - \delta^j_i g_{lm}. \quad (2.4)$$

Using the condition $\mathcal{S}^j_{lmi}$=0 and contracting with g^{lj} equation (2.4) implies that

$$\mathcal{R}_{mi} = -\left(\frac{3\nu + 4\mu}{4}\right)\mathcal{R} g_{mi}. \quad (2.5)$$

As a consequence, we state:

Theorem 2.1. *A semiconformally flat spacetime is an Einstein space and scalar curvature $\mathcal{R}$ is constant.*

Corollary 2.2. *A semiconformally flat Einstein spacetime is of constant scalar curvature $\mathcal{R}$.*

Corollary 2.3. *A conharmonically flat spacetime is of constant scalar curvature $\mathcal{R}$.*

Corollary 2.4. *A conformally flat Einstein spacetime is of constant scalar curvature $\mathcal{R}$.*

Corollary 2.5. *A conharmonically flat Einstein spacetime is of constant scalar curvature $\mathcal{R}$.*

In view of Theorem 2.1 , the field equation (1.9) in $f(R)$-gravity possess the form

$$\mathcal{R}_{lm} - \frac{f}{2f'} g_{lm} = \frac{\kappa}{f'}\mathcal{T}_{lm}. \quad (2.6)$$

In vacuum case (that is, $\mathcal{T}_{lm}$=0), it gives

$$\mathcal{R}_{lm} - \frac{f}{2f'} g_{lm} = 0.$$

After contracting with g^{lm} and integrating, we get

$$f = \psi\mathcal{R}^{-\frac{2\nu}{(\nu+2\mu)}}, \quad (2.7)$$

where ψ=$constant$. Conversely, if (2.6) holds, then we yields $\mathcal{T}_{lm}$=0.
Thus we state:

Theorem 2.6. *A semiconformally flat spacetime in $f(R)$ gravity is vacuum if and only if f=$\psi\mathcal{R}^{-\frac{2\nu}{(\nu+2\mu)}}$.*

Corollary 2.7. *A semiconformally flat Eistein spacetime in $f(R)$ gravity is vacuum if and only if f=$\psi\mathcal{R}^{-\frac{2}{(3\nu+4\mu)}}$.*

Corollary 2.8. *A conharmonically flat spacetime in $f(R)$ gravity is vacuum if and only if f=$\psi\mathcal{R}^{-2}$*

Corollary 2.9. *A conformally flat Einstein spacetime in $f(R)$ gravity is vacuum if and only if* $f=\psi\mathcal{R}^{-1}$.

Corollary 2.10. *A conharmonically flat Einstein spacetime in $f(R)$ gravity is vacuum if and only if* $f=\psi\mathcal{R}^{-\frac{2}{3}}$

A spacetime is said to admit a matter collineation with respect to a vector field ζ if the Lie derivative of the energy momentum tensor $\mathcal{T}$ with respect to ζ satisfies

$$\mathfrak{L}_{\zeta}\mathcal{T}_{lm} = 0. \tag{2.8}$$

Thus, every Killing vector field is a matter collineation, but the converse is not generally true. The energy-momentum tensor $\mathcal{T}_{lm}$ has the Lie inheritance property along the flow lines of the vector field ζ if the Lie derivative of $\mathcal{T}_{lm}$ with respect to ζ satisfies (see, [3, 25]:

$$\mathfrak{L}_{\zeta}\mathcal{T}_{lm} = 2\psi^{\flat}\mathcal{T}_{lm}. \tag{2.9}$$

With the help of (2.3) and (2.6), we have

$$-\left[\left(\frac{\nu+2\mu}{\nu}\right)\frac{\mathcal{R}}{4} + \frac{f}{2f'}\right]g_{lm} = \frac{\kappa}{f'}\mathcal{T}_{lm}. \tag{2.10}$$

We suppose that a non-vacuum semiconformally flat spacetime. Then the Lie derivative $\mathfrak{L}_{\zeta}$ of (2.10) indicate that

$$-\left[\left(\frac{\nu+2\mu}{\nu}\right)\frac{\mathcal{R}}{4} + \frac{f}{2f'}\right]\mathfrak{L}_{\zeta}g_{lm} = \frac{\kappa}{f'}\mathfrak{L}_{\zeta}\mathcal{T}_{lm}. \tag{2.11}$$

Let that vector field ζ is Killing on Θ, then (2.11) implies (2.8). Conversely, if (2.8) holds then it follows from (2.11) that $\mathfrak{L}_{\zeta}g_{lm}$=0.
So, we state the following theorem:

Theorem 2.11. *Let Θ be a non-vacuum semiconformally flat spacetime satisfying $f(R)$-gravity, then the vector field ζ is Killing if and only if Θ admits matter collineation with respect to ζ.*

Corollary 2.12. *Let Θ be a non-vacuum semiconformally flat Einstein spacetime satisfying $f(R)$-gravity, then the vector field ζ is Killing if and only if Θ admits matter collineation with respect to ζ.*

The isometry of spacetimes prescriped by Killing vector fields represents a very important type of spacetime symmetry. Spacetimes of constant curvature are known to have maximum such symmetry, that is, they admit the maximum number of linearly independent Killing vector fields. The maximum numer of linearly independent Killing vector fields in an n-dimensional spacetime is $\frac{n(n+1)}{2}$ [21]. Due to this fact and the above discussion, we state

Corollary 2.13. *A non-vacuum semiconformally flat spacetime satisfying $f(R)$-gravity admits the maximum number of matter collineations* 10.

Again, if ζ be a conformal Killing vector field, that is, $\mathfrak{L}_{\zeta}g_{lm}$=$2\psi^{\flat}g_{lm}$ holds on Θ. Therefore (2.11) implies (2.9). Conversely, assume that (2.9) holds, then from (2.11) we obtain $\mathfrak{L}_{\zeta}g_{lm}$=$2\psi^{\flat}g_{lm}$. As per the consequence we state the result.

Theorem 2.14. *Let Θ be a non-vacuum semiconformally flat spacetime satisfying $f(R)$-gravity, then Θ has a conformal Killing vector filed ζ if only if the energy-momentum tensor $\mathcal{T}_{lm}$ has the Lie inheritance property along ζ.*

Also, taking covariant derivative of both sides of (2.10) implies that $\nabla_k \mathcal{T}_{lm}$=0. Since in a semiconformally flat spacetime $\mathcal{R}$ is constant, so f and $f^{'}$ are also constant. Thus from (2.6) we yields $\nabla_k \mathcal{R}_{lm}$=0. At this sequel we state the results:

Theorem 2.15. *Let Θ be a non-vacuum semiconformally flat spacetime satisfying $f(R)$-gravity is Ricci symmetric.*

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3. Semiconformally flat perfect fulid spacetime in $f(R)$-Gravity

We consider the EMT for a perfect fluid spacetime:

$$\mathcal{T}_{lm} = (\widetilde{p} + \widetilde{\sigma})v_l v_m + \widetilde{p} g_{lm}, \tag{3.1}$$

where $\widetilde{p}$ is the isotropic pressure, $\widetilde{\sigma}$ is the energy density, and u_l is a unit timelike vector field [20, 7].
By virtue of (3.1) and (2.3), one can get from (2.6) that

$$-\left(\frac{\nu + 2\mu}{\nu}\right)\frac{\mathcal{R}}{4} g_{lm} = \frac{\kappa}{f'}[(\widetilde{p} + \widetilde{\sigma})v_l v_m + \widetilde{p} g_{lm}] + \frac{f}{2f'} g_{lm}. \tag{3.2}$$

After contracting (3.2) with v^l, we yields

$$\widetilde{\sigma} = \frac{(\nu + 2\mu)f^{'}\mathcal{R} + 2\nu f}{4\nu\kappa}. \tag{3.3}$$

Again taking transvecting (3.2) with g^{lm} and using (3.3), we get

$$\widetilde{p} = -\frac{(\nu + 2\mu)f^{'}\mathcal{R} + 2\nu f}{4\nu\kappa}. \tag{3.4}$$

So, we state the result:

Theorem 3.1. *In a semiconformally flat perfect fluid spacetime satisfying $f(R)$-gravity the energy density $\widetilde{\sigma}$ and the isotropic pressure $\widetilde{p}$ are constants and given by* (3.3) *and* (3.4) *respectively.*

In view of (3.3) and (3.4), we conclude that $\widetilde{p}$+$\widetilde{\sigma}$=0 that is, the spacetime behaves dark matter era or the perfect fluid reduces to as a cosmological constant[23].
We finalize the corollary:

Corollary 3.2. *A semiconformally flat perfect fluid spacetime satisfying $f(R)$-gravity is dark matter era.*

Again, in radiation era $\widetilde{\sigma}$=3$\widetilde{p}$, the EMT $\mathcal{T}_{lm}$ has the form

$$\mathcal{T}_{lm} = 4\widetilde{p} v_l v_m + \widetilde{p} g_{lm}. \tag{3.5}$$

Keep in mind this fact $\widetilde{p}$+$\widetilde{\sigma}$=0, we have $\widetilde{p}$=0, it follows from (3.5) that $\mathcal{T}_{lm}$=0.
In this sequel we state:

Corollary 3.3. *Let Θ be a semiconformally flat spacetime admitting $f(R)$-gravity, then the Radiation era in Θ is vacuum.*

Further, for pressureless fluid spacetime $\widetilde{p}$=0, the EMT takes the form[22]:

$$\mathcal{T}_{lm} = \widetilde{\sigma} v_l v_m. \tag{3.6}$$

Due to this $\widetilde{p}$+$\widetilde{\sigma}$=0, we get $\widetilde{\sigma}$=0, thus from (3.6), we obtain $\mathcal{T}_{lm}$=0.
Thus we have the result:

Corollary 3.4. *Let Θ be a semiconformally flat dust fulid spacetime satisfying $f(R)$-gravity, then Θ is vacuum.*

In addition, we assume that $\widetilde{p}$ and $\widetilde{\sigma}$ are related by an equation of the form $\widetilde{p}$=$\varpi\widetilde{\sigma}$. Then from (3.3) and (3.4), we get either ϖ=-1 or f=$\psi\mathcal{R}^{-\frac{2\nu}{(\nu+2\mu)}}$, where ψ is an integrating constant.
This leads to the following result:

Theorem 3.5. *The matter content in a semiconformally flat perfect fluid spacetime satisfying $f(R)$-gravity obeys the simple barotropic equation of state $\widetilde{p}$=$\varpi\widetilde{\sigma}$ if and only if it represents a dark matter or f=$\psi\mathcal{R}^{-\frac{2\nu}{(\nu+2\mu)}}$.*

Again we consider the EMT $\mathcal{T}_{lm}$ of a Viscus fluid spacetime [20]:

$$\mathcal{T}_{lm} = (\widetilde{p} + \widetilde{\sigma})v_l v_m + \widetilde{p}g_{lm} + \mathcal{H}_{lm}, \tag{3.7}$$

where $\mathcal{H}_{lm}$ denotes the anisotropic pressure of the fluid.
Using (3.7) and (2.3) in (2.6), we get

$$-\left(\frac{\nu + 2\mu}{\nu}\right)\frac{\mathcal{R}}{4}g_{lm} = \frac{\kappa}{f'}[(\widetilde{p} + \widetilde{\sigma})v_l v_m + \widetilde{p}g_{lm} + \mathcal{H}_{lm}] + \frac{f}{2f'}g_{lm}. \tag{3.8}$$

Taking contraction of (3.8) with g^{lm}, we obtain

$$3\widetilde{p} - \widetilde{\sigma} = -\left[\frac{(\nu + 2\mu)\mathcal{R}f' - 2\nu f'}{\nu\mathcal{R}}\right] - \mathcal{I}. \tag{3.9}$$

where $\mathcal{I}$=$g^{lm}\mathcal{H}_{lm}$.
Thus we can state:

Theorem 3.6. *A semiconformally flat viscous fluid spacetime satisfying $f(R)$-gravity, the isotropic pressure and the energy density are given by the relations*(3.9)*.*

Corollary 3.7. *If a semiconformally flat viscous fluid spacetime satisfying $f(R)$-gravity, then the trace of anisotropic pressure for radiation era is*

$$\mathcal{I} = -\left[\frac{(\nu + 2\mu)\mathcal{R}f' - 2\nu f'}{\nu\mathcal{R}}\right]$$

Next, we investigate whether or not a viscous fluid in semiconformally flat spacetime obeying $f(R)$-gravity may permit heat flux. In order to do this, we take the EMT, $\mathcal{T}_{lm}$[20]:

$$\mathcal{T}_{lm} = (\widetilde{p} + \widetilde{\sigma})v_l v_m + \widetilde{p}g_{lm} + v_m\mathcal{F}_l + \mathcal{F}_m v_l, \tag{3.10}$$

where $\mathcal{F}_l$=$g(\mathcal{B}_1, \xi)$ and v_l=$g(\mathcal{B}_1, \mathcal{V})$ for all vector fields $\mathcal{B}_1$; ξ being the heat flux vector field and $g(\mathcal{V}, \xi)$=0, that is, $\mathcal{F}(\mathcal{V})$=0. In view of (3.10), (2.3) and (2.6), we get

$$-\left[\left(\frac{\nu + 2\mu}{\nu}\right)\frac{\mathcal{R}}{4} + \frac{f}{2f'}\right]g_{lm} = \frac{\kappa}{f'}[(\widetilde{p} + \widetilde{\sigma})v_l v_m + \widetilde{p}g_{lm} + v_m\mathcal{F}_l + \mathcal{F}_m v_l]. \tag{3.11}$$

On contracting (3.11) with v^l, we get

$$\mathcal{F}_m = -(\widetilde{p} + \widetilde{\sigma}) + \frac{f'}{4\kappa}\left[4\kappa + (1 + 2\mu)\mathcal{R} + 2ff'\right]. \tag{3.12}$$

Therefore, we have:

Theorem 3.8. *A viscous fluid in semiconformally flat spacetime obeying $f(R)$-gravity, admits heat flux, provided*

$$\widetilde{p}+\widetilde{\sigma}\neq\frac{f^{'}}{4\kappa}\left[4\kappa+(1+2\mu)\mathcal{R}+2ff^{'}\right]$$

.

3.1. **Energy conditions in semiconformally flat spacetime admitting $f(R)$-gravity.** In both standard and modified theories of gravity, the energy conditions act as a filtration system for the energy-momentum tensor (see, [13, 14, 15]). The authors of [8] investigated weak energy conditions (WEC), dominant energy conditions (DEC), null energy conditions (NEC), and strong energy conditions (SEC) in two extended theories of gravity. In order to state the energy condition, we need to determine the effective isotropic pressure $\widetilde{p}^{\text{eff}}$ and the effective energy density $\widetilde{\sigma}^{\text{eff}}$ in semiconformally flat spacetime admitting $f(R)$-gravity.

Since (2.6) takes the form

$$\mathcal{R}_{lm}-\frac{\mathcal{R}}{2}g_{lm}=\frac{\kappa}{f^{'}}\mathcal{T}^{\text{eff}}_{lm}, \tag{3.13}$$

where

$$\mathcal{T}^{\text{eff}}_{lm}=\mathcal{T}_{lm}+\frac{f-\mathcal{R}f^{'}}{2\kappa}g_{lm}.$$

Also equation (3.1) can be written as follows

$$T^{\text{eff}}_{lm}=(\widetilde{p}^{\text{eff}}+\widetilde{\sigma}^{\text{eff}})v_lv_m+\widetilde{p}^{\text{eff}}g_{lm}, \tag{3.14}$$

where

$$\widetilde{p}^{\text{eff}}=\widetilde{p}+\frac{f-\mathcal{R}f^{'}}{2\kappa},\quad \widetilde{\sigma}^{\text{eff}}=\widetilde{\sigma}-\frac{f-\mathcal{R}f^{'}}{2\kappa}.$$

By using (3.2) and (3.3) it gives

$$\widetilde{p}^{\text{eff}}=-\frac{f^{'}\mathcal{R}}{4\kappa}[1-\frac{2\mu}{\nu}],\quad \widetilde{\sigma}^{\text{eff}}=\frac{f^{'}\mathcal{R}}{4\kappa}[3+\frac{2\mu}{\nu}].$$

As per above consequence, the energy conditions are given by:

(i) Null energy condition (NEC): $\widetilde{p}^{\text{eff}}+\widetilde{\sigma}^{\text{eff}}\geq 0$

(ii) Weak energy condition (WEC): $\widetilde{\sigma}^{\text{eff}}\geq 0$ and $\widetilde{p}^{\text{eff}}+\widetilde{\sigma}^{\text{eff}}\geq 0$

(iii) Strong energy condition (SEC): $\widetilde{\sigma}^{\text{eff}}\geq 0$ and $\widetilde{p}^{\text{eff}}\pm\widetilde{\sigma}^{\text{eff}}\geq 0$

(vi) Dominant energy condition (DEC): $3\widetilde{p}^{\text{eff}}+\widetilde{\sigma}^{\text{eff}}\geq 0$ and $\widetilde{p}^{\text{eff}}+\widetilde{\sigma}^{\text{eff}}\geq 0$.

4. Spacetime with divergence of semiconformal curvature tensor in $f(R)$-gravity

Taking covariant derivative on both sides of (2.1), we get

$$\begin{aligned}\nabla_h\mathcal{S}^j_{lmi}=\nu[\nabla_h\mathcal{R}^j_{lmi}+\tfrac{1}{2}(\delta^j_m\nabla_h\mathcal{R}_{li}-\delta^j_i\nabla_h\mathcal{R}_{lm}+g_{li}\nabla_h\mathcal{R}^j_m-g_{lm}\nabla_h\mathcal{R}^j_i]\\-\tfrac{\mu\nabla_h\mathcal{R}}{3}(\delta^j_ig_{lm}-\delta^j_mg_{li}).\end{aligned} \tag{4.1}$$

After contracting (4.1) over h and j it leads to

$$\begin{aligned}\nabla_j\mathcal{S}^j_{lmi}=\nu[\nabla_j\mathcal{R}^j_{lmi}+\tfrac{1}{2}(\nabla_m\mathcal{R}_{li}-\nabla_i\mathcal{R}_{lm}+g_{li}\nabla_m\mathcal{R}-g_{lm}\nabla_i\mathcal{R}]\\-\tfrac{\mu}{3}(g_{lm}\nabla_i\mathcal{R}-g_{li}\nabla_m\mathcal{R}).\end{aligned} \tag{4.2}$$

It is well-known that

$$\nabla_j\mathcal{R}^j_{lmi}=\nabla_i\mathcal{R}_{lm}-\nabla_m\mathcal{R}_{li}. \tag{4.3}$$

With the help of (4.3), equation (4.2), reduces to

$$\nabla_j \mathcal{S}^j_{lmi} = \nu \left[\nabla_i \left(\mathcal{R}_{lm} - \frac{1}{2}\mathcal{R}g_{lm}\right) - \nabla_m \left(\mathcal{R}_{li} - \frac{1}{2}\mathcal{R}g_{li}\right)\right] - \frac{\mu}{3}(g_{lm}\nabla_i\mathcal{R} - g_{li}\nabla_m\mathcal{R}). \tag{4.4}$$

For fix $\nabla_j \mathcal{S}^j_{lmi}$=0, equation (4.4) implies that

$$\nabla_i\mathcal{R}_{lm} - \nabla_m\mathcal{R}_{li} = \left(1 + \frac{2\mu}{3\nu}\right)[g_{lm}\nabla_i\mathcal{R} - g_{li}\nabla_m\mathcal{R}]. \tag{4.5}$$

Taking contracting (4.5) with g^{lm}, we get

$$\nabla_i\mathcal{R} = 0. \tag{4.6}$$

So, from (4.5) and (4.5), we obtain

$$\nabla_i\mathcal{R}_{lm} = \nabla_m\mathcal{R}_{li}. \tag{4.7}$$

Thus we can state:

Theorem 4.1. *Let Θ be a spacetime with semiconformal curvature tensor, then Θ has Codazzi type of Ricci tensor if and only if the semiconformal curvature tensor is divergence free.*

By virtue of (4.6), the field (1.9) in $f(R)$-gravity implies

$$\mathcal{R}_{lm} - \frac{f}{2f'}g_{lm} = \frac{\kappa}{f'}\mathcal{T}_{lm}. \tag{4.8}$$

So, from (4.7) and (4.8), we yields $\nabla_i\mathcal{T}_{lm}$=$\nabla_m\mathcal{T}_{li}$.
This motivated us for the result:

Corollary 4.2. *The energy-momentum tensor of a spacetime with divergence free semi-conformal curvature tensor obeying $f(R)$-gravity is of Codazzi type.*

The spacetime is called Ricci semi-symmetric [30] if

$$(\nabla_h\nabla_i - \nabla_i\nabla_h)\mathcal{R}_{lm} = 0. \tag{4.9}$$

From (4.8), one can get

$$(\nabla_h\nabla_i - \nabla_i\nabla_h)\mathcal{R}_{lm} = (\nabla_h\nabla_i - \nabla_i\nabla_h)\mathcal{T}_{lm}. \tag{4.10}$$

So, we can state

Theorem 4.3. *Let Θ be a spacetime with divergence free semiconformal curvature satisfying $f(R)$-gravity, then Θ is Ricci semi-symmetric if and only if the energy-momentum tensor is semi-symmetric.*

The EMT $\mathcal{T}_{lm}$ is called (i) recurrent if there exists a non-zero 1-form ε_h such that

$$\nabla_h\mathcal{T}_{lm} = \varepsilon_h\mathcal{T}_{lm}. \tag{4.11}$$

(ii) bi-recurrent if there exists a non-zero tensor υ_{ki} such that

$$\nabla_h\nabla_i\mathcal{T}_{lm} = \upsilon_{ki}\mathcal{T}_{lm}.$$

After contracting (4.11) with g^{lm}, we have

$$\varepsilon_h = \frac{1}{\mathcal{T}}\nabla_h\mathcal{T}, \text{ where } \mathcal{T} = g^{lm}\mathcal{T}_{lm}. \tag{4.12}$$

Finally, taking the covariant derivative of (4.12) and (4.11), using (4.12), we yields

$$(\nabla_h\nabla_i - \nabla_i\nabla_h)\mathcal{T}_{lm} = 0. \tag{4.13}$$

This implies the result.

Theorem 4.4. *Let Θ be a spacetime with divergence free semiconformal curvature tensor obeying $f(R)$-gravity. If the energy-momentum (Ricci) tensor is recurrent or bi-recurrent, then the Ricci (energymomentum) tensor is semi-symmetric.*

Finally, we consider perfect fluid spacetime with divergence free semiconformal curvature tensor, whose energymomentum tensor is recurrent or bi-recurrent. Then from (3.1) and (2.6), we get

$$\mathcal{R}_{lm} = \gamma_1 g_{lm} + \gamma_2 v_l v_m, \tag{4.14}$$

where

$$\gamma_1 = \frac{1}{2f'}(2\kappa\widetilde{p} + f) \text{ and } \gamma_2 = \frac{\kappa}{f'}(\widetilde{p} + \widetilde{\sigma}). \tag{4.15}$$

On contracting (4.14) with g^{lm}, we yields

$$\mathcal{R} = \frac{1}{f'}(3\kappa\widetilde{p} - \kappa\widetilde{\sigma} + 2f). \tag{4.16}$$

If the EMT, $\mathcal{T}_{lm}$ is recurrent or bi-recurrent, then $(\nabla_h\nabla_i - \nabla_i\nabla_h)\mathcal{T}_{lm}$=0 implies that $(\nabla_h\nabla_i - \nabla_i\nabla_h)\mathcal{R}_{lm}$=0. So from (4.14), we have

$$\gamma_2 v_l(\nabla_h\nabla_i - \nabla_i\nabla_h)v_m + \gamma_2 v_m(\nabla_h\nabla_i - \nabla_i\nabla_h)v_l = 0. \tag{4.17}$$

After contracting (4.17) with v^l, it gives

$$\gamma_2(\nabla_h\nabla_i - \nabla_i\nabla_h)v_m = 0 \Rightarrow \gamma_2\mathcal{R}^s_{him}v_s = 0,$$

which implies the following cases:
Case (i): If $\mathcal{R}^s_{him}v_s \neq 0$, then γ_2=0, that is, $\widetilde{p}$+$\widetilde{\sigma}$=0 thus the spacetime indicates inflation and the fluid behaves as a cosmological constant.
Case (ii): If $\gamma_2 \neq 0$ then $\mathcal{R}^s_{him}v_s$=0 so from (4.14), we get $(\gamma_1 - \gamma_2)v_l$=0. Therefore from (4.15), we obtain

$$\widetilde{\sigma} = \frac{f}{2\kappa}, \quad \widetilde{p} = \frac{2\mathcal{R}f' - f}{6\kappa}. \tag{4.18}$$

Theorem 4.5. *If the EMT of a perfect fluid spacetime with divergence free semiconformal curvature tensor satisfying $f(R)$-gravity is recurrent or bi-recurrent then either the spacetime be an inflation, or the isotropic pressure and the energy density are constants.*

5. CONCLUSIONS

In light of the recent studies (see, [13, 14, 15, 16]). We have clarified the idea of semiconformally flat spacetime and perfect fluid spacetime in $f(R)$-gravity. We have also obtained the strong, weak, null, and the dominant energy conditions in semiconformally flat spacetime admitting $f(R)$-gravity. Moreover, the divergence-free semiconformal curvature tensors in $f(R)$-gravity are also taken into account.

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