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ON NEW APPROACH TOWARDS CUBIC VAGUE SUBBISEMIRINGS IN BISEMIRINGS

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ABSTRACT. From the nature of subbisemiring, we develop a new generalized hybrid structure of vague subbisemiring known as cubic vague subbisemiring (shortly CVSBS). We talk about the CVSBS and level sets CVSBS of bisemiring. At first we define some basic operation such as intersection, cartesian product on them and use these to obtain some of its basic properties under CVSBS. Let $\mathcal{L} = \langle \bar{\mathbb{A}}_{\mathcal{L}}, \mathbb{V}_{\mathcal{L}} \rangle$ be the cubic vague subset of S. It is shown that $\mathcal{L}$ is a CVSBS if and only if all non empty level set $\mathcal{L}_{(\alpha,\beta)}(\alpha, \beta \in D[0,1])$ is a SBS. Let $\mathcal{L}$ be the CVSBS and $\mathcal{W}$ be the strongest vague relation of S. We show that $\mathcal{L}$ is a CVSBS if and only if $\mathcal{W}$ is a CVSBS of $S \times S$. After we define homomorphic image and preimage of bisemiring. It will be shown that the homomorphic image and preimage of CVSBS is a CVSBS. To strengthen our results with examples are indicated.

1. INTRODUCTION

In 1965, Zadeh [23] introduced the study of fuzzy sets. A fuzzy set on a set X is a mapping μ into the interval $[0,1]$ for $x \in X$, $\mu(x)$ is called the membership of x belonging to X. The membership function gives only an approximation for belonging but it does not give any information of not belonging. In 1993, Gau et al. expanded the fuzzy set to Vague set [8], which consists of two membership degree namely true and false membership degree. Vague sets have a strong ability than fuzzy sets to process fuzzy information to some degree. Human cognition is usually a gradual process. As a result, how to characterize a vague concept and further measure its uncertainty becomes an interesting issue worth studying. Fuzzy algebraic structure beginning by Rosenfeld [21]. Ranjit Biswas [7] discussed the theory of vague algebra with vague groups, normal groups etc. The idea of cubic set and it was characterized by interval valued fuzzy set and fuzzy set by Jun et al. [11], which is a more general tool, since fuzzy set deals with single value membership while interval valued fuzzy set ranges the membership in the form of intervals. The cubic set has the main advantage since it contains fuzzy set and an interval valued fuzzy set. In 1993, Ahsan et al [1] introduced the theory of fuzzy semirings. In 2001, Sen et al. introduced in the research of bisemirings [22]. The purpose of this paper is to further extend the concept

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of vague subbisemiring theory to CVSBS theory and deriving its some properties. The paper is organized five sections as follows. Section 1 is the introduction followed by Section 2 which is preliminaries of vague set and cubic vague set. Section 3 presents CVSBS of its properties with examples. Section 4 introduces the notion of CVSBS homomorphism and discusses its properties. Concluding for further investigation is provided in Section 5. Also insert some numerical example is given to evaluate that the cubic vague subbisemirings.

2. Preliminaries

We recall some basic concepts of vague sets, cubic sets and its some algebraic properties which are needed in the later sections.

Definition 2.1. [8] The vague set $\mathcal{L}$ in the universe U is a pair $(T_{\mathcal{L}}, F_{\mathcal{L}})$ it consists of two membership functions called as truth membership function $\mathbb{T}_{\mathcal{L}} : U \to [0,1]$ and false membership function $\mathbb{F}_{\mathcal{L}} : U \to [0,1]$ are mappings such that $0 \leq \mathbb{T}_{\mathcal{L}}(u) + \mathbb{F}_{\mathcal{L}}(u) \leq 1$, for all $u \in U$, where $\mathbb{T}_{\mathcal{L}}(u)$ is a lower bound of the grade of membership of u derived from the "evidence for u", and $\mathbb{F}_{\mathcal{L}}(u)$ is a lower bound of the negation of u derived from the "evidence against u". The vague set $\mathcal{L}$ as $\mathcal{L} = \{u, [\mathbb{T}_{\mathcal{L}}(u), 1 - \mathbb{F}_{\mathcal{L}}(u)] | u \in U\}$, the interval $[\mathbb{T}_{\mathcal{L}}(u), 1 - \mathbb{F}_{\mathcal{L}}(u)]$ is called the vague value of u in $\mathcal{L}$. That is $\mathbb{V}_{\mathcal{L}}(u) = [\mathbb{T}_{\mathcal{L}}(u), 1 - \mathbb{F}_{\mathcal{L}}(u)]$.

Definition 2.2. [8] The complement of a vague set $\mathcal{L}$ is defined by $\mathbb{T}_{\mathcal{L}^c} = \mathbb{F}_{\mathcal{L}}$ and $1 - \mathbb{F}_{\mathcal{L}^c} = 1 - \mathbb{T}_{\mathcal{L}}$.

Definition 2.3. [8] Let $\mathcal{L}$ and $\mathcal{M}$ be any two vague sets in U.

(1) A vague set $\mathcal{L}$ is contained in the other vague set $\mathcal{M}$, $\mathcal{L} \subseteq \mathcal{M}$ if and only if $\mathbb{V}_{\mathcal{L}}(u) \leq \mathbb{V}_{\mathcal{M}}(u)$ i.e. $\mathbb{T}_{\mathcal{L}}(u) \leq \mathbb{T}_{\mathcal{M}}(u)$ and $1 - \mathbb{F}_{\mathcal{L}}(u) \leq 1 - \mathbb{F}_{\mathcal{M}}(u)$, for all $u \in U$.
(2) The union of two vague sets $\mathcal{L}$ and $\mathcal{M}$, as $\mathcal{N} = \mathcal{L} \cup \mathcal{M}$, $\mathbb{T}_{\mathcal{N}} = max\{\mathbb{T}_{\mathcal{L}}, \mathbb{T}_{\mathcal{M}}\}$ and $1 - \mathbb{F}_{\mathcal{N}} = max\{1 - \mathbb{F}_{\mathcal{L}}, 1 - \mathbb{F}_{\mathcal{M}}\} = 1 - min\{\mathbb{F}_{\mathcal{L}}, \mathbb{F}_{\mathcal{M}}\}$.
(3) The intersection of two vague sets $\mathcal{L}$ and $\mathcal{M}$ as $\mathcal{N} = \mathcal{L} \cap \mathcal{M}$, $\mathbb{T}_{\mathcal{N}} = min\{\mathbb{T}_{\mathcal{L}}, \mathbb{T}_{\mathcal{M}}\}$ and $1 - \mathbb{F}_{\mathcal{N}} = min\{1 - \mathbb{F}_{\mathcal{L}}, 1 - \mathbb{F}_{\mathcal{M}}\} = 1 - max\{\mathbb{F}_{\mathcal{L}}, \mathbb{F}_{\mathcal{M}}\}$.

Definition 2.4. [7] Let $\mathcal{L}$ be a vague set of U, $\alpha, \beta \in [0,1]$ with $\alpha \leq \beta$, the (α, β)- cut or vague cut of a vague set $\mathcal{L}$ is the crisp subset of U is given by $\mathcal{L}_{(\alpha,\beta)} = \{u \in U | \mathbb{V}_{\mathcal{L}}(u) \geq [\alpha, \beta]\}$. That is, $\mathcal{L}_{(\alpha,\beta)} = \{u \in U | \mathbb{T}_{\mathcal{L}}(u) \geq \alpha,\ \ 1 - \mathbb{F}_{\mathcal{L}}(u) \geq \beta]\}$.

Definition 2.5. [7] Let $\mathcal{L}$ and $\mathcal{M}$ be any two vague sets in U.

(1) $\mathcal{L} \cap \mathcal{M} = \Big\{ \langle u, \min\{\mathbb{T}_{\mathcal{L}}(u), \mathbb{T}_{\mathcal{M}}(u)\}, \min\{1 - \mathbb{F}_{\mathcal{L}}(u), 1 - \mathbb{F}_{\mathcal{M}}(u)\}\rangle \Big\}$
(2) $\mathcal{L} \cup \mathcal{M} = \Big\{ \langle u, \max\{\mathbb{T}_{\mathcal{L}}(u), \mathbb{T}_{\mathcal{M}}(u)\}, \max\{1 - \mathbb{F}_{\mathcal{L}}(u), 1 - \mathbb{F}_{\mathcal{M}}(u)\}\rangle \Big\}$
(3) $\square\mathcal{L} = \Big\{ \langle u, \mathbb{T}_{\mathcal{L}}(u), 1 - \mathbb{T}_{\mathcal{L}}(u)\rangle \,|u \in U \Big\}$
(4) $\diamond\mathcal{L} = \Big\{ \langle u, 1 - \mathbb{F}_{\mathcal{L}}(u), \mathbb{F}_{\mathcal{L}}(u)\rangle \,|u \in U \Big\}$ for all $u \in U$.

Definition 2.6. Let $\mathcal{L}$ and $\mathcal{M}$ be fuzzy subset of G and H respectively. The product of $\mathcal{L}$ and $\mathcal{M}$ denoted by $\mathcal{L} \times \mathcal{M}$ is defined as $\mathcal{L} \times \mathcal{M} = \{\mu_{\mathcal{L}\times\mathcal{M}}(q_1, q_2) | \ for\ all\ q_1 \in G\ and\ q_2 \in H\}$, where $\mu_{\mathcal{L}\times\mathcal{M}}(q_1, q_2) = \min\{\mu_{\mathcal{L}}(q_1), \mu_{\mathcal{M}}(q_2)\}$.

Definition 2.7. [11] Let X be a non-empty set. The structure $\mathcal{L} = \{(x, \bar{\mathbb{A}}_{\mathcal{L}}(x), \mathbb{V}_{\mathcal{L}}(x)) | x \in X\}$ is called a cubic set in X, since $\bar{\mathbb{A}}_{\mathcal{L}}$ is an interval valued fuzzy set and $\mathbb{V}_{\mathcal{L}}$ is a fuzzy set in X.

Definition 2.8. [22] A bisemiring $(S, \circledast_1, \circledast_2, \circledast_3)$ is an algebraic structure in which $(S, \circledast_1, \circledast_2)$ and $(S, \circledast_2, \circledast_3)$ are semirings in which $(S, \circledast_1)$, $(S, \circledast_2)$ and $(S, \circledast_3)$ are semigroups such that

(1) $s_1 \circledast_2 (s_2 \circledast_1 s_3) = (s_1 \circledast_2 s_2) \circledast_1 (s_1 \circledast_2 s_3)$,
(2) $(s_2 \circledast_1 s_3) \circledast_2 s_1 = (s_2 \circledast_2 s_1) \circledast_1 (s_3 \circledast_2 s_1)$,
(3) $s_1 \circledast_3 (s_2 \circledast_2 s_3) = (s_1 \circledast_3 s_2) \circledast_2 (s_1 \circledast_3 s_3)$,
(4) $(s_2 \circledast_2 s_3) \circledast_3 s_1 = (s_2 \circledast_3 s_1) \circledast_2 (s_3 \circledast_3 s_1), \forall\ s_1, s_2, s_3 \in S$.

Definition 2.9. [10] A non-empty subset A of S is called a subbisemiring(shortly SBS) if $s_1 \circledast_1 s_2, s_1 \circledast_2 s_2, s_1 \circledast_3 s_2 \in A$ for all $s_1, s_2 \in A$.

Definition 2.10. [10] The two bisemirings namely $(S_1, +, \cdot, \times)$ and $(S_2, \oplus, \circ, \otimes)$. Let $\phi : S_1 \to S_2$ is called a homomorphism if

(1) $\phi(q_1 + q_2) = \phi(q_1) \oplus \phi(q_2)$
(2) $\phi(q_1 \cdot q_2) = \phi(q_1) \circ \phi(q_2)$
(3) $\phi(q_1 \times q_2) = \phi(q_1) \otimes \phi(q_2)$ for all $q_1, q_2 \in S_1$.

Definition 2.11. [15] A fuzzy subset $\mathcal{L}$ of S is said to be fuzzy subbisemiring(shortly FSBS) if

$$\left\{\begin{array}{l} \mu_{\mathcal{L}}(q_1 \circledast_1 q_2) \geq \min\{\mu_{\mathcal{L}}(q_1), \mu_{\mathcal{L}}(q_2)\} \\ \mu_{\mathcal{L}}(q_1 \circledast_2 q_2) \geq \min\{\mu_{\mathcal{L}}(q_1), \mu_{\mathcal{L}}(q_2)\} \\ \mu_{\mathcal{L}}(q_1 \circledast_3 q_2) \geq \min\{\mu_{\mathcal{L}}(q_1), \mu_{\mathcal{L}}(q_2)\} \end{array}\right\}$$

$\forall\ q_1, q_2 \in S$.

3. Cubic Vague Subbisemirings

In this section we briefly present the CVSBS and obtain its various characterizations. Also stated illustrate examples under finite case and infinite case. Here S denotes a bisemiring unless otherwise noted.

Definition 3.1. A cubic vague set $\mathcal{L} = \langle \bar{\mathbb{A}}_{\mathcal{L}}, \mathbb{V}_{\mathcal{L}} \rangle$ of S is called a CVSBS if the following conditions are satisfied : For all $q_1, q_2 \in S$,

$$\left\{\begin{array}{l} \bar{\mathbb{A}}_{\mathcal{L}}(q_1 \circledast_1 q_2) \geq Min^i\{\bar{\mathbb{A}}_{\mathcal{L}}(q_1), \bar{\mathbb{A}}_{\mathcal{L}}(q_2)\} \\ \bar{\mathbb{A}}_{\mathcal{L}}(q_1 \circledast_2 q_2) \geq Min^i\{\bar{\mathbb{A}}_{\mathcal{L}}(q_1)\bar{\mathbb{A}}_{\mathcal{L}}(q_2)\} \\ \bar{\mathbb{A}}_{\mathcal{L}}(q_1 \circledast_3 q_2) \geq Min^i\{\bar{\mathbb{A}}_{\mathcal{L}}(q_1), \bar{\mathbb{A}}_{\mathcal{L}}(q_2)\} \end{array}\right\}$$

and

$$\left\{\begin{array}{l} \mathbb{V}_{\mathcal{L}}(q_1 \circledast_1 q_2) \geq min\{\mathbb{V}_{\mathcal{L}}(q_1), \mathbb{V}_{\mathcal{L}}(q_2)\} \\ \mathbb{V}_{\mathcal{L}}(q_1 \circledast_2 q_2) \geq min\{\mathbb{V}_{\mathcal{L}}(q_1), \mathbb{V}_{\mathcal{L}}(q_2)\} \\ \mathbb{V}_{\mathcal{L}}(q_1 \circledast_3 q_2) \geq min\{\mathbb{V}_{\mathcal{L}}(q_1), \mathbb{V}_{\mathcal{L}}(q_2)\} \end{array}\right\}$$

ie)

$$\left\{\begin{array}{l} \bar{\mathbb{T}}_{\mathbb{A}_{\mathcal{L}}}(q_1 \circledast_1 q_2) \geq Min^i\{\bar{\mathbb{T}}_{\mathbb{A}_{\mathcal{L}}}(q_1), \bar{\mathbb{T}}_{\mathbb{A}_{\mathcal{L}}}(q_2)\} \\ \bar{\mathbb{T}}_{\mathbb{A}_{\mathcal{L}}}(q_1 \circledast_2 q_2) \geq Min^i\{\bar{\mathbb{T}}_{\mathbb{A}_{\mathcal{L}}}(q_1), \bar{\mathbb{T}}_{\mathbb{A}_{\mathcal{L}}}(q_2)\} \\ \bar{\mathbb{T}}_{\mathbb{A}_{\mathcal{L}}}(q_1 \circledast_3 q_2) \geq Min^i\{\bar{\mathbb{T}}_{\mathbb{A}_{\mathcal{L}}}(q_1), \bar{\mathbb{T}}_{\mathbb{A}_{\mathcal{L}}}(q_2)\} \end{array}\right\}$$

$$\left\{\begin{array}{l} 1 - \bar{\mathbb{F}}_{\mathbb{A}_{\mathcal{L}}}(q_1 \circledast_1 q_2) \geq Min^i\{1 - \bar{\mathbb{F}}_{\mathbb{A}_{\mathcal{L}}}(q_1), 1 - \bar{\mathbb{F}}_{\mathbb{A}_{\mathcal{L}}}(q_2)\} \\ 1 - \bar{\mathbb{F}}_{\mathbb{A}_{\mathcal{L}}}(q_1 \circledast_2 q_2) \geq Min^i\{1 - \bar{\mathbb{F}}_{\mathbb{A}_{\mathcal{L}}}(q_1), 1 - \bar{\mathbb{F}}_{\mathbb{A}_{\mathcal{L}}}(q_2)\} \\ 1 - \bar{\mathbb{F}}_{\mathbb{A}_{\mathcal{L}}}(q_1 \circledast_3 q_2) \geq Min^i\{1 - \bar{\mathbb{F}}_{\mathbb{A}_{\mathcal{L}}}(q_1), 1 - \bar{\mathbb{F}}_{\mathbb{A}_{\mathcal{L}}}(q_2)\} \end{array}\right\}$$

and

$$\left\{\begin{array}{l} \mathbb{T}_{\mathbb{V}_{\mathcal{L}}}(q_1 \circledast_1 q_2) \geq min\{\mathbb{T}_{\mathbb{V}_{\mathcal{L}}}(q_1), \mathbb{T}_{\mathbb{V}_{\mathcal{L}}}(q_2)\} \\ \mathbb{T}_{\mathbb{V}_{\mathcal{L}}}(q_1 \circledast_2 q_2) \geq min\{\mathbb{T}_{\mathbb{V}_{\mathcal{L}}}(q_1), \mathbb{T}_{\mathbb{V}_{\mathcal{L}}}(q_2)\} \\ \mathbb{T}_{\mathbb{V}_{\mathcal{L}}}(q_1 \circledast_3 q_2) \geq min\{\mathbb{T}_{\mathbb{V}_{\mathcal{L}}}(q_1), \mathbb{T}_{\mathbb{V}_{\mathcal{L}}}(q_2)\} \end{array}\right\}$$

$$\left\{\begin{array}{l} 1-\mathbb{F}_{\mathbb{V}_{\mathcal{L}}}(q_1 \circledast_1 q_2) \geq min\{1-\mathbb{F}_{\mathbb{V}_{\mathcal{L}}}(q_1), 1-\mathbb{F}_{\mathbb{V}_{\mathcal{L}}}(q_2)\} \\ 1-\mathbb{F}_{\mathbb{V}_{\mathcal{L}}}(q_1 \circledast_2 q_2) \geq min\{1-\mathbb{F}_{\mathbb{V}_{\mathcal{L}}}(q_1), 1-\mathbb{F}_{\mathbb{V}_{\mathcal{L}}}(q_2)\} \\ 1-\mathbb{F}_{\mathbb{V}_{\mathcal{L}}}(q_1 \circledast_3 q_2) \geq min\{1-\mathbb{F}_{\mathbb{V}_{\mathcal{L}}}(q_1), 1-\mathbb{F}_{\mathbb{V}_{\mathcal{L}}}(q_2)\} \end{array}\right\}$$

Example 3.2. If $\mathbb{N}$ be the set of natural numbers, then $(\mathbb{N}, min, max, \cdot)$ is a SBS, where " $\cdot$ " is a usual multiplication. Let $\mathcal{L} = \langle \bar{\mathbb{A}}_{\mathcal{L}}, \mathbb{V}_{\mathcal{L}} \rangle$, where $\bar{\mathbb{A}}_{\mathcal{L}} : \mathbb{N} \to D[0,1]$ and $\mathbb{V}_{\mathcal{L}} : \mathbb{N} \to [0,1]$ defined by

Table 1

$\mathbb{N}$	$\bar{\mathbb{A}}_{\mathcal{L}}$	$\mathbb{V}_{\mathcal{L}}$
q_1	$\langle [0.45, 0.50], [0.55, 0.60] \rangle$	$(0.30, 0.40)$
q_2	$\langle [0.65, 0.70], [0.75, 0.80] \rangle$	$(0.40, 0.45)$
q_3	$\langle [0.75, 0.80], [0.85, 0.90] \rangle$	$(0.50, 0.50)$

Clearly $\mathcal{L}$ is a CVSBS of S.

Example 3.3. Let $S = \{q_1, q_2, q_3, q_4\}$ be the bisemirings with the Cayley tables:

$\circledast_1$	q_1	q_2	q_3	q_4
q_1	q_1	q_1	q_1	q_1
q_2	q_1	q_2	q_1	q_2
q_3	q_1	q_1	q_3	q_3
q_4	q_1	q_2	q_3	q_4

$\circledast_2$	q_1	q_2	q_3	q_4
q_1	q_1	q_2	q_3	q_4
q_2	q_2	q_2	q_4	q_4
q_3	q_3	q_4	q_3	q_4
q_4	q_4	q_4	q_4	q_4

$\circledast_3$	q_1	q_2	q_3	q_4
q_1	q_1	q_1	q_1	q_1
q_2	q_1	q_2	q_3	q_4
q_3	q_4	q_4	q_4	q_4
q_4	q_4	q_4	q_4	q_4

Table 2

S	$\bar{\mathbb{A}}_{\mathcal{L}}$	$\mathbb{V}_{\mathcal{L}}$
q_1	$\langle [0.35, 0.45], [0.50, 0.55] \rangle$	$(0.45, 0.50)$
q_2	$\langle [0.30, 0.40], [0.45, 0.50] \rangle$	$(0.35, 0.55)$
q_3	$\langle [0.20, 0.25], [0.35, 0.40] \rangle$	$(0.15, 0.65)$
q_4	$\langle [0.25, 0.30], [0.40, 0.45] \rangle$	$(0.25, 0.60)$

Clearly $\mathcal{L}$ is a CVSBS of S.

Theorem 3.1. *The arbitrary intersection of a family of CVSBS's is a CVSBS of S.*

Proof. Let $\{\mathcal{W}_j : j \in J\}$ be a family of CVSBS and $\mathcal{L} = \bigcap_{j \in J} \mathcal{W}_j$. Let $q_1, q_2 \in S$. Now,

$$\begin{aligned} \bar{\mathbb{T}}_{\mathbb{A}_{\mathcal{L}}}(q_1 \circledast_1 q_2) &= \inf_{j \in J} \bar{\mathbb{T}}_{\mathbb{A}_{\mathcal{W}_j}}(q_1 \circledast_1 q_2) \\ &\geq \inf_{j \in J} Min^i\{\bar{\mathbb{T}}_{\mathbb{A}_{\mathcal{W}_j}}(q_1), \bar{\mathbb{T}}_{\mathbb{A}_{\mathcal{W}_j}}(q_2)\} \\ &= Min^i\Big\{\inf_{j \in J} \bar{\mathbb{T}}_{\mathbb{A}_{\mathcal{W}_j}}(q_1), \inf_{j \in J} \bar{\mathbb{T}}_{\mathbb{A}_{\mathcal{W}_j}}(q_2)\Big\} \\ &= Min^i\{\bar{\mathbb{T}}_{\mathbb{A}_{\mathcal{L}}}(q_1), \bar{\mathbb{T}}_{\mathbb{A}_{\mathcal{L}}}(q_2)\} \end{aligned}$$

$$\begin{aligned} 1-\bar{\mathbb{F}}_{\mathbb{A}_{\mathcal{L}}}(q_1 \circledast_1 q_2) &= \inf_{j \in J} [1-\bar{\mathbb{F}}_{\mathbb{A}_{\mathcal{W}_j}}(q_1 \circledast_1 q_2)] \\ &\geq \inf_{j \in J} Min^i\{1-\bar{\mathbb{F}}_{\mathbb{A}_{\mathcal{W}_j}}(q_1), 1-\bar{\mathbb{F}}_{\mathbb{A}_{\mathcal{W}_j}}(q_2)\} \\ &= Min^i\Big\{\inf_{j \in J} 1-\bar{\mathbb{F}}_{\mathbb{A}_{\mathcal{W}_j}}(q_1), \inf_{j \in J} 1-\bar{\mathbb{F}}_{\mathbb{A}_{\mathcal{W}_j}}(q_2)\Big\} \\ &= Min^i\{1-\bar{\mathbb{F}}_{\mathbb{A}_{\mathcal{L}}}(q_1), 1-\bar{\mathbb{F}}_{\mathbb{A}_{\mathcal{L}}}(q_2)\} \end{aligned}$$

Thus, $\bar{\mathbb{A}}_{\mathcal{L}}(q_1 \circledast_1 q_2) \geq Min^i\{\bar{\mathbb{A}}_{\mathcal{L}}(q_1), \bar{\mathbb{A}}_{\mathcal{L}}(q_2)\}$. Now,

$$\begin{aligned}
\mathbb{T}_{\mathbb{V}_{\mathcal{L}}}(q_1 \circledast_1 q_2) &= \inf_{j\in J} \mathbb{T}_{\mathbb{V}_{\mathcal{W}_j}}(q_1 \circledast_1 q_2) \\
&\geq \inf_{j\in J} \min\{\mathbb{T}_{\mathbb{V}_{\mathcal{W}_j}}(q_1), \mathbb{T}_{\mathbb{V}_{\mathcal{W}_j}}(q_2)\} \\
&= \min\left\{\inf_{j\in J} \mathbb{T}_{\mathbb{V}_{\mathcal{W}_j}}(q_1), \inf_{j\in J} \mathbb{T}_{\mathbb{V}_{\mathcal{W}_j}}(q_2)\right\} \\
&= \min\{\mathbb{T}_{\mathbb{V}_{\mathcal{L}}}(q_1), \mathbb{T}_{\mathbb{V}_{\mathcal{L}}}(q_2)\} \\
1 - \mathbb{F}_{\mathbb{V}_{\mathcal{L}}}(q_1 \circledast_1 q_2) &= \inf_{j\in J} [1 - \mathbb{F}_{\mathbb{V}_{\mathcal{W}_j}}(q_1 \circledast_1 q_2)] \\
&\geq \inf_{j\in J} \min\{1 - \mathbb{F}_{\mathbb{V}_{\mathcal{W}_j}}(q_1), 1 - \mathbb{F}_{\mathbb{V}_{\mathcal{W}_j}}(q_2)\} \\
&= \min\left\{\inf_{j\in J} 1 - \mathbb{F}_{\mathbb{V}_{\mathcal{W}_j}}(q_1), \inf_{j\in J} 1 - \mathbb{F}_{\mathbb{V}_{\mathcal{W}_j}}(q_2)\right\} \\
&= \min\{1 - \mathbb{F}_{\mathbb{V}_{\mathcal{L}}}(q_1), 1 - \mathbb{F}_{\mathbb{V}_{\mathcal{L}}}(q_2)\}
\end{aligned}$$

Thus, $\mathbb{V}_{\mathcal{L}}(q_1 \circledast_1 q_2) \geq \min\{\mathbb{V}_{\mathcal{L}}(q_1), \mathbb{V}_{\mathcal{L}}(q_2)\}$. Hence, $\mathcal{L}(q_1 \circledast_1 q_2) \geq \min\{\mathcal{L}(q_1), \mathcal{L}(q_2)\}$. Similarly, $\mathcal{L}(q_1 \circledast_2 q_2) \geq \min\{\mathcal{L}(q_1), \mathcal{L}(q_2)\}$ and $\mathcal{L}(q_1 \circledast_3 q_2) \geq \min\{\mathcal{L}(q_1), \mathcal{L}(q_2)\}$. Hence $\mathcal{L}$ is a CVSBS.

Theorem 3.2. *If $\mathcal{N}$ and $\mathcal{M}$ are any two CVSBS's of S_1 and S_2 respectively, then the cartesian product $\mathcal{N} \times \mathcal{M}$ is a CVSBS.*

Proof. Let $\mathcal{N}$ and $\mathcal{M}$ be two CVSBS's of S_1 and S_2 respectively. Let $p_1, p_2 \in S_1$ and $q_1, q_2 \in S_2$. Then (p_1, p_2) and (q_1, q_2) are in $S_1 \times S_2$. Now $\bar{\mathbb{T}}_{\mathbb{A}_{\mathcal{N}\times\mathcal{M}}}[(p_1, q_1) \circledast_1 (p_2, q_2)]$

$$\begin{aligned}
&= \bar{\mathbb{T}}_{\mathbb{A}_{\mathcal{N}\times\mathcal{M}}}(p_1 \circledast_1 p_2, q_1 \circledast_1 q_2) \\
&= Min^i\{\bar{\mathbb{T}}_{\mathbb{A}_{\mathcal{N}}}(p_1 \circledast_1 p_2), \bar{\mathbb{T}}_{\mathbb{A}_{\mathcal{M}}}(q_1 \circledast_1 q_2)\} \\
&\geq Min^i\{Min^i\{\bar{\mathbb{T}}_{\mathbb{A}_{\mathcal{N}}}(p_1), \bar{\mathbb{T}}_{\mathbb{A}_{\mathcal{N}}}(p_2)\}, Min^i\{\bar{\mathbb{T}}_{\mathbb{A}_{\mathcal{M}}}(q_1), \bar{\mathbb{T}}_{\mathbb{A}_{\mathcal{M}}}(q_2)\}\} \\
&= Min^i\{Min^i\{\bar{\mathbb{T}}_{\mathbb{A}_{\mathcal{N}}}(p_1), \bar{\mathbb{T}}_{\mathbb{A}_{\mathcal{M}}}(q_1)\}, Min^i\{\bar{\mathbb{T}}_{\mathbb{A}_{\mathcal{N}}}(p_2), \bar{\mathbb{T}}_{\mathbb{A}_{\mathcal{M}}}(q_2)\}\} \\
&= Min^i\{\bar{\mathbb{T}}_{\mathbb{A}_{\mathcal{N}\times\mathcal{M}}}(p_1, q_1), \bar{\mathbb{T}}_{\mathbb{A}_{\mathcal{N}\times\mathcal{M}}}(p_2, q_2)\}
\end{aligned}$$

and $1 - \bar{\mathbb{F}}_{\mathbb{A}_{\mathcal{N}\times\mathcal{M}}}[(p_1, q_1) \circledast_1 (p_2, q_2)]$

$$\begin{aligned}
&= 1 - \bar{\mathbb{F}}_{\mathbb{A}_{\mathcal{N}\times\mathcal{M}}}(p_1 \circledast_1 p_2, q_1 \circledast_1 q_2) \\
&\geq Min^i\{1 - \bar{\mathbb{F}}_{\mathbb{A}_{\mathcal{N}}}(p_1 \circledast_1 p_2), 1 - \bar{\mathbb{F}}_{\mathbb{A}_{\mathcal{M}}}(q_1 \circledast_1 q_2)\} \\
&\geq Min^i\{Min^i\{1 - \bar{\mathbb{F}}_{\mathbb{A}_{\mathcal{N}}}(p_1), 1 - \bar{\mathbb{F}}_{\mathbb{A}_{\mathcal{N}}}(p_2)\}, Min^i\{1 - \bar{\mathbb{F}}_{\mathbb{A}_{\mathcal{M}}}(q_1), 1 - \bar{\mathbb{F}}_{\mathbb{A}_{\mathcal{M}}}(q_2)\}\} \\
&= Min^i\{Min^i\{1 - \bar{\mathbb{F}}_{\mathbb{A}_{\mathcal{N}}}(p_1), 1 - \bar{\mathbb{F}}_{\mathbb{A}_{\mathcal{M}}}(q_1)\}, Min^i\{1 - \bar{\mathbb{F}}_{\mathbb{A}_{\mathcal{N}}}(p_2), 1 - \bar{\mathbb{F}}_{\mathbb{A}_{\mathcal{M}}}(q_2)\}\} \\
&= Min^i\{1 - \bar{\mathbb{F}}_{\mathbb{A}_{\mathcal{N}\times\mathcal{M}}}(p_1, q_1), 1 - \bar{\mathbb{F}}_{\mathbb{A}_{\mathcal{N}\times\mathcal{M}}}(p_2, q_2)\}
\end{aligned}$$

Thus, $\bar{\mathbb{A}}_{\mathcal{N}\times\mathcal{M}}[(p_1, q_1) \circledast_1 (p_2, q_2)] \geq Min^i\{\bar{\mathbb{A}}_{\mathcal{N}\times\mathcal{M}}(p_1, q_1), \bar{\mathbb{A}}_{\mathcal{N}\times\mathcal{M}}(p_2, q_2)\}$. Now,

$$\begin{aligned}
\mathbb{T}_{\mathbb{V}_{\mathcal{N}\times\mathcal{M}}}[(p_1, q_1) \circledast_1 (p_2, q_2)] &= \mathbb{T}_{\mathbb{V}_{\mathcal{N}\times\mathcal{M}}}(p_1 \circledast_1 p_2, q_1 \circledast_1 q_2) \\
&= \min\{\mathbb{T}_{\mathbb{V}_{\mathcal{N}}}(p_1 \circledast_1 p_2), \mathbb{T}_{\mathbb{V}_{\mathcal{M}}}(q_1 \circledast_1 q_2)\} \\
&\geq \min\{\min\{\mathbb{T}_{\mathbb{V}_{\mathcal{N}}}(p_1), \mathbb{T}_{\mathbb{V}_{\mathcal{N}}}(p_2)\}, \min\{\mathbb{T}_{\mathbb{V}_{\mathcal{M}}}(q_1), \mathbb{T}_{\mathbb{V}_{\mathcal{M}}}(q_2)\}\} \\
&= \min\{\min\{\mathbb{T}_{\mathbb{V}_{\mathcal{N}}}(p_1), \mathbb{T}_{\mathbb{V}_{\mathcal{M}}}(q_1)\}, \min\{\mathbb{T}_{\mathbb{V}_{\mathcal{N}}}(p_2), \mathbb{T}_{\mathbb{V}_{\mathcal{M}}}(q_2)\}\} \\
&= \min\{\mathbb{T}_{\mathbb{V}_{\mathcal{N}\times\mathcal{M}}}(p_1, q_1), \mathbb{T}_{\mathbb{V}_{\mathcal{N}\times\mathcal{M}}}(p_2, q_2)\}
\end{aligned}$$

$1 - \mathbb{F}_{\mathbb{V}_{\mathcal{N}\times\mathcal{M}}}[(p_1, q_1) \circledast_1 (p_2, q_2)]$

$$= 1 - \mathbb{F}_{\mathbb{V}_{\mathcal{N}\times\mathcal{M}}}(p_1 \circledast_1 p_2, q_1 \circledast_1 q_2)$$

$\geq \min\{1-\mathbb{F}_{\mathbb{V}_{\mathcal{N}}}(p_1 \circledast_1 p_2), 1-\mathbb{F}_{\mathbb{V}_{\mathcal{M}}}(q_1 \circledast_1 q_2)\}$
$\geq \min\{\min\{1-\mathbb{F}_{\mathbb{V}_{\mathcal{N}}}(p_1), 1-\mathbb{F}_{\mathbb{V}_{\mathcal{N}}}(p_2)\}, \min\{1-\mathbb{F}_{\mathbb{V}_{\mathcal{M}}}(q_1), 1-\mathbb{F}_{\mathbb{V}_{\mathcal{M}}}(q_2)\}\}$
$= \min\{\min\{1-\mathbb{F}_{\mathbb{V}_{\mathcal{N}}}(p_1), 1-\mathbb{F}_{\mathbb{V}_{\mathcal{M}}}(q_1)\}, \min\{1-\mathbb{F}_{\mathbb{V}_{\mathcal{N}}}(p_2), 1-\mathbb{F}_{\mathbb{V}_{\mathcal{M}}}(q_2)\}\}$
$= \min\{1-\mathbb{F}_{\mathbb{V}_{\mathcal{N}\times\mathcal{M}}}(p_1, q_1), 1-\mathbb{F}_{\mathbb{V}_{\mathcal{N}\times\mathcal{M}}}(p_2, q_2)\}$

Thus, $\mathbb{V}_{\mathcal{N}\times\mathcal{M}}[(p_1, q_1) \circledast_1 (p_2, q_2)] \geq \min\{\mathbb{V}_{\mathcal{N}\times\mathcal{M}}(p_1, q_1), \mathbb{V}_{\mathcal{N}\times\mathcal{M}}(p_2, q_2)\}$.
Hence, $\mathcal{L}_{\mathcal{N}\times\mathcal{M}}[(p_1, q_1) \circledast_1 (p_2, q_2)] \geq \min\{\mathcal{L}_{\mathcal{N}\times\mathcal{M}}(p_1, q_1), \mathcal{L}_{\mathcal{N}\times\mathcal{M}}(p_2, q_2)\}$.
Similarly, $\mathcal{L}_{\mathcal{N}\times\mathcal{M}}[(p_1, q_1) \circledast_2 (p_2, q_2)] \geq \min\{\mathcal{L}_{\mathcal{N}\times\mathcal{M}}(p_1, q_1), \mathcal{L}_{\mathcal{N}\times\mathcal{M}}(p_2, q_2)\}$ and $\mathcal{L}_{\mathcal{N}\times\mathcal{M}}[(p_1, q_1) \circledast_3 (p_2, q_2)] \geq \min\{\mathcal{L}_{\mathcal{N}\times\mathcal{M}}(p_1, q_1), \mathcal{L}_{\mathcal{N}\times\mathcal{M}}(p_2, q_2)\}$.
Hence $\mathcal{N}\times\mathcal{M}$ is a CVSBS.

Theorem 3.3. *Let $\mathcal{L}$ be cubic vague subset of S. Then $\mathcal{L} = \langle \bar{\mathbb{A}}_{\mathcal{L}}, \mathbb{V}_{\mathcal{L}}\rangle$ is a CVSBS if and only if all non empty level sets $\mathcal{L}_{(\alpha,\beta)}(\alpha, \beta \in D[0,1])$ is a SBS.*

Proof. Assume that $\mathcal{L} = \langle \bar{\mathbb{A}}_{\mathcal{L}}, \mathbb{V}_{\mathcal{L}}\rangle$ is a CVSBS. For each $\alpha, \beta \in D[0,1]$ and $s_1, s_2 \in \mathcal{L}_{(\alpha,\beta)}$. We have $\bar{\mathbb{T}}_{\mathbb{A}_{\mathcal{L}}}(s_1) \geq \bar{\alpha}$, $\bar{\mathbb{T}}_{\mathbb{A}_{\mathcal{L}}}(s_2) \geq \bar{\alpha}$, $1-\bar{\mathbb{F}}_{\mathbb{A}_{\mathcal{L}}}(s_1) \geq \bar{\beta}$ and $1-\bar{\mathbb{F}}_{\mathbb{A}_{\mathcal{L}}}(s_2) \geq \bar{\beta}$. Since $\mathcal{L}$ is a CVSBS, $\bar{\mathbb{T}}_{\mathbb{A}_{\mathcal{L}}}(s_1 \circledast_1 s_2) \geq \min\{\bar{\mathbb{T}}_{\mathbb{A}_{\mathcal{L}}}(s_1), \bar{\mathbb{T}}_{\mathbb{A}_{\mathcal{L}}}(s_2)\} \geq \bar{\alpha}$. Similarly, $1-\bar{\mathbb{F}}_{\mathbb{A}_{\mathcal{L}}}(s_1 \circledast_1 s_2) \geq \min\{1-\bar{\mathbb{F}}_{\mathbb{A}_{\mathcal{L}}}(s_1), 1-\bar{\mathbb{F}}_{\mathbb{A}_{\mathcal{L}}}(s_2)\} \geq \bar{\beta}$. Now, $\mathbb{T}_{\mathbb{V}_{\mathcal{L}}}(s_1) \geq \alpha$, $\mathbb{T}_{\mathbb{V}_{\mathcal{L}}}(s_2) \geq \alpha$, $1-\mathbb{F}_{\mathbb{V}_{\mathcal{L}}}(s_1) \geq \beta$ and $1-\mathbb{F}_{\mathbb{V}_{\mathcal{L}}}(s_2) \geq \beta$. Since $\mathcal{L}$ is a CVSBS, $\mathbb{T}_{\mathbb{V}_{\mathcal{L}}}(s_1 \circledast_1 s_2) \geq \min\{\mathbb{T}_{\mathbb{V}_{\mathcal{L}}}(s_1), \mathbb{T}_{\mathbb{V}_{\mathcal{L}}}(s_2)\} \geq \alpha$. Similarly, $1-\mathbb{F}_{\mathbb{V}_{\mathcal{L}}}(s_1 \circledast_1 s_2) \geq \min\{1-\mathbb{F}_{\mathbb{V}_{\mathcal{L}}}(s_1), 1-\mathbb{F}_{\mathbb{V}_{\mathcal{L}}}(s_2)\} \geq \beta$. This implies that $s_1 \circledast_1 s_2 \in \mathcal{L}_{(\alpha,\beta)}$. Similarly, $s_1 \circledast_2 s_2 \in \mathcal{L}_{(\alpha,\beta)}$ and $s_1 \circledast_3 s_2 \in \mathcal{L}_{(\alpha,\beta)}$. Therefore $\mathcal{L}_{(\alpha,\beta)}$ is a SBS for each $\alpha, \beta \in D[0,1]$.
Conversely, let us assume $\mathcal{L}_{(\alpha,\beta)}$ is a SBS for each $\alpha, \beta \in D[0,1]$. Let $s_1, s_2 \in S$. Then $s_1, s_2 \in \mathcal{L}_{(\alpha,\beta)}$, where $\bar{\alpha} = \min\{\bar{\mathbb{T}}_{\mathbb{A}_{\mathcal{L}}}(s_1), \bar{\mathbb{T}}_{\mathbb{A}_{\mathcal{L}}}(s_2)\}$ and $\bar{\beta} = \min\{1-\bar{\mathbb{F}}_{\mathbb{A}_{\mathcal{L}}}(s_1), 1-\bar{\mathbb{F}}_{\mathbb{A}_{\mathcal{L}}}(s_2)\}$. Thus, $s_1 \circledast_1 s_2 \in \mathcal{L}_{(\alpha,\beta)}$ imply $\bar{\mathbb{T}}_{\mathbb{A}_{\mathcal{L}}}(s_1 \circledast_1 s_2) \geq \bar{\alpha} = \min\{\bar{\mathbb{T}}_{\mathbb{A}_{\mathcal{L}}}(s_1), \bar{\mathbb{T}}_{\mathbb{A}_{\mathcal{L}}}(s_2)\}$ and $1-\bar{\mathbb{F}}_{\mathbb{A}_{\mathcal{L}}}(s_1 \circledast_1 s_2) \geq \bar{\beta} = \min\{1-\bar{\mathbb{F}}_{\mathbb{A}_{\mathcal{L}}}(s_1), 1-\bar{\mathbb{F}}_{\mathbb{A}_{\mathcal{L}}}(s_2)\}$. Similarly, $\bar{\mathbb{T}}_{\mathbb{A}_{\mathcal{L}}}(s_1 \circledast_2 s_2) \geq \bar{\alpha} = \min\{\bar{\mathbb{T}}_{\mathbb{A}_{\mathcal{L}}}(s_1), \bar{\mathbb{T}}_{\mathbb{A}_{\mathcal{L}}}(s_2)\}$ and $1-\bar{\mathbb{F}}_{\mathbb{A}_{\mathcal{L}}}(s_1 \circledast_2 s_2) \geq \bar{\beta} = \min\{1-\bar{\mathbb{F}}_{\mathbb{A}_{\mathcal{L}}}(s_1), 1-\bar{\mathbb{F}}_{\mathbb{A}_{\mathcal{L}}}(s_2)\}$ and $\bar{\mathbb{T}}_{\mathbb{A}_{\mathcal{L}}}(s_1 \circledast_3 s_2) \geq \bar{\alpha} = \min\{\bar{\mathbb{T}}_{\mathbb{A}_{\mathcal{L}}}(s_1), \bar{\mathbb{T}}_{\mathbb{A}_{\mathcal{L}}}(s_2)\}$ and $1-\bar{\mathbb{F}}_{\mathbb{A}_{\mathcal{L}}}(s_1 \circledast_3 s_2) \geq \bar{\beta} = \min\{1-\bar{\mathbb{F}}_{\mathbb{A}_{\mathcal{L}}}(s_1), 1-\bar{\mathbb{F}}_{\mathbb{A}_{\mathcal{L}}}(s_2)\}$. Now, $\alpha = \min\{\mathbb{T}_{\mathbb{V}_{\mathcal{L}}}(s_1), \mathbb{T}_{\mathbb{V}_{\mathcal{L}}}(s_2)\}$ and $\beta = \min\{1-\mathbb{F}_{\mathbb{V}_{\mathcal{L}}}(s_1), 1-\mathbb{F}_{\mathbb{V}_{\mathcal{L}}}(s_2)\}$. Thus, $s_1 \circledast_1 s_2 \in \mathcal{L}_{(\alpha,\beta)}$ imply $\mathbb{T}_{\mathbb{V}_{\mathcal{L}}}(s_1 \circledast_1 s_2) \geq \alpha = \min\{\mathbb{T}_{\mathbb{V}_{\mathcal{L}}}(s_1), \mathbb{T}_{\mathbb{V}_{\mathcal{L}}}(s_2)\}$ and $1-\mathbb{F}_{\mathbb{V}_{\mathcal{L}}}(s_1 \circledast_1 s_2) \geq \beta = \min\{1-\mathbb{F}_{\mathbb{V}_{\mathcal{L}}}(s_1), 1-\mathbb{F}_{\mathbb{V}_{\mathcal{L}}}(s_2)\}$. Similarly, $\mathbb{T}_{\mathbb{V}_{\mathcal{L}}}(s_1 \circledast_2 s_2) \geq \alpha = \min\{\mathbb{T}_{\mathbb{V}_{\mathcal{L}}}(s_1), \mathbb{T}_{\mathbb{V}_{\mathcal{L}}}(s_2)\}$ and $1-\mathbb{F}_{\mathbb{V}_{\mathcal{L}}}(s_1 \circledast_2 s_2) \geq \beta = \min\{1-\mathbb{F}_{\mathbb{V}_{\mathcal{L}}}(s_1), 1-\mathbb{F}_{\mathbb{V}_{\mathcal{L}}}(s_2)\}$ and $\mathbb{T}_{\mathbb{V}_{\mathcal{L}}}(s_1 \circledast_3 s_2) \geq \alpha = \min\{\mathbb{T}_{\mathbb{V}_{\mathcal{L}}}(s_1), \mathbb{T}_{\mathbb{V}_{\mathcal{L}}}(s_2)\}$ and $1-\mathbb{F}_{\mathbb{V}_{\mathcal{L}}}(s_1 \circledast_3 s_2) \geq \beta = \min\{1-\mathbb{F}_{\mathbb{V}_{\mathcal{L}}}(s_1), 1-\mathbb{F}_{\mathbb{V}_{\mathcal{L}}}(s_2)\}$. Hence $\mathcal{L} = \langle \bar{\mathbb{A}}_{\mathcal{L}}, \mathbb{V}_{\mathcal{L}}\rangle$ is a CVSBS.

Theorem 3.4. *Let $\mathcal{L}$ be the CVSBS and $\mathcal{W}$ be the strongest Vague relation of S. Then $\mathcal{L}$ is a CVSBS if and only if $\mathcal{W}$ is a CVSBS of $S \times S$.*

Proof. Let $\mathcal{L}$ is a CVSBS and $\mathcal{W}$ be the strongest Vague relation of S. For $p = (p_1, p_2)$ and $q = (q_1, q_2)$ are in $S \times S$.
Now,

$$\begin{aligned}
\bar{\mathbb{T}}_{\mathbb{A}_{\mathcal{W}}}(p \circledast_1 q) &= \bar{\mathbb{T}}_{\mathbb{A}_{\mathcal{W}}}[((p_1, p_2) \circledast_1 (q_1, q_2)] \\
&= \bar{\mathbb{T}}_{\mathbb{A}_{\mathcal{W}}}(p_1 \circledast_1 q_1, p_2 \circledast_1 q_2) \\
&= Min^i\{\bar{\mathbb{T}}_{\mathbb{A}_{\mathcal{L}}}(p_1 \circledast_1 q_1), \bar{\mathbb{T}}_{\mathbb{A}_{\mathcal{L}}}(p_2 \circledast_1 q_2)\} \\
&\geq Min^i\{Min^i\{\bar{\mathbb{T}}_{\mathbb{A}_{\mathcal{L}}}(p_1), \bar{\mathbb{T}}_{\mathbb{A}_{\mathcal{L}}}(q_1)\}, Min^i\{\bar{\mathbb{T}}_{\mathbb{A}_{\mathcal{L}}}(p_2), \bar{\mathbb{T}}_{\mathbb{A}_{\mathcal{L}}}(q_2)\}\} \\
&= Min^i\{Min^i\{\bar{\mathbb{T}}_{\mathbb{A}_{\mathcal{L}}}(p_1), \bar{\mathbb{T}}_{\mathbb{A}_{\mathcal{L}}}(p_2)\}, Min^i\{\bar{\mathbb{T}}_{\mathbb{A}_{\mathcal{L}}}(q_1), \bar{\mathbb{T}}_{\mathbb{A}_{\mathcal{L}}}(q_2)\}\}
\end{aligned}$$

$$= Min^i\{\bar{\mathbb{T}}_{\mathbb{A}_{\mathcal{W}}}(p_1,p_2), \bar{\mathbb{T}}_{\mathbb{A}_{\mathcal{W}}}(q_1,q_2)\}$$
$$= Min^i\{\bar{\mathbb{T}}_{\mathbb{A}_{\mathcal{W}}}(p), \bar{\mathbb{T}}_{\mathbb{A}_{\mathcal{W}}}(q)\}$$

and $1-\bar{\mathbb{F}}_{\mathbb{A}_{\mathcal{W}}}(p \circledast_1 q)$

$$= 1-\bar{\mathbb{F}}_{\mathbb{A}_{\mathcal{W}}}[((p_1,p_2)\circledast_1 (q_1,q_2)]$$
$$= 1-\bar{\mathbb{F}}_{\mathbb{A}_{\mathcal{W}}}(p_1\circledast_1 q_1, p_2\circledast_1 q_2)$$
$$= Min^i\{1-\bar{\mathbb{F}}_{\mathbb{A}_{\mathcal{L}}}(p_1\circledast_1 q_1), 1-\bar{\mathbb{F}}_{\mathbb{A}_{\mathcal{L}}}(p_2\circledast_1 q_2)\}$$
$$\geq Min^i\{Min^i\{1-\bar{\mathbb{F}}_{\mathbb{A}_{\mathcal{L}}}(p_1), 1-\bar{\mathbb{F}}_{\mathbb{A}_{\mathcal{L}}}(q_1)\}, Min^i\{1-\bar{\mathbb{F}}_{\mathbb{A}_{\mathcal{L}}}(p_2), 1-\bar{\mathbb{F}}_{\mathbb{A}_{\mathcal{L}}}(q_2)\}\}$$
$$= Min^i\{Min^i\{1-\bar{\mathbb{F}}_{\mathbb{A}_{\mathcal{L}}}(p_1), 1-\bar{\mathbb{F}}_{\mathbb{A}_{\mathcal{L}}}(p_2)\}, Min^i\{1-\bar{\mathbb{F}}_{\mathbb{A}_{\mathcal{L}}}(q_1), 1-\bar{\mathbb{F}}_{\mathbb{A}_{\mathcal{L}}}(q_2)\}\}$$
$$= Min^i\{1-\bar{\mathbb{F}}_{\mathbb{A}_{\mathcal{W}}}(p_1,p_2), 1-\bar{\mathbb{F}}_{\mathbb{A}_{\mathcal{W}}}(q_1,q_2)\}$$
$$= Min^i\{1-\bar{\mathbb{F}}_{\mathbb{A}_{\mathcal{W}}}(p), 1-\bar{\mathbb{F}}_{\mathbb{A}_{\mathcal{W}}}(q)\}$$

Thus, $\bar{\mathbb{A}}_{\mathcal{W}}(p\circledast_1 q) \geq Min^i\{\bar{\mathbb{A}}_{\mathcal{W}}(p), \bar{\mathbb{A}}_{\mathcal{W}}(q)\}$.
Now,

$$\begin{aligned}
\mathbb{T}_{\mathbb{V}_{\mathcal{W}}}(p\circledast_1 q) &= \mathbb{T}_{\mathbb{V}_{\mathcal{W}}}[((p_1,p_2)\circledast_1 (q_1,q_2)]\\
&= \mathbb{T}_{\mathbb{V}_{\mathcal{W}}}(p_1\circledast_1 q_1, p_2\circledast_1 q_2)\\
&= \min\{\mathbb{T}_{\mathbb{V}_{\mathcal{L}}}(p_1\circledast_1 q_1), \mathbb{T}_{\mathbb{V}_{\mathcal{L}}}(p_2\circledast_1 q_2)\}\\
&\geq \min\{\min\{\mathbb{T}_{\mathbb{V}_{\mathcal{L}}}(p_1), \mathbb{T}_{\mathbb{V}_{\mathcal{L}}}(q_1)\}, \min\{\mathbb{T}_{\mathbb{V}_{\mathcal{L}}}(p_2), \mathbb{T}_{\mathbb{V}_{\mathcal{L}}}(q_2)\}\}\\
&= \min\{\min\{\mathbb{T}_{\mathbb{V}_{\mathcal{L}}}(p_1), \mathbb{T}_{\mathbb{V}_{\mathcal{L}}}(p_2)\}, \min\{\mathbb{T}_{\mathbb{V}_{\mathcal{L}}}(q_1), \mathbb{T}_{\mathbb{V}_{\mathcal{L}}}(q_2)\}\}\\
&= \min\{\mathbb{T}_{\mathbb{V}_{\mathcal{W}}}(p_1,p_2), \mathbb{T}_{\mathbb{V}_{\mathcal{W}}}(q_1,q_2)\}\\
&= \min\{\mathbb{T}_{\mathbb{V}_{\mathcal{W}}}(p), \mathbb{T}_{\mathbb{V}_{\mathcal{W}}}(q)\}
\end{aligned}$$

$$\begin{aligned}
1-\mathbb{F}_{\mathbb{V}_{\mathcal{W}}}(p\circledast_1 q) &= 1-\mathbb{F}_{\mathbb{V}_{\mathcal{W}}}[((p_1,p_2)\circledast_1 (q_1,q_2)]\\
&= 1-\mathbb{F}_{\mathbb{V}_{\mathcal{W}}}(p_1\circledast_1 q_1, p_2\circledast_1 q_2)\\
&= \min\{1-\mathbb{F}_{\mathbb{V}_{\mathcal{L}}}(p_1\circledast_1 q_1), 1-\mathbb{F}_{\mathbb{V}_{\mathcal{L}}}(p_2\circledast_1 q_2)\}\\
&\geq \min\{\min\{1-\mathbb{F}_{\mathbb{V}_{\mathcal{L}}}(p_1), 1-\mathbb{F}_{\mathbb{V}_{\mathcal{L}}}(q_1)\}, \min\{1-\mathbb{F}_{\mathbb{V}_{\mathcal{L}}}(p_2), 1-\mathbb{F}_{\mathbb{V}_{\mathcal{L}}}(q_2)\}\}\\
&= \min\{\min\{1-\mathbb{F}_{\mathbb{V}_{\mathcal{L}}}(p_1), 1-\mathbb{F}_{\mathbb{V}_{\mathcal{L}}}(p_2)\}, \min\{1-\mathbb{F}_{\mathbb{V}_{\mathcal{L}}}(q_1), 1-\mathbb{F}_{\mathbb{V}_{\mathcal{L}}}(q_2)\}\}\\
&= \min\{1-\mathbb{F}_{\mathbb{V}_{\mathcal{W}}}(p_1,p_2), 1-\mathbb{F}_{\mathbb{V}_{\mathcal{W}}}(q_1,q_2)\}\\
&= \min\{1-\mathbb{F}_{\mathbb{V}_{\mathcal{W}}}(p), 1-\mathbb{F}_{\mathbb{V}_{\mathcal{W}}}(q)\}
\end{aligned}$$

Thus, $\mathbb{V}_{\mathcal{W}}(p\circledast_1 q) \geq \min\{\mathbb{V}_{\mathcal{W}}(p), \mathbb{V}_{\mathcal{W}}(q)\}$.
Hence, $\mathcal{L}_{\mathcal{W}}(p\circledast_1 q) \geq \min\{\mathcal{L}_{\mathcal{W}}(p), \mathcal{L}_{\mathcal{W}}(q)\}$.
Similarly, $\mathcal{L}_{\mathcal{W}}(p\circledast_2 q) \geq \min\{\mathcal{L}_{\mathcal{W}}(p), \mathcal{L}_{\mathcal{W}}(q)\}$ and $\mathcal{L}_{\mathcal{W}}(p\circledast_3 q) \geq \min\{\mathcal{L}_{\mathcal{W}}(p), \mathcal{L}_{\mathcal{W}}(q)\}$.
Hence $\mathcal{W}$ is a CVSBS of $S\times S$.
Conversely, assume that $\mathcal{W}$ is a CVSBS of $S\times S$, $p=(p_1,p_2)$ and $q=(q_1,q_2)$ are in $S\times S$. Now, $Min^i\{\bar{\mathbb{T}}_{\mathbb{A}_{\mathcal{L}}}(p_1\circledast_1 q_1), \bar{\mathbb{T}}_{\mathbb{A}_{\mathcal{L}}}(p_2\circledast_1 q_2)\}$

$$= \bar{\mathbb{T}}_{\mathbb{A}_{\mathcal{W}}}(p_1\circledast_1 q_1, p_2\circledast_1 q_2)$$
$$= \bar{\mathbb{T}}_{\mathbb{A}_{\mathcal{W}}}[(p_1,p_2)\circledast_1 (q_1,q_2)]$$
$$= \bar{\mathbb{T}}_{\mathbb{A}_{\mathcal{W}}}(p\circledast_1 q)$$
$$\geq Min^i\{\bar{\mathbb{T}}_{\mathbb{A}_{\mathcal{W}}}(p), \bar{\mathbb{T}}_{\mathbb{A}_{\mathcal{W}}}(q)\}$$
$$= Min^i\{\bar{\mathbb{T}}_{\mathbb{A}_{\mathcal{W}}}(p_1,p_2)\}, \bar{\mathbb{T}}_{\mathbb{A}_{\mathcal{W}}}(q_1,q_2)\}$$
$$= Min^i\{Min^i\{\bar{\mathbb{T}}_{\mathbb{A}_{\mathcal{L}}}(p_1), \bar{\mathbb{T}}_{\mathbb{A}_{\mathcal{L}}}(p_2)\}, Min^i\{\bar{\mathbb{T}}_{\mathbb{A}_{\mathcal{L}}}(q_1), \bar{\mathbb{T}}_{\mathbb{A}_{\mathcal{L}}}(q_2)\}\}$$

If $\bar{\mathbb{T}}_{\mathbb{A}_{\mathcal{L}}}(p_1 \circledast_1 q_1) \leq \bar{\mathbb{T}}_{\mathbb{A}_{\mathcal{L}}}(p_2 \circledast_1 q_2)$, then $\bar{\mathbb{T}}_{\mathbb{A}_{\mathcal{L}}}(p_1) \leq \bar{\mathbb{T}}_{\mathbb{A}_{\mathcal{L}}}(p_2)$ and $\bar{\mathbb{T}}_{\mathbb{A}_{\mathcal{L}}}(q_1) \leq \bar{\mathbb{T}}_{\mathbb{A}_{\mathcal{L}}}(q_2)$.
We get $\bar{\mathbb{T}}_{\mathbb{A}_{\mathcal{L}}}(p_1 \circledast_1 q_1) \geq Min^i\{\bar{\mathbb{T}}_{\mathbb{A}_{\mathcal{L}}}(p_1), \bar{\mathbb{T}}_{\mathbb{A}_{\mathcal{L}}}(q_1)\}$ for all $p_1, q_1 \in S$.
And $Min^i\{1 - \bar{\mathbb{F}}_{\mathbb{A}_{\mathcal{L}}}(p_1 \circledast_1 q_1), 1 - \bar{\mathbb{F}}_{\mathbb{A}_{\mathcal{L}}}(p_2 \circledast_1 q_2)\}$

$$\begin{aligned}
&= 1 - \bar{\mathbb{F}}_{\mathbb{A}_{\mathcal{W}}}(p_1 \circledast_1 q_1, p_2 \circledast_1 q_2) \\
&= 1 - \bar{\mathbb{F}}_{\mathbb{A}_{\mathcal{W}}}[(p_1, p_2) \circledast_1 (q_1, q_2)] \\
&= 1 - \bar{\mathbb{F}}_{\mathbb{A}_{\mathcal{W}}}(p \circledast_1 q) \\
&\geq Min^i\{1 - \bar{\mathbb{F}}_{\mathbb{A}_{\mathcal{W}}}(p), 1 - \bar{\mathbb{F}}_{\mathbb{A}_{\mathcal{W}}}(q)\} \\
&= Min^i\{1 - \bar{\mathbb{F}}_{\mathbb{A}_{\mathcal{W}}}(p_1, p_2)\}, 1 - \bar{\mathbb{F}}_{\mathbb{A}_{\mathcal{W}}}(q_1, q_2)\}
\end{aligned}$$

$$= Min^i\{Min^i\{1 - \bar{\mathbb{F}}_{\mathbb{A}_{\mathcal{L}}}(p_1), 1 - \bar{\mathbb{F}}_{\mathbb{A}_{\mathcal{L}}}(p_2)\}, Min^i\{1 - \bar{\mathbb{F}}_{\mathbb{A}_{\mathcal{L}}}(q_1), 1 - \bar{\mathbb{F}}_{\mathbb{A}_{\mathcal{L}}}(q_2)\}\}$$

If $1 - \bar{\mathbb{F}}_{\mathbb{A}_{\mathcal{L}}}(p_1 \circledast_1 q_1) \leq 1 - \bar{\mathbb{F}}_{\mathbb{A}_{\mathcal{L}}}(p_2 \circledast_1 q_2)$, then $1 - \bar{\mathbb{F}}_{\mathbb{A}_{\mathcal{L}}}(p_1) \leq 1 - \bar{\mathbb{F}}_{\mathbb{A}_{\mathcal{L}}}(p_2)$ and $1-\bar{\mathbb{F}}_{\mathbb{A}_{\mathcal{L}}}(q_1) \leq 1-\bar{\mathbb{F}}_{\mathbb{A}_{\mathcal{L}}}(q_2)$. We get $1-\bar{\mathbb{F}}_{\mathbb{A}_{\mathcal{L}}}(p_1 \circledast_1 q_1) \geq Min^i\{1-\bar{\mathbb{F}}_{\mathbb{A}_{\mathcal{L}}}(p_1), 1-\bar{\mathbb{F}}_{\mathbb{A}_{\mathcal{L}}}(q_1)\}$ for all $p_1, q_1 \in S$.
Thus, $\bar{\mathbb{A}}_{\mathcal{L}}(p_1 \circledast_1 q_1) \geq Min^i\{\bar{\mathbb{A}}_{\mathcal{L}}(p_1), \bar{\mathbb{A}}_{\mathcal{L}}(q_1)\}$.
Now, $\min\{\mathbb{T}_{\mathbb{V}_{\mathcal{L}}}(p_1 \circledast_1 q_1), \mathbb{T}_{\mathbb{V}_{\mathcal{L}}}(p_2 \circledast_1 q_2)\}$

$$\begin{aligned}
&= \mathbb{T}_{\mathbb{V}_{\mathcal{W}}}(p_1 \circledast_1 q_1, p_2 \circledast_1 q_2) \\
&= \mathbb{T}_{\mathbb{V}_{\mathcal{W}}}[(p_1, p_2) \circledast_1 (q_1, q_2)] \\
&= \mathbb{T}_{\mathbb{V}_{\mathcal{W}}}(p \circledast_1 q) \\
&\geq \min\{\mathbb{T}_{\mathbb{V}_{\mathcal{W}}}(p), \mathbb{T}_{\mathbb{V}_{\mathcal{W}}}(q)\} \\
&= \min\{\mathbb{T}_{\mathbb{V}_{\mathcal{W}}}(p_1, p_2)\}, \mathbb{T}_{\mathbb{V}_{\mathcal{W}}}(q_1, q_2)\} \\
&= \min\{\min\{\mathbb{T}_{\mathbb{V}_{\mathcal{L}}}(p_1), \mathbb{T}_{\mathbb{V}_{\mathcal{L}}}(p_2)\}, \min\{\mathbb{T}_{\mathbb{V}_{\mathcal{L}}}(q_1), \mathbb{T}_{\mathbb{V}_{\mathcal{L}}}(q_2)\}\}
\end{aligned}$$

If $\mathbb{T}_{\mathbb{V}_{\mathcal{L}}}(p_1 \circledast_1 q_1) \leq \mathbb{T}_{\mathbb{V}_{\mathcal{L}}}(p_2 \circledast_1 q_2)$, then $\mathbb{T}_{\mathbb{V}_{\mathcal{L}}}(p_1) \leq \mathbb{T}_{\mathbb{V}_{\mathcal{L}}}(p_2)$ and $\mathbb{T}_{\mathbb{V}_{\mathcal{L}}}(q_1) \leq \mathbb{T}_{\mathbb{V}_{\mathcal{L}}}(q_2)$.
We get $\mathbb{T}_{\mathbb{V}_{\mathcal{L}}}(p_1 \circledast_1 q_1) \geq \min\{\mathbb{T}_{\mathbb{V}_{\mathcal{L}}}(p_1), \mathbb{T}_{\mathbb{V}_{\mathcal{L}}}(q_1)\}$ for all $p_1, q_1 \in S$.
And $\min\{1 - \mathbb{F}_{\mathbb{V}_{\mathcal{L}}}(p_1 \circledast_1 q_1), 1 - \mathbb{F}_{\mathbb{V}_{\mathcal{L}}}(p_2 \circledast_1 q_2)\}$

$$\begin{aligned}
&= 1 - \mathbb{F}_{\mathbb{V}_{\mathcal{W}}}(p_1 \circledast_1 q_1, p_2 \circledast_1 q_2) \\
&= 1 - \mathbb{F}_{\mathbb{V}_{\mathcal{W}}}[(p_1, p_2) \circledast_1 (q_1, q_2)] \\
&= 1 - \mathbb{F}_{\mathbb{V}_{\mathcal{W}}}(p \circledast_1 q) \\
&\geq \min\{1 - \mathbb{F}_{\mathbb{V}_{\mathcal{W}}}(p), 1 - \mathbb{F}_{\mathbb{V}_{\mathcal{W}}}(q)\} \\
&= \min\{1 - \mathbb{F}_{\mathbb{V}_{\mathcal{W}}}(p_1, p_2)\}, 1 - \mathbb{F}_{\mathbb{V}_{\mathcal{W}}}(q_1, q_2)\} \\
&= \min\{\min\{1 - \mathbb{F}_{\mathbb{V}_{\mathcal{L}}}(p_1), 1 - \mathbb{F}_{\mathbb{V}_{\mathcal{L}}}(p_2)\}, \min\{1 - \mathbb{F}_{\mathbb{V}_{\mathcal{L}}}(q_1), 1 - \mathbb{F}_{\mathbb{V}_{\mathcal{L}}}(q_2)\}\}
\end{aligned}$$

If $1 - \mathbb{F}_{\mathbb{V}_{\mathcal{L}}}(p_1 \circledast_1 q_1) \leq 1 - \mathbb{F}_{\mathbb{V}_{\mathcal{L}}}(p_2 \circledast_1 q_2)$, then $1 - \mathbb{F}_{\mathbb{V}_{\mathcal{L}}}(p_1) \leq 1 - \mathbb{F}_{\mathbb{V}_{\mathcal{L}}}(p_2)$ and $1-\mathbb{F}_{\mathbb{V}_{\mathcal{L}}}(q_1) \leq 1-\mathbb{F}_{\mathbb{V}_{\mathcal{L}}}(q_2)$. We get $1-\mathbb{F}_{\mathbb{V}_{\mathcal{L}}}(p_1 \circledast_1 q_1) \geq \min\{1-\mathbb{F}_{\mathbb{V}_{\mathcal{L}}}(p_1), 1-\mathbb{F}_{\mathbb{V}_{\mathcal{L}}}(q_1)\}$ for all $p_1, q_1 \in S$.
Thus, $\mathbb{V}_{\mathcal{L}}(p_1 \circledast_1 q_1) \geq \min\{\mathbb{V}_{\mathcal{L}}(p_1), \mathbb{V}_{\mathcal{L}}(q_1)\}$. Hence, $\mathcal{L}(p_1 \circledast_1 q_1) \geq \min\{\mathcal{L}(p_1), \mathcal{L}(q_1)\}$.
Similarly, $\mathcal{L}(p_1 \circledast_2 q_1) \geq \min\{\mathcal{L}(p_1), \mathcal{L}(q_1)\}$ and $\mathcal{L}(p_1 \circledast_3 q_1) \geq \min\{\mathcal{L}(p_1), \mathcal{L}(q_1)\}$.
Hence $\mathcal{L}$ is a CVSBS.

Theorem 3.5. *If $\mathcal{L}$ is a CVSBS of $(S, \circledast_1, \circledast_2, \circledast_3)$, then $\Box\mathcal{L}$ is a CVSBS.*

Proof. Let $\mathcal{L}$ be an CVSBS of a bisemiring S. Consider $\mathcal{L} = \{\langle q_1, \bar{\mathbb{A}}_{\mathcal{L}}(q_1), \mathbb{V}_{\mathcal{L}}(q_1)\rangle\}$, for all $q_1 \in S$. Take $\Box\mathcal{L} = \mathcal{M} = \{\langle q_1, \bar{\mathbb{A}}_{\mathcal{M}}(q_1), \mathbb{V}_{\mathcal{M}}(q_1)\rangle\}$, where $\bar{\mathbb{T}}_{\mathbb{A}_{\mathcal{M}}}(q_1) = \bar{\mathbb{T}}_{\mathbb{A}_{\mathcal{L}}}(q_1)$, $\bar{\mathbb{F}}_{\mathbb{A}_{\mathcal{M}}}(q_1) = 1 - \bar{\mathbb{T}}_{\mathbb{A}_{\mathcal{L}}}(q_1)$, $\mathbb{T}_{\mathbb{V}_{\mathcal{M}}}(q_1) = \mathbb{T}_{\mathbb{V}_{\mathcal{L}}}(q_1)$ and $\mathbb{F}_{\mathbb{V}_{\mathcal{M}}}(q_1) = 1 - \mathbb{T}_{\mathbb{V}_{\mathcal{L}}}(q_1)$. Clearly $\bar{\mathbb{T}}_{\mathbb{A}_{\mathcal{M}}}(q_1 \circledast_1 q_2) \geq Min^i\{\bar{\mathbb{T}}_{\mathbb{A}_{\mathcal{M}}}(q_1), \bar{\mathbb{T}}_{\mathbb{A}_{\mathcal{M}}}(q_2)\}$ and $\mathbb{T}_{\mathbb{V}_{\mathcal{M}}}(q_1 \circledast_1 q_2) \geq \min\{\mathbb{T}_{\mathbb{V}_{\mathcal{M}}}(q_1), \mathbb{T}_{\mathbb{V}_{\mathcal{M}}}(q_2)\}$, for all $q_1, q_2 \in S$. Since $\mathcal{L}$ is a CVSBS. Then $\bar{\mathbb{T}}_{\mathbb{A}_{\mathcal{L}}}(q_1 \circledast_1 q_2) \geq Min^i\{\bar{\mathbb{T}}_{\mathbb{A}_{\mathcal{L}}}(q_1), \bar{\mathbb{T}}_{\mathbb{A}_{\mathcal{L}}}(q_2)\}$

and $\mathbb{T}_{\mathbb{V}_{\mathcal{L}}}(q_1 \circledast_1 q_2) \geq \min\{\mathbb{T}_{\mathbb{V}_{\mathcal{L}}}(q_1), \mathbb{T}_{\mathbb{V}_{\mathcal{L}}}(q_2)\}$ imply $1 - \bar{\mathbb{F}}_{\mathbb{A}_{\mathcal{M}}}(q_1 \circledast_1 q_2) \geq Min^i\{(1 - \bar{\mathbb{F}}_{\mathbb{A}_{\mathcal{M}}}(q_1)), (1-\bar{\mathbb{F}}_{\mathbb{A}_{\mathcal{M}}}(q_2))\}$ and $1-\mathbb{F}_{\mathbb{V}_{\mathcal{M}}}(q_1 \circledast_1 q_2) \geq \min\{(1-\mathbb{F}_{\mathbb{V}_{\mathcal{M}}}(q_1)), (1-\mathbb{F}_{\mathbb{V}_{\mathcal{M}}}(q_2))\}$.
Hence $\Box\mathcal{L}$ is a CVSBS.

Remark. *The reverse of the Theorem 3.5 fails by the Example 3.3. Here* $\Box\mathcal{L} = \langle \Box\bar{\mathbb{A}}_{\mathcal{L}}, \Box\mathbb{V}_{\mathcal{L}} \rangle$.

Table 3

S	$\Box\bar{\mathbb{A}}_{\mathcal{L}}$	$\bar{\mathbb{A}}_{\mathcal{L}}$
q_1	$\langle [0.25, 0.30], [0.70, 0.75] \rangle$	$\langle [0.25, 0.30], [0.40, 0.45] \rangle$
q_2	$\langle [0.15, 0.20], [0.80, 0.85] \rangle$	$\langle [0.15, 0.20], [0.20, 0.25] \rangle$
q_3	$\langle [0.10, 0.13], [0.87, 0.90] \rangle$	$\langle [0.10, 0.13], [0.30, 0.55] \rangle$
q_4	$\langle [0.12, 0.15], [0.85, 0.88] \rangle$	$\langle [0.12, 0.15], [0.25, 0.30] \rangle$

Clearly $\Box\bar{\mathbb{A}}_{\mathcal{L}}$ is interval valued FSBS, but $\bar{\mathbb{A}}_{\mathcal{L}}$ is not a interval valued FSBS . And,

	$\Box\mathbb{V}_{\mathcal{L}}$	$\mathbb{V}_{\mathcal{L}}$
q_1	$(0.30, 0.70)$	$(0.30, 0.60)$
q_2	$(0.20, 0.80)$	$(0.20, 0.40)$
q_3	$(0.10, 0.90)$	$(0.10, 0.55)$
q_4	$(0.15, 0.85)$	$(0.15, 0.45)$

Clearly $\Box\mathbb{V}_{\mathcal{L}}$ is FSBS, but $\mathbb{V}_{\mathcal{L}}$ is not a FSBS.
Hence $\Box\mathcal{L}$ is CVSBS, but $\mathcal{L}$ is not a CVSBS.

Theorem 3.6. *If $\mathcal{L}$ is a CVSBS of $(S, \circledast_1, \circledast_2, \circledast_3)$ then $\Diamond\mathcal{L}$ is a CVSBS.*

Proof. Let $\mathcal{L}$ be an CVSBS of a bisemiring S. Consider $\mathcal{L} = \{\langle q_1, \bar{\mathbb{A}}_{\mathcal{L}}(q_1), \mathbb{V}_{\mathcal{L}}(q_1) \rangle\}$, for all $q_1 \in S$. Take $\Diamond\mathcal{L} = \mathcal{M} = \{\langle q_1, \bar{\mathbb{A}}_{\mathcal{M}}(q_1), \mathbb{V}_{\mathcal{M}}(q_1) \rangle\}$, where $\bar{\mathbb{T}}_{\mathbb{A}_{\mathcal{M}}}(q_1) = 1 - \bar{\mathbb{F}}_{\mathbb{A}_{\mathcal{L}}}(q_1), \bar{\mathbb{F}}_{\mathbb{A}_{\mathcal{M}}}(q_1) = \bar{\mathbb{F}}_{\mathbb{A}_{\mathcal{L}}}(q_1), \mathbb{T}_{\mathbb{V}_{\mathcal{M}}}(q_1) = 1 - \mathbb{F}_{\mathbb{V}_{\mathcal{L}}}(q_1)$ and $\mathbb{F}_{\mathbb{V}_{\mathcal{M}}}(q_1) = \mathbb{F}_{\mathbb{V}_{\mathcal{L}}}(q_1)$ and the applying Theorem 3.5. Hence $\Diamond\mathcal{L}$ is an CVSBS.

Remark. *The inversion of Theorem 3.6 fails by the Example 3.3. Here* $\Diamond\mathcal{L} = \langle \Diamond\bar{\mathbb{A}}_{\mathcal{L}}, \Diamond\mathbb{V}_{\mathcal{L}} \rangle$.

Table 4

S	$\Diamond\bar{\mathbb{A}}_{\mathcal{L}}$	$\bar{\mathbb{A}}_{\mathcal{L}}$
q_1	$\langle [0.65, 0.75], [0.25, 0.35] \rangle$	$\langle [0.20, 0.22], [0.65, 0.75] \rangle$
q_2	$\langle [0.55, 0.65], [0.35, 0.45] \rangle$	$\langle [0.14, 0.16], [0.55, 0.65] \rangle$
q_3	$\langle [0.30, 0.35], [0.65, 0.70] \rangle$	$\langle [0.18, 0.20], [0.30, 0.35] \rangle$
q_4	$\langle [0.50, 0.55], [0.45, 0.50] \rangle$	$\langle [0.16, 0.18], [0.50, 0.55] \rangle$

Clearly $\Diamond\bar{\mathbb{A}}_{\mathcal{L}}$ is interval valued FSBS, but $\bar{\mathbb{A}}_{\mathcal{L}}$ is not a interval valued FSBS . And,

	$\Diamond\mathbb{V}_{\mathcal{L}}$	$\mathbb{V}_{\mathcal{L}}$
q_1	$(0.70, 0.30)$	$(0.25, 0.70)$
q_2	$(0.65, 0.35)$	$(0.10, 0.65)$
q_3	$(0.40, 0.60)$	$(0.20, 0.40)$
q_4	$(0.55, 0.45)$	$(0.15, 0.55)$

Clearly $\Diamond\mathbb{V}_{\mathcal{L}}$ is a FSBS, but $\mathbb{V}_{\mathcal{L}}$ is not a FSBS.
Hence $\Diamond\mathcal{L}$ is a CVSBS, but $\mathcal{L}$ is not a CVSBS.

4. CUBIC VAGUE SUBBISEMIRINGS HOMOMORPHISM

We introduce homomorphism of CVSBS and level set homomorphism of CVSBS.

Definition 4.1. Let $(S_1, \oplus_1, \oplus_2, \oplus_3)$ and $(S_2, \odot_1, \odot_2, \odot_3)$ be any two bisemirings. Let $\phi : S_1 \to S_2$ be any function and L be an CVSBS in S_1, W be an CVSBS in $\phi(S_1) = S_2$, the image of cubic vague set can be defined by $\bar{\mathbb{A}}_{\phi(W)}(s_2) = [\bar{\mathbb{T}}_{\phi(W)}(s_2), \bar{\mathbb{F}}_{\phi(W)}(s_2)]$ and $\mathbb{V}_{\phi(W)}(s_2) = [\mathbb{T}_{\phi(W)}(s_2), \mathbb{F}_{\phi(W)}(s_2)]$, where $\bar{\mathbb{T}}_{\phi(W)}(s_2) = \bar{\mathbb{T}}_W\phi(s_2)$, $\bar{\mathbb{F}}_{\phi(W)}(s_2) = \bar{\mathbb{F}}_W\phi(s_2)$, $\mathbb{T}_{\phi(W)}(s_2) = \mathbb{T}_W\phi(s_2)$ and $\mathbb{F}_{\phi(W)}(s_2) = \mathbb{F}_W\phi(s_2)$.

Definition 4.2. Let $(S_1, \oplus_1, \oplus_2, \oplus_3)$ and $(S_2, \odot_1, \odot_2, \odot_3)$ be any two bisemirings. Let $\phi : S_1 \to S_2$ be any function. Let W be a cubic vague set in $\phi(S_1) = S_2$. Then the inverse image of W, ϕ^{-1} is the cubic vague set in S_1 by $\bar{\mathbb{A}}_{\phi^{-1}(W)}(s_1) = [\bar{\mathbb{T}}_{\phi^{-1}(W)}(s_1), \bar{\mathbb{F}}_{\phi^{-1}(W)}(s_1)]$ and $\mathbb{V}_{\phi^{-1}(W)}(s_1) = [\mathbb{T}_{\phi^{-1}(W)}(s_1), \mathbb{F}_{\phi^{-1}(W)}(s_1)]$, where $\bar{\mathbb{T}}_{\phi^{-1}(W)}(s_1) = \bar{\mathbb{T}}_W(\phi^{-1}(s_1))$, $\bar{\mathbb{F}}_{\phi^{-1}(W)}(s_1) = \bar{\mathbb{F}}_W(\phi^{-1}(s_1))$, $\mathbb{T}_{\phi^{-1}(W)}(s_1) = \mathbb{T}_W(\phi^{-1}(s_1))$ and $\mathbb{F}_{\phi^{-1}(W)}(s_1) = \mathbb{F}_W(\phi^{-1}(s_1))$.

Theorem 4.1. *Let $(S_1, \oplus_1, \oplus_2, \oplus_3)$ and $(S_2, \odot_1, \odot_2, \odot_3)$ be any two bisemirings. The homomorphic image of CVSBS of S_1 is also CVSBS of S_2.*

Proof. Let $\phi : S_1 \to S_2$ be a homomorphism. Then $\phi(s_1 \oplus_1 s_2) = \phi(s_1) \odot_1 \phi(s_2)$, $\phi(s_1 \oplus_2 s_2) = \phi(s_1) \odot_2 \phi(s_2)$ and $\phi(s_1 \oplus_3 s_2) = \phi(s_1) \odot_3 \phi(s_2)$ for all $s_1, s_2 \in S_1$. Let $W = \phi(L)$, L is a CVSBS of S_1. Let $\phi(s_1), \phi(s_2) \in S_2$, $\mathbb{T}^+_W(\phi(s_1) \odot_1 \phi(s_2)) \geq \mathbb{T}^+_{\mathbb{A}_L}(s_1 \oplus_1 s_2) \geq \min\{\mathbb{T}^+_{\mathbb{A}_L}(s_1), \mathbb{T}^+_{\mathbb{A}_L}(s_2)\} = \min\{\mathbb{T}^+_W\phi(s_1), \mathbb{T}^+_W\phi(s_2)\}$ and $\mathbb{T}^-_W(\phi(s_1) \odot_1 \phi(s_2)) \geq \mathbb{T}^-_{\mathbb{A}_L}(s_1 \oplus_1 s_2) \geq \min\{\mathbb{T}^-_{\mathbb{A}_L}(s_1), \mathbb{T}^-_{\mathbb{A}_L}(s_2)\}$ $= \min\{\mathbb{T}^-_W\phi(s_1), \mathbb{T}^-_W\phi(s_2)\}$. Thus, $\bar{\mathbb{T}}_W(\phi(s_1) \odot_1 \phi(s_2)) \geq \min\{\bar{\mathbb{T}}_W\phi(s_1), \bar{\mathbb{T}}_W\phi(s_2)\}$. Now, $1-\mathbb{F}^+_W(\phi(s_1)\odot_1\phi(s_2)) \geq 1-\mathbb{F}^+_{\mathbb{A}_L}(s_1\oplus_1 s_2) \geq \min\{1-\mathbb{F}^+_{\mathbb{A}_L}(s_1), 1-\mathbb{F}^+_{\mathbb{A}_L}(s_2)\} = \min\{1-\mathbb{F}^+_W\phi(s_1), 1-\mathbb{F}^+_W\phi(s_2)\}$ and $1-\mathbb{F}^-_W(\phi(s_1) \odot_1 \phi(s_2)) \geq 1-\mathbb{F}^-_{\mathbb{A}_L}(s_1 \oplus_1 s_2) \geq \min\{1-\mathbb{F}^-_{\mathbb{A}_L}(s_1), 1-\mathbb{F}^-_{\mathbb{A}_L}(s_2)\} = \min\{1-\mathbb{F}^-_W\phi(s_1), 1-\mathbb{F}^-_W\phi(s_2)\}$. Thus, $1-\bar{\mathbb{F}}_W(\phi(s_1)\odot_1\phi(s_2)) \geq \min\{1-\bar{\mathbb{F}}_W\phi(s_1), 1-\bar{\mathbb{F}}_W\phi(s_2)\}$. Hence, $\bar{\mathbb{A}}_W(\phi(s_1)\odot_1\phi(s_2)) \geq Min^i\{\bar{\mathbb{A}}_W\phi(s_1), \bar{\mathbb{A}}_W\phi(s_2)\}$, Similarly, $\bar{\mathbb{A}}_W(\phi(s_1)\odot_2\phi(s_2)) \geq Min^i\{\bar{\mathbb{A}}_W\phi(s_1), \bar{\mathbb{A}}_W\phi(s_2)\}$ and $\bar{\mathbb{A}}_W(\phi(s_1) \odot_3 \phi(s_2)) \geq Min^i\{\bar{\mathbb{A}}_W\phi(s_1), \bar{\mathbb{A}}_W\phi(s_2)\}$. Now, $\mathbb{T}_W(\phi(s_1) \odot_1 \phi(s_2)) \geq \mathbb{T}_{\mathbb{V}_L}(s_1 \oplus_1 s_2) \geq \min\{\mathbb{T}_{\mathbb{V}_L}(s_1), \mathbb{T}_{\mathbb{V}_L}(s_2)\} = \min\{\mathbb{T}_W\phi(s_1), \mathbb{T}_W\phi(s_2)\}$ and $1-\mathbb{F}_W(\phi(s_1) \odot_1 \phi(s_2)) \geq 1-\mathbb{F}_{\mathbb{V}_L}(s_1 \oplus_1 s_2) \geq \min\{1-\mathbb{F}_{\mathbb{V}_L}(s_1), 1-\mathbb{F}_{\mathbb{V}_L}(s_2)\} = \min\{1-\mathbb{F}_W\phi(s_1), 1-\mathbb{F}_W\phi(s_2)\}$. Thus, $\mathbb{V}_W(\phi(s_1)\odot_1\phi(s_2)) \geq \min\{\mathbb{V}_W\phi(s_1), \mathbb{V}_W\phi(s_2)\}$. Similarly, $\mathbb{V}_W(\phi(s_1)\odot_2\phi(s_2)) \geq \min\{\mathbb{V}_W\phi(s_1), \mathbb{V}_W\phi(s_2)\}$ and $\mathbb{V}_W(\phi(s_1)\odot_3\phi(s_2)) \geq \min\{\mathbb{V}_W\phi(s_1), \mathbb{V}_W\phi(s_2)\}$. Hence W is a CVSBS of S_2.

Theorem 4.2. *Let $(S_1, \oplus_1, \oplus_2, \oplus_3)$ and $(S_2, \odot_1, \odot_2, \odot_3)$ be any two bisemirings. The homomorphic preimage of CVSBS of S_2 is also CVSBS of S_1.*

Proof. Let $\phi : S_1 \to S_2$ be a homomorphism. Then $\phi(s_1 \oplus_1 s_2) = \phi(s_1) \odot_1 \phi(s_2)$, $\phi(s_1 \oplus_2 s_2) = \phi(s_1) \odot_2 \phi(s_2)$ and $\phi(s_1 \oplus_3 s_2) = \phi(s_1) \odot_3 \phi(s_2)$ $\forall$ $s_1, s_2 \in S_1$. Let $W = \phi(L)$, where W is an CVSBS of S_2. Now, $\mathbb{T}^-_{\mathbb{A}_L}(s_1 \oplus_1 s_2) = \mathbb{T}^-_W(\phi(s_1) \odot_1 \phi(s_2)) \geq \min\{\mathbb{T}^-_W(\phi(s_1)), \mathbb{T}^-_W(\phi(s_2))\} = \min\{\mathbb{T}^-_{\mathbb{A}_L}(s_1), \mathbb{T}^-_{\mathbb{A}_L}(s_2)\}$ and $\mathbb{T}^+_{\mathbb{A}_L}(s_1 \oplus_1 s_2) = \mathbb{T}^+_W(\phi(s_1)\odot_1\phi(s_2)) \geq \min\{\mathbb{T}^+_W(\phi(s_1)), \mathbb{T}^+_W(\phi(s_2))\} = \min\{\mathbb{T}^+_{\mathbb{A}_L}(s_1), \mathbb{T}^+_{\mathbb{A}_L}(s_2)\}$. Thus, $\bar{\mathbb{T}}_{\mathbb{A}_L}(s_1 \oplus_1 s_2) \geq \min\{\bar{\mathbb{T}}_{\mathbb{A}_L}(s_1), \bar{\mathbb{T}}_{\mathbb{A}_L}(s_2)\}$. Now, $1-\mathbb{F}^-_{\mathbb{A}_L}(s_1 \oplus_1 s_2) = 1-\mathbb{F}^-_W(\phi(s_1) \odot_1 \phi(s_2)) \geq \min\{1-(\mathbb{F}^-_W\phi(s_1)), 1-(\mathbb{F}^-_W\phi(s_2))\} = \min\{1-\mathbb{F}^-_{\mathbb{A}_L}(s_1), 1-\mathbb{F}^-_{\mathbb{A}_L}(s_2)\}$ and $1-\mathbb{F}^+_{\mathbb{A}_L}(s_1 \oplus_1 s_2) = 1-\mathbb{F}^+_W(\phi(s_1) \odot_1 \phi(s_2)) \geq \min\{1-(\mathbb{F}^+_W\phi(s_1)), 1-(\mathbb{F}^+_W\phi(s_2))\} = \min\{1-\mathbb{F}^+_{\mathbb{A}_L}(s_1), 1-\mathbb{F}^+_{\mathbb{A}_L}(s_2)\}$. Thus, $1-\bar{\mathbb{F}}_{\mathbb{A}_L}(s_1 \oplus_1 s_2) \geq \min\{1-$

$\bar{\mathbb{F}}_{\mathbb{A}_L}(s_1), 1-\bar{\mathbb{F}}_{\mathbb{A}_L}(s_2)\}$. Hence, $\bar{\mathbb{A}}_L(s_1 \oplus_1 s_2) \geq Min^i\{\bar{\mathbb{A}}_L(s_1), \bar{\mathbb{A}}_L(s_2)\}$. Similarly, $\bar{\mathbb{A}}_L(s_1 \oplus_2 s_2) \geq Min^i\{\bar{\mathbb{A}}_L(s_1), \bar{\mathbb{A}}_L(s_2)\}$ and $\bar{\mathbb{A}}_L(s_1 \oplus_3 s_2) \geq Min^i\{\bar{\mathbb{A}}_L(s_1), \bar{\mathbb{A}}_L(s_2)\}$. Now, $\mathbb{T}_{\mathbb{V}_L}(s_1 \oplus_1 s_2) = \mathbb{T}_W(\phi(s_1) \odot_1 \phi(s_2)) \geq \min\{\mathbb{T}_W(\phi(s_1)), \mathbb{T}_W(\phi(s_2))\} = \min\{\mathbb{T}_{\mathbb{V}_L}(s_1), \mathbb{T}_{\mathbb{V}_L}(s_2)\}$. And $1-\mathbb{F}_{\mathbb{V}_L}(s_1 \oplus_1 s_2) = 1-\mathbb{F}_W(\phi(s_1) \odot_1 \phi(s_2)) \geq \min\{1-(\mathbb{F}_W\phi(s_1)), 1-(\mathbb{F}_W\phi(s_2))\} = \min\{1-\mathbb{F}_{\mathbb{V}_L}(s_1), 1-\mathbb{F}_{\mathbb{V}_L}(s_2)\}$. Hence, $\mathbb{V}_L(s_1 \oplus_1 s_2) \geq \min\{\mathbb{V}_L(s_1), \mathbb{V}_L(s_2)\}$. Similarly, $\mathbb{V}_L(s_1 \oplus_2 s_2) \geq \min\{\mathbb{V}_L(s_1), \mathbb{V}_L(s_2)\}$ and $\mathbb{V}_L(s_1 \oplus_3 s_2) \geq \min\{\mathbb{V}_L(s_1), \mathbb{V}_L(s_2)\}$. Hence L is a CVSBS of S_1.

Theorem 4.3. *Let $(S_1, \oplus_1, \oplus_2, \oplus_3)$ and $(S_2, \odot_1, \odot_2, \odot_3)$ be any two bisemirings. If $\phi : S_1 \to S_2$ is a homomorphism, then $L_{(\alpha,\beta)}$ is a level SBS of an CVSBS L of S_1.*

Proof. Let $\phi : S_1 \to S_2$ be a homomorphism. Then $\phi(s_1 \oplus_1 s_2) = \phi(s_1) \odot_1 \phi(s_2), \phi(s_1 \oplus_2 s_2) = \phi(s_1) \odot_2 \phi(s_2)$ and $\phi(s_1 \oplus_3 s_2) = \phi(s_1) \odot_3 \phi(s_2)$ for all $s_1, s_2 \in S_1$. Let $W = \phi(L)$, W is an CVSBS of S_2. By Theorem 4.2, L is a CVSBS of S_1. Let $\phi(L_{(\alpha,\beta)})$ be a level SBS of W. Suppose $\phi(s_1), \phi(s_2) \in \phi(L_{(\alpha,\beta)})$. Then $\phi(s_1 \oplus_1 s_2), \phi(s_1 \oplus_2 s_2)$ and $\phi(s_1 \oplus_3 s_2) \in \phi(L_{(\alpha,\beta)})$. Now, $\bar{\mathbb{T}}_{\mathbb{A}_L}(s_1) = \bar{\mathbb{T}}_{\mathbb{A}_W}(\phi(s_1)) \geq \bar{\alpha}, \bar{\mathbb{T}}_{\mathbb{A}_L}(s_2) = \bar{\mathbb{T}}_{\mathbb{A}_W}(\phi(s_2)) \geq \bar{\alpha}$. Then $\bar{\mathbb{T}}_{\mathbb{A}_L}(s_1 \oplus_1 s_2) \geq \bar{\alpha}$ and $1-\bar{\mathbb{F}}_{\mathbb{A}_L}(s_1) = 1-\bar{\mathbb{F}}_{\mathbb{A}_W}(\phi(s_1)) \geq \bar{\beta}, 1-\bar{\mathbb{F}}_{\mathbb{A}_L}(s_2) = 1-\bar{\mathbb{F}}_{\mathbb{A}_W}(\phi(s_2)) \geq \bar{\beta}$. Then $1-\bar{\mathbb{F}}_{\mathbb{A}_L}(s_1 \oplus_1 s_2) \geq \bar{\beta}$. Thus, $\bar{\mathbb{A}}_L(s_1 \oplus_1 s_2) \geq [\bar{\alpha}, \bar{\beta}]$. Similarly, $\bar{\mathbb{A}}_L(s_1 \oplus_2 s_2) \geq [\bar{\alpha}, \bar{\beta}]$ and $\bar{\mathbb{A}}_L(s_1 \oplus_3 s_2) \geq [\bar{\alpha}, \bar{\beta}]$. Now, $\mathbb{T}_{\mathbb{V}_L}(s_1) = \mathbb{T}_{\mathbb{V}_W}(\phi(s_1)) \geq \alpha, \mathbb{T}_{\mathbb{V}_L}(s_2) = \mathbb{T}_{\mathbb{V}_W}(\phi(s_2)) \geq \alpha$. Then $\mathbb{T}_{\mathbb{V}_L}(s_1 \oplus_1 s_2) \geq \alpha$ and $1-\mathbb{F}_{\mathbb{V}_L}(s_1) = 1-\mathbb{F}_{\mathbb{V}_W}(\phi(s_1)) \geq \beta, 1-\mathbb{F}_{\mathbb{V}_L}(s_2) = 1-\mathbb{F}_{\mathbb{V}_W}(\phi(s_2)) \geq \beta$. Then $1-\mathbb{F}_{\mathbb{V}_L}(s_1 \oplus_1 s_2) \geq \beta$. Thus, $\mathbb{V}_L(s_1 \oplus_1 s_2) \geq [\alpha, \beta]$. Similarly, $\mathbb{V}_L(s_1 \oplus_2 s_2) \geq [\alpha, \beta]$ and $\mathbb{V}_L(s_1 \oplus_3 s_2) \geq [\alpha, \beta]$. Hence $L_{(\alpha,\beta)}$ is a level SBS of an CVSBS L of S_1.

Theorem 4.4. *Let $(S_1, \oplus_1, \oplus_2, \oplus_3)$ and $(S_2, \odot_1, \odot_2, \odot_3)$ be any two bisemirings. If $\phi : S_1 \to S_2$ is a homomorphism, then $\phi(L_{(\alpha,\beta)})$ is a level SBS of an CVSBS W of S_2.*

Proof. Let $\phi : S_1 \to S_2$ be a homomorphism. Then $\phi(s_1 \oplus_1 s_2) = \phi(s_1) \odot_1 \phi(s_2), \phi(s_1 \oplus_2 s_2) = \phi(s_1) \odot_2 \phi(s_2)$ and $\phi(s_1 \oplus_3 s_2) = \phi(s_1) \odot_3 \phi(s_2)$ for all $s_1, s_2 \in S_1$. Let $W = \phi(L)$, L is an CVSBS of S_1. By Theorem 4.1, W is a CVSBS of S_2. Let $L_{(\alpha,\beta)}$ be a level SBS of L. Suppose $s_1, s_2 \in L_{(\alpha,\beta)}$. Then $\phi(s_1 \oplus_1 s_2), \phi(s_1 \oplus_2 s_2)$ and $\phi(s_1 \oplus_3 s_2) \in L_{(\alpha,\beta)}$. Now, $\bar{\mathbb{T}}_{\mathbb{A}_W}(\phi(s_1)) \geq \bar{\mathbb{T}}_{\mathbb{A}_L}(s_1) \geq \bar{\alpha}, \bar{\mathbb{T}}_{\mathbb{A}_W}(\phi(s_2)) \geq \bar{\mathbb{T}}_{\mathbb{A}_L}(s_2) \geq \bar{\alpha}$. Then $\bar{\mathbb{T}}_{\mathbb{A}_W}(\phi(s_1) \odot_1 \phi(s_2)) \geq \bar{\mathbb{T}}_{\mathbb{A}_L}(s_1 \oplus_1 s_2) \geq \bar{\alpha}$, for all $\phi(s_1), \phi(s_2) \in S_2$. And $1-\bar{\mathbb{F}}_{\mathbb{A}_W}(\phi(s_1)) \geq 1-\bar{\mathbb{F}}_{\mathbb{A}_L}(s_1) \geq \bar{\beta}, 1-\bar{\mathbb{F}}_{\mathbb{A}_W}(\phi(s_2)) \geq 1-\bar{\mathbb{F}}_{\mathbb{A}_L}(s_2) \geq \bar{\beta}$. Then $1-\bar{\mathbb{F}}_{\mathbb{A}_W}(\phi(s_1) \odot_1 \phi(s_2)) \geq 1-\bar{\mathbb{F}}_{\mathbb{A}_L}(s_1 \oplus_1 s_2) \geq \bar{\beta}$, for all $\phi(s_1), \phi(s_2) \in S_2$. Thus, $\bar{\mathbb{A}}_W(\phi(s_1) \odot_1 \phi(s_2)) \geq [\bar{\alpha}, \bar{\beta}]$. Similarly, $\bar{\mathbb{A}}_W(\phi(s_1) \odot_2 \phi(s_2)) \geq [\bar{\alpha}, \bar{\beta}]$ and $\bar{\mathbb{A}}_W(\phi(s_1) \odot_3 \phi(s_2)) \geq [\bar{\alpha}, \bar{\beta}]$. Now, $\mathbb{T}_{\mathbb{V}_W}(\phi(s_1)) \geq \mathbb{T}_{\mathbb{V}_L}(s_1) \geq \alpha, \mathbb{T}_{\mathbb{V}_W}(\phi(s_2)) \geq \mathbb{T}_{\mathbb{V}_L}(s_2) \geq \alpha$. Then $\mathbb{T}_{\mathbb{V}_W}(\phi(s_1) \odot_1 \phi(s_2)) \geq \mathbb{T}_{\mathbb{V}_L}(s_1 \oplus_1 s_2) \geq \alpha$, for all $\phi(s_1), \phi(s_2) \in S_2$. And $1-\mathbb{F}_{\mathbb{V}_W}(\phi(s_1)) \geq 1-\mathbb{F}_{\mathbb{V}_L}(s_1) \geq \beta, 1-\mathbb{F}_{\mathbb{V}_W}(\phi(s_2)) \geq 1-\mathbb{F}_{\mathbb{V}_L}(s_2) \geq \beta$. Then $1-\mathbb{F}_{\mathbb{V}_W}(\phi(s_1) \odot_1 \phi(s_2)) \geq 1-\mathbb{F}_{\mathbb{V}_L}(s_1 \oplus_1 s_2) \geq \beta$, for all $\phi(s_1), \phi(s_2) \in S_2$. Thus, $\mathbb{V}_W(\phi(s_1) \odot_1 \phi(s_2)) \geq [\alpha, \beta]$. Similarly, $\mathbb{V}_W(\phi(s_1) \odot_2 \phi(s_2)) \geq [\alpha, \beta]$ and $\mathbb{V}_W(\phi(s_1) \odot_3 \phi(s_2)) \geq [\alpha, \beta]$. Hence $\phi(L_{(\alpha,\beta)})$ is a level SBS of an CVSBS W of S_2.

5. CONCLUSION

The main goal of this work is to present a vague subsemirings of semirings to CVSBS of bisemirings. We find out the cubic vague fuzzy set $\mathcal{L}$ is a CVSBS if and only if $\mathcal{W}$ is a CVSBS of $S \times S$. Also defined homomorphic image and preimage of bisemiring. We determined that homomorphic image and preimage of CVSBS is a CVSBS. So in future, we should consider the ordered CVSBS, cubic soft subbisemirings and cubic vague soft subbisemirings of bisemirings and its applications.

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