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# NEW APPROACH TOWARDS $\mathscr{A}$-IDEALS IN TERNARY SEMIRINGS

M. PALANIKUMAR* AND K. ARULMOZHI

ABSTRACT. To interact somebody with someone various almost ideals (shortly $\mathscr{A}$-ideals), quasi $\mathscr{A}$-ideals, bi quasi $\mathscr{A}$-ideals, tri $\mathscr{A}$-ideals and tri quasi $\mathscr{A}$-ideals in ternary semiring and give some characterizations. We develop the implications ideal $\implies$ quasi ideal $\implies$ two sided bi quasi ideal $\implies$ two sided tri quasi ideal $\implies$ two sided tri quasi $\mathscr{A}$-ideal $\implies$ two sided bi quasi $\mathscr{A}$-ideal $\implies$ bi $\mathscr{A}$-ideal $\implies$ quasi $\mathscr{A}$-ideal $\implies$ $\mathscr{A}$-ideal and reverse implications do not holds with examples. We show that the union of $\mathscr{A}$-ideals (bi $\mathscr{A}$-ideals, quasi $\mathscr{A}$-ideals, bi quasi $\mathscr{A}$-ideals) is a $\mathscr{A}$-ideal (bi $\mathscr{A}$-ideal, quasi $\mathscr{A}$-ideal, bi quasi $\mathscr{A}$-ideal) in ternary semiring.

## 1. INTRODUCTION

The notion of semiring was introduced by Vandiver in 1934 as a generalization of rings [22]. In 1962, Hestenes [8] studied the notion of ternary algebra with application to matrices and linear transformation. In 1971, Lister characterized those additive subgroups of rings which are closed under the triple ring product and he called this algebraic system a ternary ring [12]. The results in ordinary semirings may be extended to $n$-ary semirings for arbitrary $n$ but the transition from $n = 3$ to arbitrary n entails a great degree of complexity that makes it undesirable for exposition. The ring of integers $\mathbb{Z}$ which plays a role in the ring theory. The subset $\mathbb{Z}^+$ of $\mathbb{Z}$ is an additive semigroup which is closed under the ring product, that is $\mathbb{Z}^+$ is a semiring. Now, if we consider the subset $\mathbb{Z}^-$ of $\mathbb{Z}$, then we see that $\mathbb{Z}^-$ is an additive semigroup which is closed under the triple ring product, that is $\mathbb{Z}^-$ forms a ternary semiring. The notion of quasi ideal was introduced by Otto Steinfeld both in semigroups and rings [21]. Shabir et al [20] characterized the semirings by the properties of quasi-ideals. Quasi-ideals of different classes of semirings have been characterized by many authors in [6, 9]. The notion of bi-ideals in semigroups introduced by Lajos [10]. Bi-ideal is a generalization of left ideal and right ideal in semiring. Many mathematicians proved important results and characterizations of algebraic structures by using various ideals [1, 5, 15, 17, 18]. Rao introduced bi-quasi-ideals of semigroups. The notion of tri-ideal is a generalization of quasi ideal, bi-ideal, ideal and properties of tri ideals of a semiring [19]. Grosek and Satko introduced the notion of $\mathscr{A}$-ideal of semigroup [7]. In

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*Corresponding author.

this paper, we give some properties of various $\mathscr{A}$-ideals and tri $\mathscr{A}$-ideals in ternary semiring. Our aim in this paper is threefold.
(1) To study the relationship between quasi $\mathscr{A}$-ideal and bi quasi $\mathscr{A}$-ideal.
(2) To characterize tri $\mathscr{A}$-ideal.
(3) To characterize two sided bi quasi ideal and two sided tri quasi ideal.

## 2. PRELIMINARIES

In this section we review some definitions and results which will be used in later sections.

**Definition 2.1.** [5] A non empty set $\mathcal{T}$ together with a binary operation called addition and ternary multiplication, denoted by juxtaposition([]) is said to be a ternary semiring if $\mathcal{T}$ is an additive commutative semigroup satisfying

(i) $[[abc]de] = [a[bcd]e] = [ab[cde]]$,
(ii) $[(a+b)cd] = [acd] + [bcd]$,
(iii) $[a(b+c)d] = [abd] + [acd]$,
(iv) $[ab(c+d)] = [abc] + [abd], \forall a, b, c, d, e \in \mathcal{T}$.

**Definition 2.2.** [5] An additive subsemigroup $A$ of $\mathcal{T}$ is called a ternary subsemiring if $[a_1a_2a_3] \in A$ for all $a_1, a_2, a_3 \in A$ .

**Definition 2.3.** [5, 17] A non-empty subsets $I, B$ and $Q$ of $\mathcal{T}$ is called a

(i) right ideal of $\mathcal{T}$ if $I$ is a ternary subsemiring (shortly TSS) of $\mathcal{T}$ and $I\mathcal{T}\mathcal{T} \subseteq I$.
(ii) lateral ideal of $\mathcal{T}$ if $I$ is a TSS of $\mathcal{T}$ and $\mathcal{T}I\mathcal{T} \subseteq I$.
(iii) left ideal of $\mathcal{T}$ if $I$ is a TSS of $\mathcal{T}$ and $\mathcal{T}\mathcal{T}I \subseteq I$.
(iv) two sided ideal of $\mathcal{T}$ if $I$ is a right ideal and left ideal of $\mathcal{T}$.
(v) ideal of $\mathcal{T}$ if $I$ is a right ideal, lateral ideal and left ideal of $\mathcal{T}$.
(vi) A TSS $B$ of $\mathcal{T}$ is called a bi ideal if $B\mathcal{T}B\mathcal{T}B \subseteq B$.
(vii) A TSS $Q$ of $\mathcal{T}$ is called a quasi ideal if $[Q\mathcal{T}\mathcal{T} \cap (\mathcal{T}Q\mathcal{T} \cup \mathcal{T}\mathcal{T}Q\mathcal{T}\mathcal{T}) \cap \mathcal{T}\mathcal{T}Q] \subseteq Q$.

**Definition 2.4.** [18] A non-empty subsets $I, B$ and $Q$ of a semiring $\mathcal{S}$ is called a

(i) right $\mathscr{A}$-ideal of $\mathcal{S}$ if $I$ is a subsemiring of $\mathcal{S}$ and $I\mathcal{S} \cap I \neq \phi$.
(ii) left $\mathscr{A}$-ideal of $\mathcal{S}$ if $I$ is a subsemiring of $\mathcal{S}$ and $\mathcal{S}I \cap I \neq \phi$.
(iii) $\mathscr{A}$-ideal of $\mathcal{S}$ if $I$ is a right $\mathscr{A}$-ideal and left $\mathscr{A}$-ideal of $\mathcal{S}$.
(iv) A subsemiring $B$ of $\mathcal{S}$ is called a bi $\mathscr{A}$-ideal if $B\mathcal{S}B \cap B \neq \phi$.
(v) A subsemiring $Q$ of $\mathcal{S}$ is called a quasi $\mathscr{A}$-ideal if $[Q\mathcal{S} \cap \mathcal{S}Q] \cap Q \neq \phi$.

**Definition 2.5.** [18] A non-empty subset $Q$ of $\mathcal{S}$ is called a

(i) right bi quasi $\mathscr{A}$- ideal of $\mathcal{S}$ if $Q$ is a subsemiring of $\mathcal{S}$ and $[Q\mathcal{S} \cap Q\mathcal{S}Q] \cap Q \neq \phi$.
(ii) left bi quasi $\mathscr{A}$- ideal of $\mathcal{S}$ if $Q$ is a subsemiring of $\mathcal{S}$ and $[\mathcal{S}Q \cap Q\mathcal{S}Q] \cap Q \neq \phi$.
(iii) bi quasi $\mathscr{A}$- ideal of $\mathcal{S}$ if $Q$ is a right bi quasi $\mathscr{A}$- ideal and left bi quasi $\mathscr{A}$- ideal of $\mathcal{S}$.

**Definition 2.6.** [18] A non-empty subsets $I$ and $Q$ of $\mathcal{S}$ is called a

(i) right tri $\mathscr{A}$- ideal of $\mathcal{S}$ if $I$ is a subsemiring of $\mathcal{S}$ and $II\mathcal{S}I \cap I \neq \phi$.
(ii) left tri $\mathscr{A}$- ideal of $\mathcal{S}$ if $I$ is a subsemiring of $\mathcal{S}$ and $I\mathcal{S}II \cap I \neq \phi$.
(iii) tri $\mathscr{A}$- ideal of $\mathcal{S}$ if $I$ is a right tri $\mathscr{A}$- ideal and left tri $\mathscr{A}$- ideal of $\mathcal{S}$.
(iv) right tri quasi $\mathscr{A}$- ideal of $\mathcal{S}$ if $Q$ is a subsemiring of $\mathcal{S}$ and $[Q\mathcal{S} \cap QQ\mathcal{S}Q] \cap Q \neq \phi$.
(v) left tri quasi $\mathscr{A}$- ideal of $\mathcal{S}$ if $Q$ is a subsemiring of $\mathcal{S}$ and $[\mathcal{S}Q \cap Q\mathcal{S}QQ] \cap Q \neq \phi$.
(vi) tri quasi $\mathscr{A}$- ideal of $\mathcal{S}$ if $Q$ is a right tri quasi $\mathscr{A}$- ideal and left tri quasi $\mathscr{A}$- ideal of $\mathcal{S}$.

**Definition 2.7.** [17] A non-empty subset $Q$ of $\mathcal{T}$ is called a

(i) right bi quasi ideal of $\mathcal{T}$ if $Q$ is a TSS of $\mathcal{T}$ and $Q\mathcal{T}\mathcal{T} \cap Q\mathcal{T}Q\mathcal{T}Q \subseteq Q$.
(ii) lateral bi quasi ideal of $\mathcal{T}$ if $Q$ is a TSS of $\mathcal{T}$ and $\mathcal{T}Q\mathcal{T} \cap Q\mathcal{T}Q\mathcal{T}Q \subseteq Q$.
(iii) left bi quasi ideal of $\mathcal{T}$ if $Q$ is a TSS of $\mathcal{T}$ and $\mathcal{T}\mathcal{T}Q \cap Q\mathcal{T}Q\mathcal{T}Q \subseteq Q$.
(iv) two sided bi quasi ideal of $\mathcal{T}$ if $Q$ is a right bi quasi ideal and left bi quasi ideal of $\mathcal{T}$.
(v) bi quasi ideal of $\mathcal{T}$ if $Q$ is a right bi quasi ideal, lateral bi quasi ideal and left bi quasi ideal of $\mathcal{T}$.

**Definition 2.8.** [17] A non-empty subset $I$ of $\mathcal{T}$ is called a

(i) right tri-ideal of $\mathcal{T}$ if $I$ is a TSS of $\mathcal{T}$ and $I^3\mathcal{T}\mathcal{T}I^2 \subseteq I$.
(ii) lateral tri-ideal of $\mathcal{T}$ if $I$ is a TSS of $\mathcal{T}$ and $I^2\mathcal{T}I\mathcal{T}I^2 \subseteq I$.
(iii) left tri-ideal of $\mathcal{T}$ if $I$ is a TSS of $\mathcal{T}$ and $I^2\mathcal{T}\mathcal{T}I^3 \subseteq I$.
(iv) two sided tri-ideal of $\mathcal{T}$ if $I$ is a right tri-ideal and left tri-ideal of $\mathcal{T}$.
(v) tri-ideal of $\mathcal{T}$ if $I$ is a right tri-ideal, lateral tri-ideal and left tri-ideal of $\mathcal{T}$.

**Definition 2.9.** [17] A non-empty subset $Q$ of $\mathcal{T}$ is called a

(i) right tri quasi- ideal of $\mathcal{T}$ if $Q$ is a TSS of $\mathcal{T}$ and $Q\mathcal{T}\mathcal{T} \cap Q^3\mathcal{T}\mathcal{T}Q^2 \subseteq Q$.
(ii) lateral tri quasi- ideal of $\mathcal{T}$ if $Q$ is a TSS of $\mathcal{T}$ and $\mathcal{T}Q\mathcal{T} \cap Q^2\mathcal{T}Q\mathcal{T}Q^2 \subseteq Q$.
(iii) left tri quasi- ideal of $\mathcal{T}$ if $Q$ is a TSS of $\mathcal{T}$ and $\mathcal{T}\mathcal{T}Q \cap Q^2\mathcal{T}\mathcal{T}Q^3 \subseteq Q$.
(iv) tri quasi- ideal of $\mathcal{T}$ if $Q$ is a right tri quasi-ideal, lateral tri quasi-ideal and left tri quasi- ideal of $\mathcal{T}$.

## 3. VARIOUS $\mathscr{A}$-IDEALS

Here $\mathcal{T}$ stands for a ternary semiring unless otherwise stated. Here we introduce different types of $\mathscr{A}$-ideals in ternary semiring.

**Definition 3.1.** A non-empty subsets $I, B$ and $Q$ of $\mathcal{T}$ is called a

(i) right $\mathscr{A}$-ideal of $\mathcal{T}$ if $I$ is a TSS of $\mathcal{T}$ and $I\mathcal{T}\mathcal{T} \cap I \neq \phi$.
(ii) lateral $\mathscr{A}$-ideal of $\mathcal{T}$ if $I$ is a TSS of $\mathcal{T}$ and $\mathcal{T}I\mathcal{T} \cap I \neq \phi$.
(iii) left $\mathscr{A}$-ideal of $\mathcal{T}$ if $I$ is a TSS of $\mathcal{T}$ and $\mathcal{T}\mathcal{T}I \cap I \neq \phi$.
(iv) two sided $\mathscr{A}$-ideal of $\mathcal{T}$ if $I$ is a right $\mathscr{A}$-ideal and left $\mathscr{A}$-ideal of $\mathcal{T}$.
(v) $\mathscr{A}$-ideal of $\mathcal{T}$ if $I$ is a right $\mathscr{A}$-ideal, lateral $\mathscr{A}$-ideal and left $\mathscr{A}$-ideal of $\mathcal{T}$.
(vi) bi $\mathscr{A}$-ideal if $B$ is a TSS and $B\mathcal{T}B\mathcal{T}B \cap B \neq \phi$.
(vii) quasi $\mathscr{A}$-ideal if $Q$ is a TSS and $[Q\mathcal{T}\mathcal{T} \cap (\mathcal{T}Q\mathcal{T} \cup \mathcal{T}\mathcal{T}Q\mathcal{T}\mathcal{T}) \cap \mathcal{T}\mathcal{T}Q] \cap Q \neq \phi$.

**Definition 3.2.** A non-empty subset $Q$ of $\mathcal{T}$ is called a

(i) right bi quasi $\mathscr{A}$-ideal of $\mathcal{T}$ if $Q$ is a TSS of $\mathcal{T}$ and $[Q\mathcal{T}\mathcal{T} \cap Q\mathcal{T}Q\mathcal{T}Q] \cap Q \neq \phi$.
(ii) lateral bi quasi $\mathscr{A}$-ideal of $\mathcal{T}$ if $Q$ is a TSS of $\mathcal{T}$ and $[\mathcal{T}Q\mathcal{T} \cap Q\mathcal{T}Q\mathcal{T}Q] \cap Q \neq \phi$.
(iii) left bi quasi $\mathscr{A}$-ideal of $\mathcal{T}$ if $Q$ is a TSS of $\mathcal{T}$ and $[\mathcal{T}\mathcal{T}Q \cap Q\mathcal{T}Q\mathcal{T}Q] \cap Q \neq \phi$.
(iv) two sided bi quasi $\mathscr{A}$-ideal of $\mathcal{T}$ if $Q$ is a right bi quasi $\mathscr{A}$-ideal and left bi quasi $\mathscr{A}$-ideal of $\mathcal{T}$.
(v) bi quasi $\mathscr{A}$-ideal of $\mathcal{T}$ if $Q$ is a right bi quasi $\mathscr{A}$-ideal, lateral bi quasi $\mathscr{A}$-ideal and left bi quasi $\mathscr{A}$-ideal of $\mathcal{T}$.

**Theorem 3.1.** *Every ideal is a $\mathscr{A}$-ideal.*

**Proof.** Suppose that $I$ is an ideal of $\mathcal{T}$, then $I\mathcal{T}\mathcal{T} \subseteq I$, $\mathcal{T}I\mathcal{T} \subseteq I$ and $\mathcal{T}\mathcal{T}I \subseteq I$. Now, $I\mathcal{T}\mathcal{T} \subseteq I \implies I\mathcal{T}\mathcal{T} \cap I \subseteq I \cap I \neq \phi$, $\mathcal{T}I\mathcal{T} \subseteq I \implies \mathcal{T}I\mathcal{T} \cap I \subseteq I \cap I \neq \phi$ and $\mathcal{T}\mathcal{T}I \subseteq I \implies \mathcal{T}\mathcal{T}I \cap I \subseteq I \cap I \neq \phi$. Hence $I$ is a $\mathscr{A}$-ideal of $\mathcal{T}$.

Converse of the Theorem 3.1 may not be true by the following counter Example.

**Example 3.3.** Consider the ternary semiring $\mathcal{T} = \{a_1, a_2, a_3, a_4, a_5, a_6\}$ with the following compositions:

| $+$ | $a_1$ | $a_2$ | $a_3$ | $a_4$ | $a_5$ | $a_6$ |
|---|---|---|---|---|---|---|
| $a_1$ | $a_1$ | $a_2$ | $a_3$ | $a_4$ | $a_5$ | $a_6$ |
| $a_2$ | $a_2$ | $a_2$ | $a_3$ | $a_4$ | $a_5$ | $a_6$ |
| $a_3$ | $a_3$ | $a_3$ | $a_3$ | $a_6$ | $a_5$ | $a_6$ |
| $a_4$ | $a_4$ | $a_4$ | $a_6$ | $a_4$ | $a_5$ | $a_6$ |
| $a_5$ | $a_5$ | $a_5$ | $a_5$ | $a_5$ | $a_5$ | $a_5$ |
| $a_6$ | $a_6$ | $a_6$ | $a_6$ | $a_6$ | $a_5$ | $a_6$ |

| $\cdot$ | $a_1$ | $a_2$ | $a_3$ | $a_4$ | $a_5$ | $a_6$ |
|---|---|---|---|---|---|---|
| $a_1$ | $a$ | $a$ | $a$ | $a$ | $a$ | $a$ |
| $a_2$ | $a$ | $b$ | $c$ | $b$ | $c$ | $c$ |
| $a_3$ | $a$ | $b$ | $c$ | $b$ | $c$ | $c$ |
| $a_4$ | $a$ | $d$ | $e$ | $d$ | $e$ | $e$ |
| $a_5$ | $a$ | $d$ | $e$ | $d$ | $e$ | $e$ |
| $a_6$ | $a$ | $d$ | $e$ | $d$ | $e$ | $e$ |

| $\cdot$ | $a_1$ | $a_2$ | $a_3$ | $a_4$ | $a_5$ | $a_6$ |
|---|---|---|---|---|---|---|
| $a$ | $a_1$ | $a_1$ | $a_1$ | $a_1$ | $a_1$ | $a_1$ |
| $b$ | $a_1$ | $a_2$ | $a_3$ | $a_2$ | $a_3$ | $a_3$ |
| $c$ | $a_1$ | $a_2$ | $a_3$ | $a_2$ | $a_3$ | $a_3$ |
| $d$ | $a_1$ | $a_4$ | $a_5$ | $a_4$ | $a_5$ | $a_5$ |
| $e$ | $a_1$ | $a_4$ | $a_5$ | $a_4$ | $a_5$ | $a_5$ |
| $f$ | $a_1$ | $a_4$ | $a_5$ | $a_4$ | $a_5$ | $a_5$ |

Clearly $I = \{a_1, a_2\}$ is a $\mathscr{A}$-ideal of $\mathcal{T}$ but $I$ is not an ideal of $\mathcal{T}$ by $I\mathcal{T}\mathcal{T} = \{a_1, a_2, a_3\} \nsubseteq I$, $\mathcal{T}I\mathcal{T} = \{a_1, a_2, a_3, a_4, a_5\} \nsubseteq I$ and $\mathcal{T}\mathcal{T}I = \{a_1, a_2, a_4\} \nsubseteq I$.

**Theorem 3.2.** *Every bi ideal is a bi $\mathscr{A}$-ideal.*

**Proof.** Suppose that $B$ is a bi ideal of $\mathcal{T}$, then $B$ is a TSS of $\mathcal{T}$ and $B\mathcal{T}B\mathcal{T}B \subseteq B$. Now, $B\mathcal{T}B\mathcal{T}B \cap B \subseteq B \implies B\mathcal{T}B\mathcal{T}B \cap B \subseteq B \cap B \neq \phi$. Hence $B$ is a bi $\mathscr{A}$-ideal of $\mathcal{T}$.

Converse of the Theorem 3.2 may not be true by the Example.

**Example 3.4.** By the Example 3.3, Clearly, $B = \{a_2, a_3\}$ is a bi $\mathscr{A}$-ideal of $\mathcal{T}$ but $B$ is not a bi ideal of $\mathcal{T}$ by $B\mathcal{T}B\mathcal{T}B = \{a_1, a_2, a_3\} \nsubseteq B$.

**Theorem 3.3.** *Every quasi ideal is a quasi $\mathscr{A}$-ideal.*

**Proof.** Straightforward.

Converse of the Theorem 3.3 is not true by the following Example.

**Example 3.5.** By the Example 3.3, Clearly, $Q = \{a_4, a_5\}$ is a quasi $\mathscr{A}$-ideal of $\mathcal{T}$ but $Q$ is not a quasi ideal of $\mathcal{T}$ by $Q\mathcal{T}\mathcal{T} \cap (\mathcal{T}Q\mathcal{T} \cup \mathcal{T}\mathcal{T}Q\mathcal{T}\mathcal{T}) \cap \mathcal{T}\mathcal{T}Q = \{a_1, a_4, a_5\} \nsubseteq Q$.

**Theorem 3.4.** *Every quasi ideal is a bi quasi ideal.*

**Proof.** Suppose that $Q$ is a quasi ideal of $\mathcal{T}$, then $Q\mathcal{T}\mathcal{T} \cap (\mathcal{T}Q\mathcal{T} \cup \mathcal{T}\mathcal{T}Q\mathcal{T}\mathcal{T}) \cap \mathcal{T}\mathcal{T}Q \subseteq Q$. Now, $Q\mathcal{T}\mathcal{T} \cap Q\mathcal{T}Q\mathcal{T}Q \subseteq Q\mathcal{T}\mathcal{T}$, $Q\mathcal{T}\mathcal{T} \cap Q\mathcal{T}Q\mathcal{T}Q \subseteq Q\mathcal{T}Q\mathcal{T}Q \subseteq \mathcal{T}\mathcal{T}Q\mathcal{T}\mathcal{T} \subseteq \mathcal{T}Q\mathcal{T} \cup \mathcal{T}\mathcal{T}Q\mathcal{T}\mathcal{T}$ and $Q\mathcal{T}\mathcal{T} \cap Q\mathcal{T}Q\mathcal{T}Q \subseteq Q\mathcal{T}Q\mathcal{T}Q \subseteq \mathcal{T}\mathcal{T}Q$ implies $Q\mathcal{T}\mathcal{T} \cap Q\mathcal{T}Q\mathcal{T}Q \subseteq [Q\mathcal{T}\mathcal{T} \cap (\mathcal{T}Q\mathcal{T} \cup \mathcal{T}\mathcal{T}Q\mathcal{T}\mathcal{T}) \cap \mathcal{T}\mathcal{T}Q] \subseteq Q$. Thus, $Q$ is a right bi quasi ideal of $\mathcal{T}$. Similarly to prove that $Q$ is a lateral and left bi quasi ideal of $\mathcal{T}$. Hence $Q$ is a bi quasi ideal of $\mathcal{T}$.

Converse of the Theorem 3.4 is not true by the following Example.

**Example 3.6.** Let $\mathcal{T} = \left\{ \begin{pmatrix} 0 & r_1 & r_2 & r_3 & r_4 & r_5 \\ 0 & 0 & r_6 & r_7 & r_8 & r_9 \\ 0 & 0 & 0 & r_{10} & r_{11} & r_{12} \\ 0 & 0 & 0 & 0 & r_{13} & r_{14} \\ 0 & 0 & 0 & 0 & 0 & r_{15} \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \middle| r_i^{'s} \text{ are non positive real numbers} \right\}$ is not a regular.

Since $a = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \in \mathcal{T}$ there exist no $x \in \mathcal{T}$ such that $a = axa$.

Clearly, $Q = \left\{ \begin{pmatrix} 0 & 0 & x_1 & 0 & 0 & 0 \\ 0 & 0 & x_2 & 0 & x_3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & x_4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \middle| x_i^{'s} \text{ are non positive real numbers} \right\}$ is a TSS of $\mathcal{T}$.

Now, $Q\mathcal{T}\mathcal{T} = \left\{ \begin{pmatrix} 0&0&0&0&y_1&y_2 \\ 0&0&0&0&y_3&y_4 \\ 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \end{pmatrix} \middle| y_i^{'s} \text{ are non positive real numbers} \right\}$,

$\mathcal{T}\mathcal{T}Q = \left\{ \begin{pmatrix} 0&0&0&0&z_1&0 \\ 0&0&0&0&z_2&0 \\ 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \end{pmatrix} \middle| z_i^{'s} \text{ are non positive real numbers} \right\}$,

$\mathcal{T}Q\mathcal{T} = \left\{ \begin{pmatrix} 0&0&0&m_1&m_2&m_3 \\ 0&0&0&0&0&m_4 \\ 0&0&0&0&0&m_5 \\ 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \end{pmatrix} \middle| m_i^{'s} \text{ are non positive real numbers} \right\}$,

$Q\mathcal{T}Q\mathcal{T} = \left\{ \begin{pmatrix} 0&0&0&0&0&l_1 \\ 0&0&0&0&0&l_2 \\ 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \end{pmatrix} \middle| l_i^{'s} \text{ are non positive real numbers} \right\}$ and

$Q\mathcal{T}Q\mathcal{T}Q = \left\{ \begin{pmatrix} 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \end{pmatrix} \right\} \subseteq Q$.

Hence $Q$ is a bi quasi-ideal, but $Q$ is not a quasi ideal by $Q\mathcal{T}\mathcal{T} \cap (\mathcal{T}Q\mathcal{T} \cup \mathcal{T}\mathcal{T}Q\mathcal{T}\mathcal{T}) \cap \mathcal{T}\mathcal{T}Q = \left\{ \begin{pmatrix} 0&0&0&0&u&0 \\ 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \end{pmatrix} \middle| u \text{ is a non positive real numbers} \right\} \not\subseteq Q$

**Theorem 3.5.** *Every bi quasi $\mathscr{A}$-ideal is a quasi $\mathscr{A}$-ideal.*

**Proof.** Suppose that $Q$ is a bi quasi $\mathscr{A}$-ideal of $\mathcal{T}$, then $Q$ is a TSS and $[Q\mathcal{T}\mathcal{T} \cap Q\mathcal{T}Q\mathcal{T}Q] \cap Q \neq \phi$, $[\mathcal{T}Q\mathcal{T} \cap Q\mathcal{T}Q\mathcal{T}Q] \cap Q \neq \phi$ and $[\mathcal{T}\mathcal{T}Q \cap Q\mathcal{T}Q\mathcal{T}Q] \cap Q \neq \phi$. Now, $\phi \neq [\mathcal{T}Q\mathcal{T} \cap Q\mathcal{T}Q\mathcal{T}Q] \cap Q \subseteq Q\mathcal{T}\mathcal{T} \cap Q$, $\phi \neq [\mathcal{T}Q\mathcal{T} \cap Q\mathcal{T}Q\mathcal{T}Q] \cap Q \subseteq \mathcal{T}Q\mathcal{T} \cap Q \subseteq (\mathcal{T}Q\mathcal{T} \cup \mathcal{T}\mathcal{T}Q\mathcal{T}\mathcal{T}) \cap Q$ and $\phi \neq [\mathcal{T}Q\mathcal{T} \cap Q\mathcal{T}Q\mathcal{T}Q] \cap Q \subseteq \mathcal{T}\mathcal{T}Q \cap Q$. Thus, $\phi \neq [\mathcal{T}Q\mathcal{T} \cap Q\mathcal{T}Q\mathcal{T}Q] \cap Q \subseteq [Q\mathcal{T}\mathcal{T} \cap (\mathcal{T}Q\mathcal{T} \cup \mathcal{T}\mathcal{T}Q\mathcal{T}\mathcal{T}) \cap \mathcal{T}\mathcal{T}Q] \cap Q$. Hence $Q$ is a quasi $\mathscr{A}$-ideal of $\mathcal{T}$.

Converse of the Theorem 3.5 is not true by the following Example.

**Example 3.7.** Consider $\mathcal{T} = \left\{ \begin{pmatrix} 0&r_1&r_2&r_3 \\ 0&0&r_4&r_5 \\ 0&0&0&r_6 \\ 0&0&0&r_7 \end{pmatrix} \middle| r_i^{'s} \text{ are non positive real numbers} \right\}$ is not a regular by $a = \begin{pmatrix} 0&0&-1&0 \\ 0&0&0&0 \\ 0&0&0&0 \\ 0&0&0&0 \end{pmatrix} \in \mathcal{T}$ there exist no $x \in \mathcal{T}$ such that $a = axa$.
Clearly, the TSS $Q = \left\{ \begin{pmatrix} 0&0&x_1&0 \\ 0&0&x_2&0 \\ 0&0&0&0 \\ 0&0&0&x_3 \end{pmatrix} \middle| x_i^{'s} \text{ are non positive real numbers} \right\}$ is a quasi $\mathscr{A}$-ideal of $\mathcal{T}$ but $Q$ is not a bi quasi $\mathscr{A}$-ideal of $\mathcal{T}$ by $[qs_1s_2 \cap qs_3qs_4q] \cap q = \phi$, where $q = \begin{pmatrix} 0&0&-1&0 \\ 0&0&-1&0 \\ 0&0&0&0 \\ 0&0&0&-1 \end{pmatrix} \in Q$ and $s_1 = \begin{pmatrix} 0&-1&0&0 \\ 0&0&0&-1 \\ 0&0&0&-1 \\ 0&0&0&-1 \end{pmatrix}$, $s_2 = \begin{pmatrix} 0&-1&-1&-1 \\ 0&0&-1&-1 \\ 0&0&0&-1 \\ 0&0&0&0 \end{pmatrix}$, $s_3 = \begin{pmatrix} 0&-1&0&-1 \\ 0&0&-1&0 \\ 0&0&0&-1 \\ 0&0&0&0 \end{pmatrix}$, $s_4 = \begin{pmatrix} 0&0&-1&0 \\ 0&0&0&-1 \\ 0&0&0&-1 \\ 0&0&0&0 \end{pmatrix}$ and $s_1, s_2, s_3, s_4 \in \mathcal{T}$.

**Theorem 3.6.** *Every bi $\mathscr{A}$ ideal is a quasi $\mathscr{A}$-ideal.*

**Proof.** Suppose that $B$ is a bi $\mathscr{A}$ ideal of $\mathcal{T}$, then $B$ is a TSS of $\mathcal{T}$ and $B\mathcal{T}B\mathcal{T}B \cap B \neq \phi$. Now, $\phi \neq B\mathcal{T}B\mathcal{T}B \cap B \subseteq B\mathcal{T}\mathcal{T} \cap B$ and $\phi \neq B\mathcal{T}B\mathcal{T}B \cap B \subseteq \mathcal{T}\mathcal{T}B \cap B$ and $\phi \neq B\mathcal{T}B\mathcal{T}B \cap B \subseteq \mathcal{T}\mathcal{T}B\mathcal{T}\mathcal{T} \cap B \subseteq (\mathcal{T}B\mathcal{T} \cup \mathcal{T}\mathcal{T}B\mathcal{T}\mathcal{T}) \cap B$. Thus, $\phi \neq B\mathcal{T}B\mathcal{T}B \cap B \subseteq [B\mathcal{T}\mathcal{T} \cap (\mathcal{T}B\mathcal{T} \cup \mathcal{T}\mathcal{T}B\mathcal{T}\mathcal{T}) \cap \mathcal{T}\mathcal{T}B] \cap B$. Hence $B$ is a quasi $\mathscr{A}$-ideal of $\mathcal{T}$

Converse of the Theorem 3.6 may not be true by the following counter Example.

**Example 3.8.** Consider $\mathcal{T} = \left\{ \begin{pmatrix} 0 & r_1 & r_2 & r_3 \\ 0 & 0 & r_4 & r_5 \\ 0 & 0 & 0 & r_6 \\ 0 & 0 & 0 & r_7 \end{pmatrix} \middle| r_i^{'s} \text{ are non positive real numbers} \right\}$ is not a regular. Let $B = \left\{ \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & x_1 & x_2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & x_3 \end{pmatrix} \middle| x_i^{'s} \text{ are are non positive real numbers} \right\}$ is a TSS of $\mathcal{T}$ and $\mathcal{T}\mathcal{T}B = \left\{ \begin{pmatrix} 0 & 0 & 0 & y_1 \\ 0 & 0 & 0 & y_2 \\ 0 & 0 & 0 & y_3 \\ 0 & 0 & 0 & y_4 \end{pmatrix} \middle| y_i^{'s} \text{ are are non positive real numbers} \right\}$ and

$B\mathcal{T}\mathcal{T} = \left\{ \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & z_1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & z_2 \end{pmatrix} \middle| z_i^{'s} \text{ are are non positive real numbers} \right\}$ and

$\mathcal{T}B\mathcal{T} = \left\{ \begin{pmatrix} 0 & 0 & 0 & l_1 \\ 0 & 0 & 0 & l_2 \\ 0 & 0 & 0 & l_3 \\ 0 & 0 & 0 & l_4 \end{pmatrix} \middle| l_i^{'s} \text{ are are non positive real numbers} \right\}$ and

$\mathcal{T}\mathcal{T}B\mathcal{T}\mathcal{T} = \left\{ \begin{pmatrix} 0 & 0 & 0 & m_1 \\ 0 & 0 & 0 & m_2 \\ 0 & 0 & 0 & m_3 \\ 0 & 0 & 0 & m_4 \end{pmatrix} \middle| m_i^{'s} \text{ are are non positive real numbers} \right\}$. Hence $[B\mathcal{T}\mathcal{T} \cap (\mathcal{T}B\mathcal{T} \cup \mathcal{T}\mathcal{T}B\mathcal{T}\mathcal{T}) \cap \mathcal{T}\mathcal{T}B] \cap B \neq \phi$. Thus $B$ is a quasi $\mathscr{A}$-ideal of $\mathcal{T}$ but $B$ is not a bi $\mathscr{A}$-ideal of $\mathcal{T}$ by $br^{'}br^{''}b \cap b = \phi$ where $b = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \in B$, $r^{'} = \begin{pmatrix} 0 & -1 & -1 & -1 \\ 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$, $r^{''} = \begin{pmatrix} 0 & -1 & 0 & -1 \\ 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \in \mathcal{T}$.

**Theorem 3.7.** *Every quasi $\mathscr{A}$-ideal is a two sided $\mathscr{A}$-ideal.*

**Proof.** Suppose that $Q$ is a quasi $\mathscr{A}$-ideal of $\mathcal{T}$, then $[Q\mathcal{T}\mathcal{T} \cap (\mathcal{T}Q\mathcal{T} \cup \mathcal{T}\mathcal{T}Q\mathcal{T}\mathcal{T}) \cap \mathcal{T}\mathcal{T}Q] \cap Q \neq \phi$. Now, $\phi \neq [Q\mathcal{T}\mathcal{T} \cap (\mathcal{T}Q\mathcal{T} \cup \mathcal{T}\mathcal{T}Q\mathcal{T}\mathcal{T}) \cap \mathcal{T}\mathcal{T}Q] \cap Q \subseteq \mathcal{T}\mathcal{T}Q \cap Q$ and $\phi \neq [Q\mathcal{T}\mathcal{T} \cap (\mathcal{T}Q\mathcal{T} \cup \mathcal{T}\mathcal{T}Q\mathcal{T}\mathcal{T}) \cap \mathcal{T}\mathcal{T}Q] \cap Q \subseteq Q\mathcal{T}\mathcal{T} \cap Q$. Hence $Q$ is a two sided $\mathscr{A}$-ideal of $\mathcal{T}$.

Converse of the Theorem 3.7 is not true by the following Example.

**Example 3.9.** Consider $\mathcal{T} = \left\{ \begin{pmatrix} 0 & r_1 & r_2 \\ 0 & 0 & r_3 \\ 0 & 0 & 0 \end{pmatrix} \middle| r_i^{'s} \text{ are non positive real numbers} \right\}$ is not a regular. Since $a = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \in \mathcal{T}$ there exist $x \in \mathcal{T}$ such that $a = axa$.

Let $Q = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & q_1 \\ 0 & 0 & 0 \end{pmatrix} \middle| q_1 \text{ is a non positive real numbers} \right\}$ is a two sided $\mathscr{A}$-ideal but $Q$ is not a quasi $\mathscr{A}$-ideal of $\mathcal{T}$ by $[qx_1x_2 \cap (x_3qx_4 \cup x_5x_6qx_7x_8) \cap x_9x_{10}q] \cap q = \phi$, where $q = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix} \in Q$, $x_1 = x_3 = x_4 = \begin{pmatrix} 0 & -1 & -1 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix} \in \mathcal{T}$ and $x_2 = x_5 = x_6 = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix}$ and $x_7 = x_8 = x_9 = x_{10} = \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix} \in \mathcal{T}$.

**Theorem 3.8.** *Every bi quasi ideal is a bi quasi $\mathscr{A}$-ideal.*

**Proof.** Straightforward.

Converse of the Theorem 3.8 is not true as by the Example.

**Example 3.10.** By the Example 3.3, Let $Q = \{a_1, a_2, a_3\}$, $Q\mathcal{T}\mathcal{T} = \mathcal{T}$, $Q\mathcal{T}Q\mathcal{T}Q = \mathcal{T}$ and $[Q\mathcal{T}\mathcal{T} \cap Q\mathcal{T}Q\mathcal{T}Q] \cap Q = Q \neq \phi$. This implies that $Q$ is a bi quasi $\mathscr{A}$-ideal of $\mathcal{T}$ but $Q$ is not a bi quasi ideal of $\mathcal{T}$ by $Q\mathcal{T}\mathcal{T} \cap Q\mathcal{T}Q\mathcal{T}Q = \mathcal{T} \nsubseteq Q$.

**Theorem 3.9.** *If $Q$ is a $\mathscr{A}$ ideal (bi $\mathscr{A}$, quasi $\mathscr{A}$, bi quasi $\mathscr{A}$) ideal of $\mathcal{T}$ and $Q \subseteq Q^{'} \subseteq \mathcal{T}$, then $Q^{'}$ is a $\mathscr{A}$ ideal (bi $\mathscr{A}$, quasi $\mathscr{A}$, bi quasi $\mathscr{A}$) ideal of $\mathcal{T}$.*

**Proof.** Suppose that $Q$ is a bi quasi $\mathscr{A}$ ideal of $\mathcal{T}$ with $Q \subseteq Q^{'} \subseteq \mathcal{T}$. Then $\phi \neq [Q\mathcal{T}\mathcal{T} \cap Q\mathcal{T}Q\mathcal{T}Q] \cap Q \subseteq [Q^{'}\mathcal{T}\mathcal{T} \cap Q^{'}\mathcal{T}Q^{'}\mathcal{T}Q^{'}] \cap Q^{'}$ and $\phi \neq [\mathcal{T}Q\mathcal{T} \cap Q\mathcal{T}Q\mathcal{T}Q] \cap Q \subseteq [\mathcal{T}Q^{'}\mathcal{T} \cap Q^{'}\mathcal{T}Q^{'}\mathcal{T}Q^{'}] \cap Q^{'}$ and $\phi \neq [\mathcal{T}\mathcal{T}Q \cap Q\mathcal{T}Q\mathcal{T}Q] \cap Q \subseteq [\mathcal{T}\mathcal{T}Q^{'} \cap Q^{'}\mathcal{T}Q^{'}\mathcal{T}Q^{'}] \cap Q^{'}$. Therefore $Q^{'}$ is a bi quasi $\mathscr{A}$ ideal of $\mathcal{T}$.

**Corollary 3.10.** *The union of $\mathscr{A}$ ideal (bi $\mathscr{A}$, quasi $\mathscr{A}$, bi quasi $\mathscr{A}$) ideal of $\mathcal{T}$ is a $\mathscr{A}$ ideal (bi $\mathscr{A}$, quasi $\mathscr{A}$, bi quasi $\mathscr{A}$) ideal of $\mathcal{T}$.*

**Proof.** Let $I_1$ and $I_2$ be any two $\mathscr{A}$-ideals of $\mathcal{T}$. Then $I_1 \subseteq I_1 \cup I_2$, by Theorem 3.9, $I_1 \cup I_2$ is a $\mathscr{A}$-ideal of $\mathcal{T}$.

## 4. VARIOUS TRI $\mathscr{A}$-IDEALS

Here we introduce different types of Tri $\mathscr{A}$-ideals in ternary semiring.

**Definition 4.1.** A non-empty subsets $I$ and $Q$ of $\mathcal{T}$ is called a

(i) right tri $\mathscr{A}$-ideal of $\mathcal{T}$ if $I$ is a TSS of $\mathcal{T}$ and $I^3\mathcal{T}\mathcal{T}I^2 \cap I \neq \phi$.
(ii) lateral tri $\mathscr{A}$-ideal of $\mathcal{T}$ if $I$ is a TSS of $\mathcal{T}$ and $I^2\mathcal{T}I\mathcal{T}I^2 \cap I \neq \phi$.
(iii) left tri $\mathscr{A}$-ideal of $\mathcal{T}$ if $I$ is a TSS of $\mathcal{T}$ and $I^2\mathcal{T}\mathcal{T}I^3 \cap I \neq \phi$.
(iv) two sided tri $\mathscr{A}$-ideal of $\mathcal{T}$ if $I$ is a right tri $\mathscr{A}$-ideal and left tri $\mathscr{A}$-ideal of $\mathcal{T}$.
(v) tri $\mathscr{A}$-ideal of $\mathcal{T}$ if $I$ is a right tri $\mathscr{A}$-ideal, lateral tri $\mathscr{A}$-ideal and left tri $\mathscr{A}$-ideal of $\mathcal{T}$.

**Definition 4.2.** A non-empty subset $Q$ of $\mathcal{T}$ is called a

(i) right tri quasi $\mathscr{A}$-ideal of $\mathcal{T}$ if $Q$ is a TSS of $\mathcal{T}$ and $Q\mathcal{T}\mathcal{T} \cap Q^3\mathcal{T}\mathcal{T}Q^2 \cap Q \neq \phi$
(ii) lateral tri quasi $\mathscr{A}$-ideal of $\mathcal{T}$ if $Q$ is a TSS of $\mathcal{T}$ and $\mathcal{T}Q\mathcal{T} \cap Q^2\mathcal{T}Q\mathcal{T}Q^2 \cap Q \neq \phi$.
(iii) left tri quasi $\mathscr{A}$-ideal of $\mathcal{T}$ if $Q$ is a TSS of $\mathcal{T}$ and $\mathcal{T}\mathcal{T}Q \cap Q^2\mathcal{T}\mathcal{T}Q^3 \cap Q \neq \phi$.
(iv) tri quasi $\mathscr{A}$-ideal of $\mathcal{T}$ if $Q$ is a right tri quasi $\mathscr{A}$-ideal, lateral tri quasi $\mathscr{A}$-ideal and left tri quasi $\mathscr{A}$-ideal of $\mathcal{T}$.

**Theorem 4.1.** *Every tri ideal is a tri $\mathscr{A}$-ideal.*

**Proof.** Suppose that $I$ is a tri ideal of $\mathcal{T}$, then $I^3\mathcal{T}\mathcal{T}I^2 \subseteq I$, $I^2\mathcal{T}I\mathcal{T}I^2 \subseteq I$ and $I^2\mathcal{T}\mathcal{T}I^3 \subseteq I$. Now, $I^3\mathcal{T}\mathcal{T}I^2 \subseteq I \implies I^3\mathcal{T}\mathcal{T}I^2 \cap I \subseteq I \cap I \neq \phi$, $I^2\mathcal{T}I\mathcal{T}I^2 \subseteq I \implies I^2\mathcal{T}I\mathcal{T}I^2 \cap I \subseteq I \cap I \neq \phi$ and $I^2\mathcal{T}\mathcal{T}I^3 \subseteq I \implies I^2\mathcal{T}\mathcal{T}I^3 \cap I \subseteq I \cap I \neq \phi$ . Hence $I$ is a tri $\mathscr{A}$-ideal of $\mathcal{T}$.

Converse of the Theorem 4.1 may not true by the following counter Example.

**Example 4.3.** In Example 3.3, Clearly $I = \{a_1, a_6\}$ is a tri $\mathscr{A}$-ideal but $I$ is not a tri ideal of $\mathcal{T}$ by $I^3\mathcal{T}\mathcal{T}I^2 = \{a_1, a_4, a_5, a_6\} \nsubseteq I$.

**Theorem 4.2.** *Every tri $\mathscr{A}$-ideal is a $\mathscr{A}$-ideal.*

**Proof.** Suppose that $I$ is a tri $\mathscr{A}$-ideal of $\mathcal{T}$, then $I$ is a TSS of $\mathcal{T}$ and $I^3\mathcal{T}\mathcal{T}I^2 \cap I \neq \phi$ and $I^2\mathcal{T}I\mathcal{T}I^2 \cap I \neq \phi$ and $I^2\mathcal{T}\mathcal{T}I^3 \cap I \neq \phi$. Now, $\phi \neq I^3\mathcal{T}\mathcal{T}I^2 \cap I \subseteq I\mathcal{T}\mathcal{T} \cap I$ and $\phi \neq I^2\mathcal{T}I\mathcal{T}I^2 \cap I \subseteq \mathcal{T}I\mathcal{T} \cap I$ and $\phi \neq I^2\mathcal{T}\mathcal{T}I^3 \cap I \subseteq \mathcal{T}\mathcal{T}I \cap I$. Hence $I$ is a $\mathscr{A}$-ideal of $\mathcal{T}$.

Converse of the Theorem 4.2 may not true as in the given Example.

**Example 4.4.** Let $\mathcal{T} = \left\{ \begin{pmatrix} 0 & r_1 & r_2 & r_3 \\ 0 & 0 & r_4 & r_5 \\ 0 & 0 & 0 & r_6 \\ 0 & 0 & 0 & 0 \end{pmatrix} \middle| r_i^{'s} \text{ are non positive real numbers} \right\}$.

Clearly $I = \left\{ \begin{pmatrix} 0 & 0 & i_1 & i_2 \\ 0 & 0 & i_3 & i_4 \\ 0 & 0 & 0 & i_5 \\ 0 & 0 & 0 & 0 \end{pmatrix} \middle| i_i^{'s} \text{ are are non positive real numbers} \right\}$ is a $\mathscr{A}$-ideal of $\mathcal{T}$ but

$I$ is not a tri $\mathscr{A}$-ideal of $\mathcal{T}$ by $i^3rr_1i^2 \cap i = \phi$ and $i^2rir_1i^2 \cap i = \phi$ and $i^2rr_1i^3 \cap i = \phi$, where
$i = \begin{pmatrix} 0 & 0 & -1 & -1 \\ 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \in I$ and $r = \begin{pmatrix} 0 & -1 & -1 & -1 \\ 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \in \mathcal{T}$ and $r_1 = \begin{pmatrix} 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \in \mathcal{T}$.

**Theorem 4.3.** *Every two sided tri $\mathscr{A}$-ideal is a bi $\mathscr{A}$-ideal.*

**Proof.** Suppose that $B$ is a two sided tri $\mathscr{A}$-ideal of $\mathcal{T}$, then $B$ is a TSS of $\mathcal{T}$ and $B^3\mathcal{T}\mathcal{T}B^2 \cap B \neq \phi$ and $B^2\mathcal{T}\mathcal{T}B^3 \cap B \neq \phi$. Since $B$ is a TSS of $\mathcal{T}$, $\phi \neq B^3\mathcal{T}\mathcal{T}B^2 \cap B \subseteq B\mathcal{T}B\mathcal{T}B \cap B$ and $\phi \neq B^2\mathcal{T}\mathcal{T}B^3 \cap B \subseteq B\mathcal{T}B\mathcal{T}B \cap B$. Hence $B$ is a bi $\mathscr{A}$-ideal of $\mathcal{T}$.

Converse of the Theorem 4.3 is not true as given Example.

**Example 4.5.** Let $\mathcal{T} = \left\{ \begin{pmatrix} 0 & r_1 & r_2 & r_3 & r_4 & r_5 \\ 0 & 0 & r_6 & r_7 & r_8 & r_9 \\ 0 & 0 & 0 & r_{10} & r_{11} & r_{12} \\ 0 & 0 & 0 & 0 & r_{13} & r_{14} \\ 0 & 0 & 0 & 0 & 0 & r_{15} \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \middle| r_i^{'s} \text{ are non positive real numbers} \right\}$ is not a regular.

Since $a = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \in \mathcal{T}$ there exist no $x \in \mathcal{T}$ such that $a = axa$.

The TSS $B = \left\{ \begin{pmatrix} 0 & b_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & b_2 & b_3 & b_4 \\ 0 & 0 & 0 & 0 & b_5 & b_6 \\ 0 & 0 & 0 & 0 & 0 & b_7 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \middle| b_i^{'s} \text{ are non positive real numbers} \right\}$ is a bi-$\mathscr{A}$-ideal but $B$ is not a two sided tri $\mathscr{A}$-ideal of $\mathcal{T}$ by $b^3rr_1b^2 \cap b = \phi$ and $b^2rr_1b^3 \cap b = \phi$,
where $b = \begin{pmatrix} 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & -1 & -1 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \in B$, $r = \begin{pmatrix} 0 & -1 & -1 & -1 & -1 & -1 \\ 0 & 0 & -1 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 & -1 & -1 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \in \mathcal{T}$ and
$r_1 = \begin{pmatrix} 0 & -1 & 0 & -1 & 0 & -1 \\ 0 & 0 & -1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \in \mathcal{T}$.

**Theorem 4.4.** *Every two sided tri $\mathscr{A}$-ideal is a quasi $\mathscr{A}$-ideal.*

**Proof.** Suppose that $Q$ is a right tri $\mathscr{A}$-ideal of $\mathcal{T}$, then $Q^3\mathcal{T}\mathcal{T}Q^2 \cap Q \neq \phi$. Since $Q$ is a TSS of $\mathcal{T}$, $\phi \neq Q^3\mathcal{T}\mathcal{T}Q^2 \cap Q \subseteq Q\mathcal{T}\mathcal{T} \cap Q$ and $\phi \neq Q^3\mathcal{T}\mathcal{T}Q^2 \cap Q \subseteq \mathcal{T}\mathcal{T}Q \cap Q$. Now, $\phi \neq Q^3\mathcal{T}\mathcal{T}Q^2 \cap Q \subseteq Q\mathcal{T}\mathcal{T}Q^2 \cap Q \subseteq \mathcal{T}Q^2 \cap Q \subseteq \mathcal{T}Q\mathcal{T} \cap Q$, also $\phi \neq Q^3\mathcal{T}\mathcal{T}Q^2 \cap Q \subseteq \mathcal{T}\mathcal{T}Q\mathcal{T}\mathcal{T} \cap Q$, implies that $\phi \neq Q^3\mathcal{T}\mathcal{T}Q^2 \cap Q \subseteq (\mathcal{T}Q\mathcal{T} \cup \mathcal{T}\mathcal{T}Q\mathcal{T}\mathcal{T}) \cap Q$. Hence $\phi \neq Q^3\mathcal{T}\mathcal{T}Q^2 \cap Q \subseteq [Q\mathcal{T}\mathcal{T} \cap (\mathcal{T}Q\mathcal{T} \cup \mathcal{T}\mathcal{T}Q\mathcal{T}\mathcal{T}) \cap \mathcal{T}\mathcal{T}Q] \cap Q$. Similarly, $Q$ is a left tri $\mathscr{A}$-ideal of $\mathcal{T}$, then prove that $\phi \neq Q^2\mathcal{T}\mathcal{T}Q^3 \cap Q \subseteq [Q\mathcal{T}\mathcal{T} \cap (\mathcal{T}Q\mathcal{T} \cup \mathcal{T}\mathcal{T}Q\mathcal{T}\mathcal{T}) \cap \mathcal{T}\mathcal{T}Q] \cap Q$. Hence $Q$ is a quasi $\mathscr{A}$-ideal of $\mathcal{T}$.

Converse of the Theorem 4.4 may not true in the given Example.

**Example 4.6.** Let $\mathcal{T} = \left\{ \begin{pmatrix} 0 & r_1 & r_2 & r_3 & r_4 & r_5 \\ 0 & 0 & r_6 & r_7 & r_8 & r_9 \\ 0 & 0 & 0 & r_{10} & r_{11} & r_{12} \\ 0 & 0 & 0 & 0 & r_{13} & r_{14} \\ 0 & 0 & 0 & 0 & 0 & r_{15} \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \middle| r_i^{'s} \text{ are non positive real numbers} \right\}$ is not a regular. The TSS $Q = \left\{ \begin{pmatrix} 0 & 0 & 0 & q_1 & q_2 & 0 \\ 0 & 0 & 0 & 0 & q_3 & q_4 \\ 0 & 0 & 0 & 0 & 0 & q_5 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \middle| q_i^{'s} \text{ are non positive real numbers} \right\}$ is a quasi $\mathscr{A}$-ideal but $Q$ is not a two sided tri $\mathscr{A}$-ideal of $\mathcal{T}$ by $q^3rr_1q^2 \cap q = \phi$ and

$q^2rr_1q^3 \cap q = \phi$, where $q = \begin{pmatrix} 0&0&0&-1&-1&0 \\ 0&0&0&0&-1&-1 \\ 0&0&0&0&0&-1 \\ 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \end{pmatrix} \in Q$ and $r = \begin{pmatrix} 0&0&-1&-1&0&0 \\ 0&0&0&-1&-1&-1 \\ 0&0&0&-1&-1&-1 \\ 0&0&0&0&-1&-1 \\ 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \end{pmatrix} \in \mathcal{T}$ and $r_1 = \begin{pmatrix} 0&0&-1&0&0&0 \\ 0&0&0&0&-1&-1 \\ 0&0&0&-1&0&-1 \\ 0&0&0&0&-1&0 \\ 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \end{pmatrix} \in \mathcal{T}$.

**Corollary 4.5.** *Every two sided tri $\mathscr{A}$-ideal is a bi quasi $\mathscr{A}$-ideal.*

**Proof.** Suppose that $Q$ is a right tri $\mathscr{A}$-ideal of $\mathcal{T}$, then $Q^3\mathcal{T}\mathcal{T}Q^2 \cap Q \neq \phi$. Since $Q$ is a TSS of $\mathcal{T}$, $\phi \neq Q^3\mathcal{T}\mathcal{T}Q^2 \cap Q \subseteq Q\mathcal{T}\mathcal{T} \cap Q$ and $\phi \neq Q^3\mathcal{T}\mathcal{T}Q^2 \cap Q \subseteq Q\mathcal{T}Q\mathcal{T}Q \cap Q$. This implies that $\phi \neq Q^3\mathcal{T}\mathcal{T}Q^2 \cap Q \subseteq [Q\mathcal{T}\mathcal{T} \cap Q\mathcal{T}Q\mathcal{T}Q] \cap Q$. Thus $Q$ is a right bi quasi $\mathscr{A}$-ideal of $\mathcal{T}$. Also, $\phi \neq Q^3\mathcal{T}\mathcal{T}Q^2 \cap Q \subseteq \mathcal{T}\mathcal{T}Q \cap Q$ implies that $\phi \neq Q^3\mathcal{T}\mathcal{T}Q^2 \cap Q \subseteq [\mathcal{T}\mathcal{T}Q \cap Q\mathcal{T}Q\mathcal{T}Q] \cap Q$. Thus $Q$ is a left bi quasi $\mathscr{A}$-ideal of $\mathcal{T}$. Also, $\phi \neq Q^3\mathcal{T}\mathcal{T}Q^2 \cap Q \subseteq \mathcal{T}Q\mathcal{T} \cap Q$ implies that $\phi \neq Q^3\mathcal{T}\mathcal{T}Q^2 \cap Q \subseteq [\mathcal{T}Q\mathcal{T} \cap Q\mathcal{T}Q\mathcal{T}Q] \cap Q$. Thus, $Q$ is a lateral bi quasi $\mathscr{A}$-ideal of $\mathcal{T}$. Hence $Q$ is a bi quasi $\mathscr{A}$-ideal of $\mathcal{T}$. Similar to prove other case.

Converse of the Corollary 4.5 may not true in the given Example.

**Example 4.7.** Let $\mathcal{T} = \left\{ \begin{pmatrix} 0&0&0&0 \\ r_1&0&0&0 \\ r_2&r_3&0&0 \\ r_4&r_5&r_6&r_7 \end{pmatrix} \middle| r_i^{'s} \text{ are non positive real numbers} \right\}$ is not a regular. Since $a = \begin{pmatrix} 0&0&0&0 \\ -1&0&0&0 \\ 0&0&0&0 \\ -1&0&0&0 \end{pmatrix} \in \mathcal{T}$ there exist no $x \in \mathcal{T}$ such that $a = axa$.

The TSS $Q = \left\{ \begin{pmatrix} 0&0&0&0 \\ 0&0&0&0 \\ q_1&q_2&0&0 \\ 0&q_3&0&q_4 \end{pmatrix} \middle| q_i^{'s} \text{ are non positive real numbers} \right\}$.

Now, $\mathcal{T}\mathcal{T}Q = \left\{ \begin{pmatrix} 0&0&0&0 \\ 0&0&0&0 \\ 0&0&0&0 \\ c_1&c_2&0&c_3 \end{pmatrix} \middle| c_i^{'s} \text{ are non positive real numbers} \right\}$,

$Q\mathcal{T}\mathcal{T} = \left\{ \begin{pmatrix} 0&0&0&0 \\ 0&0&0&0 \\ 0&0&0&0 \\ d_1&d_2&d_3&d_4 \end{pmatrix} \middle| d_i^{'s} \text{ are real numbers} \right\}$,

$\mathcal{T}Q\mathcal{T} = \left\{ \begin{pmatrix} 0&0&0&0 \\ 0&0&0&0 \\ 0&0&0&0 \\ l_1&l_2&l_3&l_4 \end{pmatrix} \middle| l_i^{'s} \text{ are real numbers} \right\}$,

$Q\mathcal{T}Q\mathcal{T}Q = \left\{ \begin{pmatrix} 0&0&0&0 \\ 0&0&0&0 \\ 0&0&0&0 \\ e_1&e_2&0&e_3 \end{pmatrix} \middle| e_i^{'s} \text{ is a real numbers} \right\}$.

Thus $[Q\mathcal{T}\mathcal{T} \cap Q\mathcal{T}Q\mathcal{T}Q] \cap Q \neq \phi$ and $[\mathcal{T}\mathcal{T}Q \cap Q\mathcal{T}Q\mathcal{T}Q] \cap Q \neq \phi$ and $[\mathcal{T}Q\mathcal{T} \cap Q\mathcal{T}Q\mathcal{T}Q] \cap Q \neq \phi$.

Hence $Q$ is a bi quasi $\mathscr{A}$-ideal of $\mathcal{T}$. But $Q$ is not a two sided tri $\mathscr{A}$-ideal of $\mathcal{T}$ by $q^3r^{'}r^{''}q^2 \cap q = \phi$ and $q^2r^{'}r^{''}q^3 \cap q = \phi$, where $q = \begin{pmatrix} 0&0&0&0 \\ 0&0&0&0 \\ -1&-1&0&0 \\ 0&-1&0&-1 \end{pmatrix} \in Q$ and $r^{'} = \begin{pmatrix} 0&0&0&0 \\ -1&0&0&0 \\ -1&-1&0&0 \\ 0&-1&-1&0 \end{pmatrix} \in \mathcal{T}$ and $r^{''} = \begin{pmatrix} 0&0&0&0 \\ 0&0&0&0 \\ 0&0&0&0 \\ 0&-1&-1&-1 \end{pmatrix} \in \mathcal{T}$.

**Theorem 4.6.** *Every two sided bi quasi ideal is a two sided tri quasi ideal.*

**Proof.** Suppose that $Q$ is a right bi quasi ideal of $\mathcal{T}$ then $Q\mathcal{T}\mathcal{T} \cap Q\mathcal{T}Q\mathcal{T}Q \subseteq Q$. Now, $Q\mathcal{T}\mathcal{T} \cap Q^3\mathcal{T}\mathcal{T}Q^2 \subseteq Q\mathcal{T}\mathcal{T} \cap Q\mathcal{T}Q\mathcal{T}Q \subseteq Q$. Also, $Q\mathcal{T}\mathcal{T} \cap Q^3\mathcal{T}\mathcal{T}Q^2 \subseteq Q^3\mathcal{T}\mathcal{T}Q^2 \subseteq \mathcal{T}\mathcal{T}Q$ and $Q\mathcal{T}\mathcal{T} \cap Q^3\mathcal{T}\mathcal{T}Q^2 \subseteq Q\mathcal{T}Q\mathcal{T}Q$ implies that $Q\mathcal{T}\mathcal{T} \cap Q^3\mathcal{T}\mathcal{T}Q^2 \subseteq \mathcal{T}\mathcal{T}Q \cap Q\mathcal{T}Q\mathcal{T}Q \subseteq Q$. Hence $Q$ is a two sided tri quasi ideal of $\mathcal{T}$. Similar to prove other part.

Converse of Theorem 4.6 not true by the following Example.

**Example 4.8.** Let $\mathcal{T} = \left\{ \begin{pmatrix} 0 & 0 & 0 & 0 \\ r_1 & 0 & 0 & 0 \\ r_2 & r_3 & 0 & 0 \\ r_4 & r_5 & r_6 & r_7 \end{pmatrix} \middle| r_i^{'s} \text{ are non positive real numbers} \right\}$ is not a regular.

Here $Q = \left\{ \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ a_1 & 0 & 0 & 0 \\ 0 & a_2 & 0 & a_3 \end{pmatrix} \middle| a_i^{'s} \text{ are are non positive real numbers} \right\}$ is a TSS of $\mathcal{T}$.

Now, $\mathcal{T}\mathcal{T}Q = \left\{ \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ c_1 & c_2 & 0 & c_3 \end{pmatrix} \middle| c_i^{'s} \text{ are non positive real numbers} \right\}$,

$Q\mathcal{T}\mathcal{T} = \left\{ \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ d_1 & d_2 & d_3 & d_4 \end{pmatrix} \middle| d_i^{'s} \text{ are non positive real numbers} \right\}$,

$Q\mathcal{T}Q\mathcal{T}Q = \left\{ \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ e_1 & e_2 & 0 & e_3 \end{pmatrix} \middle| e_i^{'s} \text{ are non positive real numbers} \right\}$,

$Q^3\mathcal{T}\mathcal{T}Q^2 = \left\{ \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & f_1 & 0 & f_2 \end{pmatrix} \middle| f_i^{'s} \text{ are non positive real numbers} \right\}$,

$Q^2\mathcal{T}\mathcal{T}Q^3 = \left\{ \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & g_1 & 0 & g_2 \end{pmatrix} \middle| g_i^{'s} \text{ are non positive real numbers} \right\}$.

Hence $Q$ is a two sided tri quasi-ideal of $\mathcal{T}$. But $Q$ is not a two sided bi quasi ideal of $\mathcal{T}$ by $Q\mathcal{T}\mathcal{T} \cap Q\mathcal{T}Q\mathcal{T}Q = \left\{ \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ h_1 & h_2 & 0 & h_3 \end{pmatrix} \middle| h_i^{'s} \text{ are non positive real numbers} \right\} \nsubseteq Q$ and

$\mathcal{T}\mathcal{T}Q \cap Q\mathcal{T}Q\mathcal{T}Q = \left\{ \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ i_1 & i_2 & 0 & i_3 \end{pmatrix} \middle| i_i^{'s} \text{ are non positive real numbers} \right\} \nsubseteq Q$.

**Corollary 4.7.** *Every two sided tri quasi $\mathscr{A}$-ideal is a two sided bi quasi $\mathscr{A}$-ideal.*

**Proof.** Suppose that $Q$ is a right tri quasi $\mathscr{A}$- ideal of $\mathcal{T}$, then $Q$ is a TSS of $\mathcal{T}$ and $[Q\mathcal{T}\mathcal{T} \cap Q^3\mathcal{T}\mathcal{T}Q^2] \cap Q \neq \phi$. Now, $\phi \neq [Q\mathcal{T}\mathcal{T} \cap Q^3\mathcal{T}\mathcal{T}Q^2] \cap Q \subseteq [Q\mathcal{T}\mathcal{T} \cap Q\mathcal{T}Q\mathcal{T}Q] \cap Q$. Also, $\phi \neq [Q\mathcal{T}\mathcal{T} \cap Q^3\mathcal{T}\mathcal{T}Q^2] \cap Q \subseteq \mathcal{T}\mathcal{T}Q \cap Q$ and $\phi \neq [Q\mathcal{T}\mathcal{T} \cap Q^3\mathcal{T}\mathcal{T}Q^2] \cap Q \subseteq Q\mathcal{T}Q\mathcal{T}Q \cap Q$ implies that $\phi \neq [Q\mathcal{T}\mathcal{T} \cap Q^3\mathcal{T}\mathcal{T}Q^2] \cap Q \subseteq [\mathcal{T}\mathcal{T}Q \cap Q\mathcal{T}Q\mathcal{T}Q] \cap Q$. Hence $Q$ is a two sided bi quasi $\mathscr{A}$-ideal of $\mathcal{T}$. Similar to prove other part.

Converse of the Corollary 4.7 may not true in the given Example.

**Example 4.9.** Consider $\mathcal{T} = \left\{ \begin{pmatrix} 0 & 0 & 0 & 0 \\ r_1 & 0 & 0 & 0 \\ r_2 & r_3 & 0 & 0 \\ r_4 & r_5 & r_6 & r_7 \end{pmatrix} \middle| r_i^{'s} \text{ are non positive real numbers} \right\}$ is not a regular. Since $a = \begin{pmatrix} 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} \in \mathcal{T}$ there exist no $x \in \mathcal{T}$ such that $a = axa$.

The TSS $Q = \left\{ \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ a_1 & 0 & 0 & 0 \\ 0 & a_2 & 0 & a_3 \end{pmatrix} \middle| a_i^{'s} \text{ are are non positive real numbers} \right\}$ is a two sided bi quasi $\mathscr{A}$-ideal of $\mathcal{T}$. But $Q$ is not a two sided tri quasi $\mathscr{A}$-ideal of $\mathcal{T}$ by $[x_1x_2q \cap q^2x_3x_4q^3] \cap q = \phi$ and $[qx_1x_2 \cap q^3x_3x_4q^2] \cap q = \phi$, where $q = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix} \in Q$ and $x_1 = x_3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \in \mathcal{T}$ and $x_2 = x_4 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \end{pmatrix} \in \mathcal{T}$.

**Theorem 4.8.** *If $Q$ is a tri $\mathscr{A}$ ideal (tri quasi $\mathscr{A}$ ideal) of $\mathcal{T}$ and $Q \subseteq Q' \subseteq \mathcal{T}$, then $Q'$ is a tri $\mathscr{A}$ ideal (tri quasi $\mathscr{A}$ ideal) of $\mathcal{T}$.*

**Proof.** Suppose that $Q$ is a tri $\mathscr{A}$ ideal of $R$ with $Q \subseteq Q^{'} \subseteq \mathcal{T}$. Then $\phi \neq Q^3\mathcal{T}\mathcal{T}Q^2 \cap Q \subseteq {Q^{'}}^3\mathcal{T}\mathcal{T}{Q^{'}}^2 \cap Q^{'}$, $\phi \neq Q^2\mathcal{T}Q\mathcal{T}Q^2 \cap Q \subseteq {Q^{'}}^2\mathcal{T}Q\mathcal{T}{Q^{'}}^2 \cap Q^{'}$ and $\phi \neq Q^2\mathcal{T}\mathcal{T}Q^3 \cap Q \subseteq {Q^{'}}^2\mathcal{T}\mathcal{T}{Q^{'}}^3 \cap Q^{'}$. Therefore $Q^{'}$ is a tri $\mathscr{A}$ ideal of $\mathcal{T}$. Similar to prove case.

**Corollary 4.9.** *The union of tri $\mathscr{A}$-ideal (tri quasi $\mathscr{A}$-ideal) of $\mathcal{T}$ is a tri $\mathscr{A}$-ideal (tri quasi $\mathscr{A}$-ideal) of $\mathcal{T}$.*

**Proof.** Let $Q_1$ and $Q_2$ be any two tri $\mathscr{A}$-ideals of $\mathcal{T}$. Then $Q_1 \subseteq Q_1 \cup Q_2$, by Theorem 4.8, $Q_1 \cup Q_2$ is a tri $\mathscr{A}$-ideal of $\mathcal{T}$.


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M. PALANIKUMAR
KINGS ENGINEERING COLLEGE, DEPARTMENT OF MATHEMATICS, CHENNAI, 602117, INDIA
*E-mail address*: palanimaths86@gmail.com

K. ARULMOZHI
BHARATH INSTITUTE OF HIGHER EDUCATION AND RESEARCH, DEPARTMENT OF MATHEMATICS, CHENNAI, 600073, INDIA
*E-mail address*: arulmozhiems@gmail.com