ANNALS OF COMMUNICATIONS IN MATHEMATICS
Volume 5, Number 1 (2022), 32-37
ISSN: 2582-0818

FUZZY MINIMAL AND MAXIMAL e-OPEN SETS

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ABSTRACT. The aim of this article is to introduce fuzzy minimal e-open and fuzzy maximal e-open sets in fuzzy topological space. Further, we investigate some properties with these new sets.

1. INTRODUCTION AND PRELIMINARIES

Zadeh[7] established fuzzy set in 1965 and Chang[2] introduced fuzzy topology in 1968. Ittanagi and Wali[3] instigated the notions of fuzzy maximal and minimal open sets. The notion of fuzzy e-open set introduced by Seenivasan and Kamala[4]. In this paper , we introduce fuzzy minimal e-open and fuzzy maximal e-open sets. Further some of their related results investigated.

The following terminologies "fuzzy e-open,fuzzy minimal e-open and fuzzy maximal e-open respectively abbreviated as Fe-O, FMIe-O and FMAe-O. "

Definition 1.1. A fuzzy subset ξ of a space X is called fuzzy regular open [1] (resp. fuzzy regular closed) if ξ =Int(Cl(ξ)) (resp.ξ =Cl(Int(ξ))).

The fuzzy δ-interior of a fuzzy subset ξ of X is the union of all fuzzy regular open sets contained in ξ. A fuzzy subset ξ is called fuzzy δ-open [5] if $\xi = \text{Int}_\delta(\xi)$. The complement of fuzzy δ-open set is called fuzzy δ-closed (i.e, $\xi = \text{Cl}_\delta(\xi)$).

Definition 1.2. A proper nonempty fuzzy open set E of $X$is said to be a FMIO[3] set if
(i)E and 0_X are only fuzzy open sets contained in E.
(ii) 1_X and E are only fuzzy open sets containing E.

Definition 1.3. A fuzzy subset ζ of a fts X is called Fe-O [4] if $\zeta \leq$cl(int$_\delta\zeta$)$\cup$int(cl$_\delta\zeta$) and fuzzy e-closed set if $\zeta \geq$cl(int$_\delta\zeta$)$\cap$int(cl$_\delta\zeta$).

Let U be a fuzzy subset of a fts X. Then the fuzzy e-closure and e-interior [4] of U are defined as follows: $\text{eCl}(U) = \bigcap\{\mu : \mu \geq U, \mu \text{ is fuzzy } e\text{-closed in } X\}$ and $\text{eInt}(U) = \bigcup\{\lambda \leq U, \lambda \text{ is F}e\text{-O in } X\}$.

2010 *Mathematics Subject Classification.* 54A40, 03E72.
Key words and phrases. Fuzzy regular open set; Fuzzy δ-open set; Fuzzy e-open set; Fuzzy minimal e-open set;Fuzzy maximal e-open set.
Received: March 18, 2022. Accepted: May 25, 2022. Published: June 30, 2022.
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2. FUZZY MINIMAL e-OPEN SETS

Definition 2.1. A proper nonzero Fe-O set E in fts (X, τ) is said to be FMIe-O iff Fe-O set contained in E is 0_X or E.

Lemma 2.1. *Let (X, τ) be a fts.*
(i) If E_1 is FMIe-O and E_2 is Fe-O in X, then $E_1 \cap E_2 = 0_X$ or $E_1 \subset E_2$.
(ii) If E_1 and E_2 are FMIe-O , then $E_1 \cap E_2 = 0_X$ or $E_1 = E_2$.

Proof. (i) Let us assume that E_2 is Fe-O in X such that $E_1 \cap E_2 \neq 0_X$. Since E_1 is FMIe-O , and $E_1 \cap E_2 \subset E_1$, then $E_1 \cap E_2 = E_1$ implies that $E_1 \subset E_2$.

(ii) Suppose that $E_1 \cap E_2 \neq 0_X$, then clearly from(ii), $E_1 \subset E_2$ and $E_2 \subset E_1$ as E_1 and E_2 are FMIe-O . Hence $E_1 = E_2$. □

Theorem 2.2. *Let E and E_i are FMIe-O sets for any $i \in M$. If $E \subseteq \bigcup_{i \in M} E_i$, then $E = E_j$ for any $j \in M$.*

Proof. Suppose $E \subseteq \bigcup_{i \in M} E_i$, then $E = E \cap \left(\bigcup_{i \in M} E_i\right) = \bigcup_{i \in M} (E \cap E_i)$. By deploying lemma 2.1(ii), $E \cap E_i = 0_X$ or $E = E_i$ as E and E_i are FMIe-O sets. If $E \cap E_i = 0_X$, then $E = 0_X$ which contradicts that E is a FMIe-O set. Hence if $E \cap E_i \neq 0_X$ then $E = E_j$ for any $j \in M$. □

Theorem 2.3. *If E and E_i are FMIe-O sets for any $i \in M$ and $E \neq E_i$, then $E \cap \left(\bigcup_{i \in M} E_i\right) = 0_X$ for any $i \in M$.*

Proof. Let $E \cap \left(\bigcup_{i \in M} E_i\right) \neq 0_X$, then $E \cap E_i \neq 0_X$ for any $i \in M$. By deploying lemma 2.1(ii), $E = E_i$ contradictory to $E \neq E_i$. Hence $E \cap \left(\bigcup_{i \in M} E_i\right) = 0_X$. □

Theorem 2.4. *If E_i is a FMIe-O for any $i \in M$ $(|M| \geq 2)$ and $E_i \neq E_j$ for any distinct $i, j \in M$, then $\left(\bigcup_{i \in M \setminus \{j\}} E_i\right) \cap E_j = 0_X$ for any $j \in M$.*

Proof. Let $\left(\bigcup_{i \in M \setminus \{j\}} E_i\right) \cap E_j \neq 0_X$. Then $\bigcup_{i \in M \setminus \{j\}} \left(E_i \cap E_j\right) \neq 0_X \Rightarrow (E_i \cap E_j) \neq 0_X$. By lemma 2.1(ii), $E_i = E_j$, a contradiction. Hence $\left(\bigcup_{i \in M \setminus \{j\}} E_i\right) \cap E_j = 0_X$ for any $j \in M$. □

Theorem 2.5. *If E_i is a FMIe-O for any $i \in M$, $(|M| \geq 2)$ and $E_i \neq E_j$ for any distinct $i, j \in M$. If K is a proper fuzzy set of M, then $\left(\bigcup_{i \in M \setminus K} E_i\right) \bigcap \left(\bigcup_{s \in K} h_s\right) = 0_X$.*

Proof. Let $\left(\bigcup_{i \in M \setminus K} E_i\right) \bigcap \left(\bigcup_{s \in K} E_s\right) \neq 0_X$. It implies that $\bigcup (E_i \cap E_s) \neq 0_X$ for $i \in M \setminus K$ and $s \in K$ implies that $E_i \cap E_s \neq 0_X$ for some $i \in M$ and $s \in K$. By lemma2.1(ii), $E_i = h_s$, which is a contradiction. Hence $\left(\bigcup_{i \in M \setminus K} E_i\right) \bigcap \left(\bigcup_{s \in K} E_s\right) = 0_X$. □

Theorem 2.6. *If E_i is a FMIe-O for any $i \in M$ such that $E_i \neq E_j$ for any distinct $i, j \in M$. If S is a proper nonzero fuzzy set of M, then $\left[\bigcup_{i \in M \setminus k} E_i\right] \cap \left[\bigcup_{k \in S} E_k\right] = 0_X$.*

Proof. Assume that $\cup [E_i \cap E_k] \neq 0_X$ for $i \in M \backslash k, k \in S$. Clearly, for some $i \in M, k \in S$ we have $[E_i \cap E_k] \neq 0_X$. By deploying lemma 2.1(ii) $E_i = E_k$, a contradiction. □

Theorem 2.7. *If E_i and E_k are FMIe-O sets for any $i \in M$ and $k \in S$ and if $\exists$ an $n \in S$ such that $E_i \neq E_n$ for any $i \in M$, then $\left[\bigcup_{n\in K} E_n\right] \not\subset \left[\bigcup_{i\in M} E_i\right]$.*

Proof. Assume that $\exists$ an $n \in S$ such that $E_i \neq E_n$ for any $i \in M$, then $\left[\bigcup_{n\in K} E_n\right] \subset \left[\bigcup_{i\in M} E_i\right]$.

$\Rightarrow E_n \subset \left[\bigcup_{i\in M} E_i\right]$ for some $n \in K$.

$\Rightarrow E_i \neq E_n$ for any $i \in M$, by theorem 2.2, which is a contradiction. Hence $\left[\bigcup_{n\in K} E_n\right] \not\subset \left[\bigcup_{i\in M} E_i\right]$. □

Theorem 2.8. *If E_i is a FMIe-O for any $i \in M$ such that $E_i \neq E_j$ for any distinct $i, j \in M$, then $\left[\bigcup_{k\in K} E_k\right] \subsetneq \left[\bigcup_{i\in M} E_i\right]$ for any proper nonzero subset K of M.*

Proof. Let $m \in M\backslash K$, then E_m is a FMI*e*-O set of the family $\{E_m | m \in M\backslash K\}$ of FMI*e*-O sets. Clearly $E_m \cap \left[\bigcup_{k\in K} E_k\right] = \bigcup_{k\in K} [E_m \cap E_k] = 0_X$. Also $E_m \cap \left[\bigcup_{i\in M} E_i\right] = \bigcup_{i\in M} [E_m \cap E_i] = E_m$.

If $\left[\bigcup_{k\in K} E_k\right] = \left[\bigcup_{i\in M} E_i\right]$, then $E_m = 0_X$ which is a contradiction that E_m is a FMI*e*-O set. Hence $\left[\bigcup_{k\in K} E_k\right] \subsetneq \left[\bigcup_{i\in M} E_i\right]$. □

Theorem 2.9. *If E_i is a FMIe-O set for any $i \in M$ such that $E_i \neq E_j$ for any distinct $i, j \in M$, then*
(i) $E_j \subset \left[\bigcup_{i\in M\backslash\{j\}} E_i\right]^c$ for some $j \in M$.
(ii) $\bigcup_{i\in M\backslash\{j\}} E_i \neq 1_X$ for any $j \in M$.

Proof. (i) By hypothesis, $E_i \neq E_j$ for any distinct $i, j \in M$.

By theorem 2.3, $\left[\bigcup_{i\in M} E_i\right] \cap E_j = 0_X$ which is true for any $j \in M$.
$\Rightarrow \bigcup_{i\in M} \left[E_i \cap E_j\right] = 0_X$
$\Rightarrow E_i \cap E_j = 0_X$ (By Lemma 2.1(ii))
$\Rightarrow E_i \subset {E_j}^c$
$\Rightarrow \bigcup_{i\in M\backslash\{j\}} E_i \subset {E_j}^c$. Hence proved.

(ii) Let $j \in M$ such that $\bigcup_{i\in M\backslash\{j\}} E_i = 1_X$
$\Rightarrow E_i = 0_X$
$\Rightarrow E_i$ is not a FMI*e*-O set, a contradiction. Hence $\bigcup_{i\in M\backslash\{j\}} E_i \neq 1_X$ for any $j \in M$. □

Corollary 2.10. *If E_i is a FMIe-O set for any $i \in M$ such that $E_i \neq E_j$ for any distinct $i, j \in M$, then $E_i \cup E_j \neq 1_X$ for any distinct $i, j \in M$.*

Proof. Similar to that of "Theorem 2.9(ii)." □

Theorem 2.11. *If E_i is a FMIe-O sets for any $i \in M$ such that $E_i \neq E_j$ for any distinct $i, j \in M$, then* $E_j = \left[\bigcup_{i\in M} E_i\right] \cap \left[\bigcup_{i\in M\setminus\{j\}} E_i\right]^c$ *for any $j \in M$.*

Proof. For any $j \in M \Rightarrow \left[\bigcup_{i\in M} E_i\right] \bigcap \left[\bigcup_{i\in M\setminus\{j\}} E_i\right]^c = \left[\bigcup_{i\in M\setminus\{j\}} E_i \bigcup E_j\right] \cap \left[\bigcup_{i\in M\setminus\{j\}} E_i\right]^c$

$= \left[\left(\bigcup_{i\in M\setminus\{j\}} E_i\right) \cap \left(\bigcup_{i\in M\setminus\{j\}} E_i\right)^c\right] \bigcup \left[E_j \cap \left(\bigcup_{i\in M\setminus\{j\}} E_i\right)^c\right]$

$= 0_X \cup E_j$

$= E_j$ for any $j \in M$. □

Proposition 2.12. *Let G be a FMIe-O set. If $x_\alpha \in G$, then $G \subset G_1$ for any fuzzy open neighbourhood G_1 of x_α.*

Proof. Let G_1 be an Fe-O neighbourhood of x_α such that $G \not\subset G_1$. Clearly $G \cap G_1$ is an Fe-O such that $G \cap G_1 \subsetneq G$ and $G \cap G_1 \neq 0_X$. This implies that G is a FMIe-O set which a contradiction. □

Proposition 2.13. *Let G be a FMIe-O set in a fuzzy topological space (X, τ). Then $G = \bigcap \{G_1 : G_1$ fuzzy e-open neighbourhood of x_α for any $x_\alpha \in G\}$*

Proof. By deploying proposition 2.12 and as G is an Fe-O neighbourhood of x_α, we have $G \subset \bigcap \{G_1 : G_1$ fuzzy e-open neighbourhood of $x_\alpha \subset G\}$. This completes the proof. □

Theorem 2.14. *Let G be a FMIe-O set. Then the following conditions are equivalent.*
(i) G is FMIe-O set.
(ii)$G \subset eCl(K)$ for any nonzero fuzzy subset K of G.
(iii)$eCl(G) = eCl(K)$ for any nonzero fuzzy subset K of G.

Proof. $(i) \Rightarrow (ii)$: By deploying “proposition 2.12”for any $x_\alpha \in G$ and Fe-O neighbourhood M of x_α, we have $K = (G \cap K) \subset (M \cap K)$ for any proper nonzero fuzzy subset $K \subset G$. Therefore, we have $(M \cap K) \neq 0_X$ and $x_\alpha \in eCl(K)$. It follows that $G \subset eCl(K)$.

$(ii) \Rightarrow (iii)$: For any proper fuzzy subset K of G, $eCl(G) \subset eCl(K)$. Also by(ii) $eCl(G) \subset eCl(eCl(K)) = eCl(K)$. Hence proved.

$(iii) \Rightarrow (i)$: Let us assume that G is not FMIe-O . Then there exists a proper Fe-O D such that $D \subset G$.Then $\exists\, y_\alpha \in G$ such that $y_\alpha \notin D$. Then $eCl(\{y_\alpha\}) \in D^c$ implies that $eCl(\{y_\alpha\}) \neq eCl(G)$, a contradiction. This completes our proof. □

3. FUZZY MAXIMAL e-OPEN SETS AND ITS PROPERTIES

Definition 3.1. A proper nonzero Fe-O set F of a fts (X, τ) is said to FMAe-O if any Fe-O set which contains F is either F or 1_X.

Example 3.2. [6] *Let $X = \{a, b, c, d\}$. Then fuzzy sets $\gamma_1 = \frac{1}{a} + \frac{0}{b} + \frac{0}{c} + \frac{0}{d}$; $\gamma_2 = \frac{0.5}{a} + \frac{0.5}{b} + \frac{0.5}{c} + \frac{0.5}{d}$; $\gamma_3 = \frac{0.4}{a} + \frac{0}{b} + \frac{0}{c} + \frac{0}{d}$; $\gamma_4 = \frac{1}{a} + \frac{0.5}{b} + \frac{0.5}{c} + \frac{0.5}{d}$ and $\gamma_5 = \frac{0.5}{a} + \frac{0}{b} + \frac{0}{c} + \frac{0}{d}$;are defined as follows: Consider the fuzzy topology $\tau = \{0_X, \gamma_1, \gamma_2, \gamma_3, \gamma_4, \gamma_5, 1_X\}$. Here γ_3 is FMIe-O and γ_4 FMAe-O set.*

Lemma 3.1. *Let (X, τ) be a fts. Then*
(i) If F_1 is a FMAe-O and F_2 is Fe-O in X, then $F_1 \cup F_2 = 1_X$ or $F_2 \subset F_1$.
(ii) If F_1 and F_3 are FMAe-O sets, then either $F_1 \cup F_3 = 1_X$ or $F_1 = F_3$.

Proof. (i) Assume that $F_2 \not\subset F_1$. Clearly, $F_1 \subset (F_1 \cup F_2)$ a contrary to F_1 is a FMA*e*-O set if $F_1 \cup F_2 \neq 1_X$. Hence, $F_1 \cup F_2 = 1_X$.
(ii) Let F_1 and F_3 are FMA*e*-O sets. Then from(i) $F_3 \subset F_1$ and $F_1 \subset F_3$ implies that $F_1 = F_3$. □

Theorem 3.2. *If F_1, F_2 and F_3 are FMAe-O sets such that $F_1 \neq F_2$ and $(F_1 \cap F_2) \subset F_3$, then either $F_1 = F_3$ or $F_2 = F_3$.*

Proof. Suppose that F_1, F_2 and F_3 are FMA*e*-O sets with $F_1 \neq F_2$, $(F_1 \cap F_2) \subset F_3$ and if $F_1 \neq F_3$, then
$(F_2 \cap F_3) = F_2 \cap (F_3 \cap 1_X)$
$= F_2 \cap [F_3 \cap (F_1 \cup F_2)]$, by lemma 3.1(ii)
$= F_2 \cap [(F_3 \cap F_1) \cup (F_3 \cap F_2)]$
$= [F_2 \cap F_3 \cap F_1] \cup [F_2 \cap F_3 \cap F_2]$
$= [F_2 \cap F_1] \cup [F_2 \cap F_3]$
$= F_2 \cap [F_1 \cup F_3]$
$= F_2 \cap 1_X$
$= F_2$
$(F_2 \cap F_3) = F_2 \Rightarrow F_2 \subset F_3$. As F_2 and F_3 are FMA*e*-O sets, $F_3 \subset F_2$. Hence $F_2 = F_3$. □

Theorem 3.3. *For any distinct FMAe-O sets F_1, F_2, F_3*
$[F_1 \cap F_2] \not\subset [F_1 \cap F_3]$.

Proof. Consider $[F_1 \cap F_2] \subset [F_1 \cap F_3]$ for any distinct FMA*e*-O sets F_1, F_2 and F_3. Then
$[F_1 \cap F_2] \cup [F_2 \cap F_3] \subset [F_1 \cap F_3] \cup [F_2 \cap F_3]$
$= [F_1 \cup F_3] \cap F_2 \subset [F_1 \cup F_2] \cap F_3$
$= 1_X \cap F_2 \subset 1_X \cap F_3$
$= F_2$ is contained in F_3
a contradiction to F_1, F_2 and F_3 are distinct . Hence $[F_1 \cap F_2] \not\subset [F_1 \cap F_3]$. □

Remark. *Proofs of "Theorem 3.4, Corollary 3.5, Theorem 3.6 and Theorem 3.7" are similar to proofs of "Theorem 2.9, Corollary 2.10, Theorem 2.11 and Theorem 2.8" respectively. Hence the proofs are omitted.*

Theorem 3.4. *If F_i is a FMAe-O sets for any $i \in M$, M is a finite set and $F_i \neq F_j$ for any distinct $i, j \in M$, then*
(i) $\left[\bigcap_{i \in M \setminus \{j\}} F_i\right]^c \subset F_j$ *for any* $j \in M$
(ii) $\bigcap_{i \in M \setminus \{j\}} F_i \neq 0_X$ *for any* $j \in M$.

Corollary 3.5. *If F_i is a FMAe-O sets for any $i \in M$, M is a finite set and $F_i \neq F_j$ for any distinct $i, j \in M$ then,$F_i \cap F_j \neq 0_X$ for any distinct $i, j \in M$.*

Theorem 3.6. *If F_i is a FMAe-O sets for any $i \in M$, M is a finite set and $F_i \neq F_j$ for any distinct $i, j \in M$, then* $F_j = \left[\bigcap_{i \in M} F_i\right] \cup \left[\bigcap_{i \in M \setminus \{j\}} F_i\right]^c$ *for any* $j \in M$.

Theorem 3.7. *If F_i is a FMAe-O sets for any $i \in M$, M is a finite set and $F_i \neq F_j$ for any distinct $i, j \in M$ and if K is a proper nonzero fuzzy subset of M, then* $\bigcap_{i \in M} F_i \subsetneq \bigcap_{k \in K} F_k$.

Theorem 3.8. *If F_i is a FMAe-O sets for any $i \in M$, M is a finite set and $F_i \neq F_j$ for any distinct $i, j \in M$ and if* $\bigcap_{i \in M} F_i$ *is a fuzzy subset, then F_j is a fuzzy subset for any $j \in M$.*

Proof. By "Theorem 3.6", we have $F_j = \left[\bigcap_{i\in M} F_i\right] \cup \left[\bigcap_{i\in M\setminus\{j\}} F_i\right]^c$ for any $j \in M$.

$F_j = \left[\bigcap_{i\in M} F_i\right] \cup \left[\bigcup_{i\in M\setminus\{j\}} F_i^c\right]$.

Since M is finite, $\bigcup_{i\in M\setminus\{j\}} F_i^c$ is fuzzy e-closed. Hence F_j is fuzzy e-closed for any $j \in M$. □

Theorem 3.9. *If F_i is a FMAe-O set for any $i \in M, M$ is a finite set and $F_i \neq F_j$ for any distinct $i, j \in M$. If $\bigcap_{i\in M} F_i = 0_X$,then $\{F_i/i \in M\}$ is the set of all FMAe-O sets of fts X.*

Proof. Suppose that $\exists$ another FMAe-O F_k of a fts X such that $F_k \neq F_i, \forall i \in M$. Clearly, $0_X = \bigcap_{i\in M} F_i = \bigcap_{i\in(M\cup k)\setminus\{k\}} F_i \neq 0_X$, by Theorem 3.4(ii), a contradiction.

Hence $\{F_i/i \in M\}$ is the family of all FMAe-O sets of fts X. □

4. ACKNOWLEDGEMENTS

The authors are thankful to the referees for their suggestions and commands to develop this manuscript.

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