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ON PYTHAGOREAN FUZZY IDEAL OF SUBTRACTION SEMIGROUP AND NEAR SUBTRACTION SEMIGROUP

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ABSTRACT. In this paper, we define the notions of Pythagorean fuzzy ideal of subtraction semigroup and near subtraction semigroup. Also, we discuss some of its properties with examples.

1. INTRODUCTION

In 1969, Abbott introduced the notion of subtraction algebra. He considered the system of the form $(\phi, \circ, \backslash)$ where ϕ is a set of functions closed under the composition $\circ$ of function where $(\phi, \circ)$ is a functional semigroup and $(\phi, \backslash)$ is a subtraction algebra [1]. Using this concept Schein[9] introduced the concept of subtraction semigroups in 1992. He proved that every subtraction semigroup is isomorphism to a different semigroup of invertible function. Zelink[13] studied an extraordinary kind of subtraction algebra called atomic subtraction algebra. A near subtraction semigroup satisfies all axioms of subtraction semigroup except one of the two distributive laws. Prince williams[8] defined the fuzzy ideal in near subtraction semigroup(NSS). Jun et al.[5] introduced the concept of ideals in subtraction algebra and gave some characterizations. Dheena et al.[3, 4] discussed and derived some properties of NSS, a generalization of subtraction semigroup. The concept of fuzzy set was initiated by Zadeh[12]. Mahalakshmi et al.[7] studied the notion of bi ideals of NSS. Chinnadurai[2] defined the concept of fuzzy ideals in algebraic structures. Kim et al.[6], studied some properties of intuitionistic fuzzy ideals of semigroups. Yager[10, 11] introduced the Pythagorean fuzzy set(PFS).
In this paper, we introduce the notions of Pythagorean fuzzy ideal in subtraction semigroup and near subtraction semigroup. Also we characterize the subtraction and near subtraction semigroup through fuzzification.

2. PRELIMINARIES

Now, we recall some new concepts of Pythagorean fuzzy ideals in NSS from the literature, which are required in the sequel.

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The basic concepts of subtraction semigroup(SS), power set(PS), NSS [6] and zero symmetric are referred respectively.

Definition 2.1. A non-empty set X together with a binary operation "$-$" is said to be a SA(subtraction algebra), if
(i) $j-(k-j)=j$
(ii) $j-(j-k)=k-(k-j)$
(iii) $(j-k)-l=(j-l)-k \ \forall j,k,l \in X$.

Definition 2.2. A subset I of SA X is called subalgebra of X if $j-k \in I \, \forall j,k \in X$. In SA the following holds:
(1) $j-0=j$ and $0-j=0$
(2) $j-(j-k) \leq k$
(3) $j \leq k$ if and only if $j=k-m$ for some $m \in X$
(4) $j \leq k$ implies $j-l \leq k-l$ and $l-k \leq l-j \, \forall l \in X$
(5) $j-(j-(j-k))=j-k$
(6) $(j-k)-j=0$
(7) $(j-k)-k=j-k$.

3. Pythagorean fuzzy ideal in subtraction semigroup

In this section X denotes subtraction semigroup(SS).

Definition 3.1. A Pythagorean fuzzy set(PFS) $D=(\pi,\vartheta)$ of X is called PFSS(Pythagorean fuzzy subtraction sub-semigroup) of X if
(1) $\pi(j-k) \geq \bigwedge\{\pi(j),\pi(k)\}$
(2) $\vartheta(j-k) \leq \bigvee\{\vartheta(j),\vartheta(k)\} \ \forall j,k \in X$.

Example 3.2. Let $X=\{0,j,k,l\}$ be a subtraction sub-semigroup with two binary operations $'-'$ and $'\cdot'$ is defined as follows.

$-$	0	j	k	l
0	0	0	0	0
j	j	0	j	0
k	k	k	0	0
l	l	k	j	0

$\cdot$	0	j	k	l
0	0	0	0	0
j	0	j	0	0
k	0	0	k	k
l	0	0	k	k

Define a Pythagorean fuzzy set $D=(\pi,\vartheta)$ where $\pi: X \to [0,1]$ by $\pi(0)=0.8, \pi(j)=0.6, \pi(k)=0.3, \pi(l)=0.2$. Then $\vartheta: X \to [0,1]$ by $\vartheta(0)=0.3, \vartheta(j)=0.4, \vartheta(k)=0.6, \vartheta(l)=0.7$ Then D is a PFSS of X. Hence $D=(\pi,\vartheta)$ is a subtraction sub-semigroup of X.

Definition 3.3. A PFS $D=(\pi,\vartheta)$ of X is called PFLI(resp. PFRI) of X, if $\forall j,k \in X$.
(1) $\pi(j) \geq \wedge\{\pi(j-k),\pi(k)\}$
(2) $\vartheta(j) \leq \vee\{\vartheta(j-k),\vartheta(k)\}$
(3) $\pi(jk) \geq \wedge\{\pi(j),\pi(k)\}$
(4) $\vartheta(jk) \leq \vee\{\vartheta(j),\vartheta(k)\}$
(5) $\pi(jk) \geq \pi(k)$ (resp.$\pi(jk) \geq \pi(j)$)
(6) $\vartheta(jk) \leq \vartheta(k)$ (resp.$\vartheta(jk) \leq \vartheta(j)$).
If π and ϑ are both PFLI and PFRI of X. Then π and ϑ are both PFI of X.

Example 3.4. Let $X = \{0, j, k, l\}$ be a SS with two binary operations $'-'$ and $'\cdot'$ is defined as follows.

$-$	0	j	k	l
0	0	0	0	0
j	j	0	j	0
k	k	k	0	0
l	l	k	j	0

$\cdot$	0	j	k	l
0	0	0	0	0
j	0	j	0	0
k	0	0	k	k
l	0	0	k	k

Define a PFS $D = (\pi, \vartheta)$ where $\pi : X \to [0, 1]$ by $\pi(0) = 0.8, \pi(j) = 0.6, \pi(k) = 0.3, \pi(l) = 0.2$. Then $\vartheta : X \to [0, 1]$ by $\vartheta(0) = 0.3, \vartheta(j) = 0.4, \vartheta(k) = 0.6, \vartheta(l) = 0.7$. Then D is a PFL (resp.PFR) ideals of X. Hence $D = (\pi, \vartheta)$ is a PFLI (PFRI) of X.

Definition 3.5. Let $D_1 = (\pi_{D_1}, \vartheta D_1)$ and $D_2 = (\pi_{D_2}, \vartheta_{D_2})$ be any two PFS of X. Then the following FSs of X are defined as follows.

$$(D_1 \star D_2)(j) = \begin{cases} (\pi_{D_1} \star \pi_{D_2})(j) = \begin{cases} \bigvee\limits_{j \le xy} \wedge\{\pi_D(x), \pi_D(y)\} & if j \le xy \\ [0,0] & otherwise. \end{cases} \\ (\vartheta_{D_1} \star \vartheta_{D_2})(j) = \begin{cases} \bigwedge\limits_{j \le xy} \vee\{\vartheta_D(x), \vartheta_D(y)\} & if j \le xy \\ [1,1] & otherwise. \end{cases} \end{cases}$$

$$(D_1 \cap D_2)(j) = \begin{cases} (\pi_{D_1} \cap \pi_{D_2})(j) \\ (\vartheta_{D_1} \cup \vartheta_{D_2})(j) \; \forall \, j \in X \end{cases}$$

$$(D_1 - D_2)(j) = \begin{cases} (\pi_{D_1} - \pi_{D_2})(j) = \begin{cases} \bigvee\limits_{j = k-l} \wedge\{\pi_D(k), \pi_D(l)\} & if j = k - l \; \forall \, j, k, l \in X \\ [0,0] & otherwise. \end{cases} \\ (\vartheta_{D_1} - \vartheta_{D_2})(j) = \begin{cases} \bigwedge\limits_{j = k-l} \vee\{\vartheta_D(k), \vartheta_D(l)\} & if j = k - l \; \forall \, j, k, l \in X \\ [1,1] & otherwise. \end{cases} \end{cases}$$

Theorem 3.1. *Every PFLI (resp. PFRI) of X is a PFSS of X.*

Proof. Let π and ϑ be an PFI of X. Then

$$\begin{aligned} \pi(j-k) &\ge \wedge\{\pi((j-k)-r), \pi(l)\} \; \forall \, l \in X \\ &\ge \wedge\{\pi((j-k)-l), \pi(j)\} \; for \; l = j \\ &= \wedge\{\pi(0), \pi(j)\} \\ &= \pi(j). \end{aligned}$$

Thus

$\pi(j-k) \ge \pi(j)$.

$$\begin{aligned} \vartheta(j-k) &\le \vee\{\vartheta((j-k)-l), \vartheta(r)\} \; \forall \, l \in X \\ &\le \vee\{\vartheta((j-k)-l), \vartheta(j)\} \; for \; l = j \\ &= \vee\{\vartheta(0), \vartheta(j)\} \\ &= \vartheta(j). \end{aligned}$$

Thus

$\vartheta(j-k) \le \vartheta(j)$.

Again consider

$\pi(j-k) \ge \wedge\{\pi((j-k)-l), \pi(l)\} \; \forall \, l \in X$

$\geq \wedge\{\pi((j-k)-k), \pi(k)\}\ for\ l = k$
$= \wedge\{\pi(j-k), \pi(k)\}$ since $(j-k)-k = j-k$
$= \wedge\{\pi(j), \pi(k)\}$.

And

$\vartheta(j-k) \leq \vee\{\vartheta((j-k)-l), \vartheta(l)\}\ \forall\ l \in X$
$\leq \vee\{\vartheta((j-k)-k), \vartheta(k)\}\ for\ l = k$
$= \vee\{\vartheta(j-k), \vartheta(k)\}$ since $(j-k)-k = j-k$
$= \vee\{\vartheta(j), \vartheta(k)\}$.

Then π and ϑ are PFSS of X.
The converse is not true. □

Theorem 3.2. *If $D = (\pi, \vartheta)$ be an PFS of a SS X. Then*
(a)(i) $\pi \star \pi \leq \pi$
(ii) $\vartheta \star \vartheta \geq \vartheta$
(b)(i) $\pi(jk) \geq \wedge\{\pi(j), \pi(k)\}$
(ii) $\vartheta(jk) \leq \vee\{\vartheta(j), \vartheta(k)\}\ \forall\ j, k \in X$.

Proof. $(a) \Rightarrow (b)$ Let $x, y \in X$.
Consider

$$(\pi \star \pi)(jk) = \bigvee_{xy \leq jk} \{\wedge\{\pi(j), \pi(k)\}\} \geq \wedge\{\pi(x), \pi(y)\}.$$

By (a) $\pi \star \pi \leq \pi$.

$$\pi(xy) \geq (\pi \star \pi)(xy) \geq \wedge\{\pi(x), \pi(y)\}.$$

Hence $\pi(xy) \geq \wedge\{\pi(x), \pi(y)\}$.
And

$$(\vartheta \star \vartheta)(jk) = \bigwedge_{xy \leq jk} \{\vee\{\vartheta(j), \vartheta(k)\}\} \leq \vee\{\vartheta(x), \vartheta(y)\}.$$

By (a) $\vartheta \star \vartheta \geq \vartheta$.

$$\vartheta(xy) \leq (\vartheta \star \vartheta)(xy) \leq \vee\{\vartheta(x), \vartheta(y)\}.$$

Hence $\vartheta(xy) \leq \vee\{\vartheta(x), \vartheta(y)\}$.
$(b) \Leftrightarrow (a)$ Let $j \in X$.
Consider

$$(\pi \star \pi)(j) = \bigvee_{j \leq xy} \wedge\{\pi(x), \pi(y)\} \leq \bigvee_{j \leq xy} \{\pi(xy)\} \leq \bigvee_{j \leq xy} \{\pi(j)\} = \pi(j).$$

Thus $\pi \star \pi \leq \pi$. If j cannot be expressed as $j \leq xy$ then $(\pi \star \pi)(x) = 0 \leq \pi(j)$.
Thus $(\pi \star \pi)(j) \leq \pi(j)\ \forall j \in X$.
And
Let $j \in X$.
Consider

$$(\vartheta \star \vartheta)(j) = \bigwedge_{j \leq xy} \vee\{\vartheta(x), \vartheta(y)\} \geq \bigwedge_{j \leq xy} \{\vartheta(xy)\}$$

$\geq \bigwedge_{j \leq xy} \{\vartheta(j)\}$
$= \vartheta(j).$
Thus $\vartheta \star \vartheta \geq \vartheta$. If j cannot be expressed as $j \leq ab$ then $(\vartheta \star \vartheta)(x) = 0 \geq \vartheta(j)$.
Thus $(\vartheta \star \vartheta)(j) \geq \vartheta(j)\ \forall j \in X$.
This implies that $\pi \star \pi \leq$ and $\vartheta \star \vartheta \geq \vartheta$. □

Theorem 3.3. *Let $D =< \pi, \vartheta >$ be an PFS of X. If $D =< \pi, \vartheta >$ is an Pythagorean fuzzy(PF) sub-semigroup (PFLI, PFRI) of X. Then $\pi - \pi = \pi$ and $\vartheta - \vartheta = \vartheta$.*

Proof. Let $D =< \pi, \vartheta >$ be an PF sub-semigroup of X.
Let $p \in X$ then.

$$(\pi - \pi)(j) = \bigvee_{j=x-y} \{\wedge\{\pi(x), \pi(y)\}\}\ x, y \in X$$
$$\geq \wedge\{\pi(j), \pi(0)\} \text{ since } j = j - 0$$
$$= \pi(j).$$
$$(\vartheta - \vartheta)(j) = \bigwedge_{j=x-y} \{\vee\{\vartheta(x), \vartheta(y)\}\}\ x, y \in X$$
$$\leq \vee\{\vartheta(j), \vartheta(0)\} \text{ since } j = j - 0$$
$$= \vartheta(j).$$

On the other hand if $j = x - y$, $x, y \in X$, then

$$\pi(j) = \pi(x - y)$$
$$\geq \wedge\{\pi(x), \pi(y)\}$$
$$\geq \bigvee_{j=x-y} \{\wedge\{\pi(x), \pi(y)\}\}$$
$$= (\pi - \pi)(j).$$
$$\vartheta(j) = \vartheta(x - y)$$
$$\leq \vee\{\vartheta(x), \vartheta(y)\}$$
$$\leq \bigwedge_{j=x-y} \{\vee\{\vartheta(x), \vartheta(y)\}\}$$
$$= (\vartheta - \vartheta)(j).$$

Hence $\pi(j) = (\pi - \pi)(j)$ and $\vartheta(j) = (\vartheta - \vartheta)(j)\ \forall\ j \in X$.
Thus $\pi = \pi - \pi$ and $\vartheta = \vartheta - \vartheta$. □

Theorem 3.4. *Let D_1 and D_2 be any two PFS of X. If D_1 and D_2 are PFLI (resp. PFRI) of X. Then D_1 and D_2 is also PFLI (resp. PFRI) of X.*

Proof. Let D_1 and D_2 be any two PFLIs of X.
Let $j, k \in X$.
Consider

$$(\pi_1 \cap \pi_2)(jk) = \wedge\{\pi_1(jk), \pi_2(jk)\}$$
$$\geq \wedge\{\wedge\{\pi_1(j), \pi_1(k)\}, \wedge\{\pi_2(j), \pi_2(k)\}\}$$
$$\geq \wedge\{\wedge\{\pi_1(j), \pi_2(j)\}, \wedge\{\pi_1(k), \pi_2(k)\}\}$$
$$= \wedge\{(\pi_1 \cap \pi_2)(j), (\pi_1 \cap \pi_2)(k)\}.$$
$$(\pi_1 \cap \pi_2)(jk) = \wedge\{\pi_1(jk), \pi_2(jk)\}$$
$$\geq \wedge\{\pi_1(k), \pi_2(k)\}$$
$$= (\pi_1 \cap \pi_2)(k).$$
$$(\vartheta_1 \cup \vartheta_2)(jk) = \vee\{\vartheta_1(jk), \vartheta_2(jk)\}$$
$$\leq \vee\{\vee\{\vartheta_1(j), \vartheta_1(k)\}, \vee\{\vartheta_2(j), \vartheta_2(k)\}\}$$
$$\leq \vee\{\vee\{\vartheta_1(j), \vartheta_2(j)\}, \vee\{\vartheta_1(k), \vartheta_2(k)\}\Big\}$$
$$= \vee\{(\vartheta_1 \cup \vartheta_2)(j), (\vartheta_1 \cup \vartheta_2)(k)\}.$$
$$(\vartheta_1 \cup \vartheta_2)(jk) = \vee\{\vartheta_1(jk), \vartheta_2(jk)\}$$

$\leq \vee\{\vartheta_1(k), \vartheta_2(k)\}$
$= (\vartheta_1 \cup \vartheta_2)(k).$

And

$$
\begin{aligned}
(\pi_1 \cap \pi_2)(j) &= \wedge\{\pi_1(j), \pi_2(j)\} \\
&\geq \wedge\{\bigwedge\{\pi_1(j-k), \pi_1(k)\}, \wedge\{\pi_2(j-k), \pi_2(k)\}\} \\
&\geq \wedge\{\wedge\{\pi_1(j-k), \pi_2(j-k)\}, \wedge\{\pi_1(k), \pi_2(k)\}\} \\
&= \wedge\{(\pi_1 \cap \pi_2)(j-k), (\pi_1 \cap \pi_2)(k)\}. \\
(\vartheta_1 \cup \vartheta_2)(j) &= \vee\{\vartheta_1(j), \vartheta_2(j)\} \\
&\leq \vee\{\vee\{\vartheta_1(j-k), \vartheta_1(k)\}, \vee\{\vartheta_2(j-k), \vartheta_2(k)\}\} \\
&\leq \vee\{\vee\{\vartheta_1(j-k), \vartheta_2(j-k)\}, \vee\{\vartheta_1(k), \vartheta_2(k)\}\} \\
&= \vee\{(\vartheta_1 \cup \vartheta_2)(j-k), (\vartheta_1 \cup \vartheta_2)(k)\}.
\end{aligned}
$$

This shows that the intersection of D_1 and D_2 two PFLI of X. □

Theorem 3.5. *If $D_i = (\pi_i, \vartheta_i | i \in \Omega)$ is a family of PFLI (resp. PFRI) of a SS X. Then $\bigcap_{i\in\Omega} D_i = (\cap_{i\in\Omega}\pi_i, \cup_{i\in\Omega}\vartheta_i)$ is also a PFLI (resp. PFRI) of X where Ω is any IS(index set).*

Proof. Let $D_i = (\pi_i, \vartheta_i | i \in \Omega)$ be a family of PFLI (resp. PFRI) of X.
Let $j, k \in X$ and $\pi(j) = \cap_{i\in\Omega}\pi_i(x) = \bigwedge \pi_i(j)$, $\vartheta(j) = \cup_{i\in\Omega}\vartheta_i(x) = \bigvee \vartheta_i(j)$.

$$
\begin{aligned}
\pi(j) &= \bigwedge \pi_i(j) \\
&\geq \bigwedge \wedge\{\pi_i(j-k), \pi_i(k)\} \\
&= \wedge\{\bigwedge \pi_i(j-k), \bigwedge \pi_i(k)\} \\
&= \wedge\{\cap\pi_i(j-k), \cap\pi_i(k)\} \\
&= \wedge\{\pi(j-k), \pi(k)\}. \\
\vartheta(j) &= \bigvee \vartheta_i(j) \\
&\leq \bigvee \vee\{\vartheta_i(j-k), \vartheta_i(k)\} \\
&= \vee\{\bigvee \vartheta_i(j-k), \bigvee \vartheta_i(k)\} \\
&= \vee\{\cup\vartheta_i(j-k), \cup\vartheta_i(k)\} \\
&= \vee\{\vartheta(j-k), \vartheta(k)\}. \\
\pi(jk) &= \bigwedge \pi_i(jk) \\
&\geq \bigwedge \wedge\{\pi_i(j), \pi_i(k)\} \\
&= \wedge\{\bigwedge \pi_i(j), \bigwedge \pi_i(k)\} \\
&= \wedge\{\cap\pi_i(j), \cap\pi_i(k)\} \\
&= \wedge\{\pi(j), \pi(k)\}. \\
\vartheta(jk) &= \bigvee \vartheta_i(jk) \\
&\leq \bigvee \vee\{\vartheta_i(j), \vartheta_i(k)\} \\
&= \vee\{\bigvee \vartheta_i(j), \bigvee \vartheta_i(k)\} \\
&= \vee\{\cup\vartheta_i(j), \cup\vartheta_i(k)\} \\
&= \vee\{\vartheta(j), \vartheta(k)\}.
\end{aligned}
$$

$\pi(jk) = \bigwedge \pi_i(jk) \geq \bigwedge \pi_i(k) \geq \pi(k)$

and

$\vartheta(jk) = \bigvee \vartheta_i(jk) \leq \bigvee \vartheta_i(k) \leq \vartheta(k)$.

Hence, $\bigcap_{i\in\Omega} D_i = (\cap_{i\in\Omega}\pi_i, \cup_{i\in\Omega}\vartheta_i)$ is also a PFLI (resp. PFRI) of X. □

Theorem 3.6. *If $D = (\pi, \vartheta)$ is any PFS of a SS X then $D = (\pi, \vartheta)$ is a PFLI (resp. PFRI) of X if and only if every Pythagorean level set $\bigcup(D, t, n)$ is a LI (resp. RI) of X when it is non-empty.*

Proof. Suppose that $D = (\pi, \vartheta)$ is a PFLI (resp. PFRI) of X. Let $j, k, j-k \in \bigcup(D, t, n)$ $\forall\ t \in [0, 1]$ and $n \in [0, 1]$. Then $\pi(j) \geq t$, $\pi(j-k) \geq t$, $\pi(k) \geq t$ and $\vartheta(j) \leq n$,

$\vartheta(j-k) \leq n, \vartheta(k) \leq n$.
Suppose $k, j-k \in \bigcup(D,t,n)$ then $\pi(j) \geq \wedge\{\pi(j-k), \pi(k)\} \geq \wedge\{t,t\} = t$ and $\vartheta(j) \vee \{\vartheta(j-k), \vartheta(k)\} \leq \vee\{n,n\} = n$. Hence , $jk \in \bigcup(D,t,n)$.
Suppose $j,k \in \bigcup(D,t,n)$ then $\pi(jk) \geq \wedge\{\pi(j), \pi(k)\} \geq \wedge\{t,t\} = t$ and $\vartheta(jk) \vee \{\vartheta(j), \vartheta(k)\} \leq \vee\{n,n\} = n$. Hence , $jk \in \bigcup(D,t,n)$.
Let $j \in X$ and $k \in \bigcup(D,t,n)$ then $\pi(jk) \geq \pi(k) \geq t$ and $\vartheta(jk) \leq \vartheta \leq n$.
This implies that $jk \in \bigcup(D,t,n)$. Hence $\bigcup(D,t,n)$ is a LI of X.
Conversely, let $t \in [0,1]$ and $n \in [0,1]$ be $\ni \bigcup(D,t,n) \neq 0$ and $\bigcup(D,t,n)$ is a LI (RI) of X.
We assume that $\pi(j) \ngeq \wedge\{\pi(j-k), \pi(k)\}$ or $\vartheta(j) \nleq \vee\{\vartheta(j-k), \vartheta(k)\}$. If $\pi(j) \ngeq \wedge\{\pi(j-k), \pi(k)\}$ then $\exists\, t \in [0,1] \ni \pi(j) < t < \wedge\{\pi(j-k), \pi(k)\}$ hence $j-k, k \in \bigcup(D, t, \vee\{\vartheta(j-k), \vartheta(k)\})$ but $j \notin \bigcup(D, t, \vee\{\vartheta(j-k), \vartheta(k)\})$ which is contradiction.
If $\vartheta(j) \nleq \vee\{\vartheta(j-k), \vartheta(k)\}$ then $\exists\, n \in [0,1] \ni \vartheta(j) > n > \vee\{\vartheta(j-k), \vartheta(k)\}$ hence $j-k, k \in \bigcup(D, \wedge\{\pi(j-k), \pi(k)\}, n)$ but $j \notin \bigcup(D, \wedge\{\pi(j-k), \pi(k)\})$ which is contradiction. Hence, $\pi(j) \geq \wedge\{\pi(j-k), \pi(k)\}$ and $\vartheta(j) \leq \vee\{\vartheta(j-k), \vartheta(k)\}$.
Let us assume that $\pi(jk) \ngeq \wedge\{\pi(j), \pi(k)\}$ or $\vartheta(jk) \nleq \vee\{\vartheta(j), \vartheta(k)\}$. If $\pi(jk) \ngeq \wedge\{\pi(j), \pi(k)\}$ then $\exists\, t \in [0,1] \ni \pi(jk) < t < \wedge\{\pi(j), \pi(k)\}$
hence $j,k \in \bigcup(D, t, \vee\{\vartheta(j), \vartheta(k)\})$ but $jk \notin \bigcup(D, t, \vee\{\vartheta(j), \vartheta(k)\})$ which is contradiction.
If $\vartheta(jk) \nleq \vee\{\vartheta(j), \vartheta(k)\}$ then $\exists\, n \in [0,1] \ni \vartheta(jk) > n > \vee\{\vartheta(j), \vartheta(k)\}$ hence $j,k \in \bigcup(D, \wedge\{\pi(j), \pi(k)\}, n)$ but $jk \notin \bigcup(D, \wedge\{\pi(j), \pi(k)\})$ which is contradiction. Hence, $\pi(jk) \geq \wedge\{\pi(j), \pi(k)\}$ and $\vartheta(jk) \leq \vee\{\vartheta(j), \vartheta(k)\}$.
Assume that $\pi(jk) \ngeq \pi(k)$ or $\vartheta(jk) \nleq \vartheta(k)$. If $\pi(jk) \ngeq \pi(k)$ then $\exists\, t \in [0,1] \ni \pi(jk) < t < \pi(k)$ hence $k \in \bigcup(D, t, \vartheta(k))$ but $jk \notin \bigcup(D, t, \vartheta(k))$ which is contradiction. If $\vartheta(jk) \nleq \vartheta(k)$ then $\exists\, n \in [0,1] \ni \vartheta(jk) > n > \vartheta(k)$ hence $k \in \bigcup(D, \pi(k), n)$ but $jk \notin \bigcup(D, \pi(k), n)$ which is contradiction.
Hence, $\pi(jk) \geq \pi(k)$ and $\vartheta(jk) \leq \vartheta(k)$.
Therefore, $D = (\pi, \vartheta)$ is a PFLI (resp. PFRI) of X. □

Theorem 3.7. *Let N be any non-empty subset of a SS X. Then N is a LI (resp. RI) of X if and only if the characteristic PFS $\chi_N = (\pi_{\chi_N}, \vartheta_{\chi_N})$ of N is X is a PFLI (resp. PFRI) of X.*

Proof. Assume that N is a LI of X. Let $j,k \in X$. Suppose that $\pi_{\chi_N}(j) < \wedge\{\pi_{\chi_N}(j-k), \pi_{\chi_N}(k)\}$ and $\vartheta_{\chi_N}(j) > \vee\{\vartheta_{\chi_N}(j-k), \vartheta_{\chi_N}(k)\}$. It follows that $\pi_{\chi_N}(j) = 0, \wedge\{\pi_{\chi_N}(j-k), \pi_{\chi_N}(k)\} = 1$ and $\vartheta_{\chi_N}(j) = 1, \vee\{\vartheta_{\chi_N}(j-k), \vartheta_{\chi_N}(k)\} = 0$. This implies that $j-k, k \in N$ but $j \notin N$, a contradicts to N being a SS of X.
Suppose that $\pi_{\chi_N}(jk) < \wedge\{\pi_{\chi_N}(j), \pi_{\chi_N}(k)\}$ and $\vartheta_{\chi_N}(jk) > \vee\{\vartheta_{\chi_N}(j), \vartheta_{\chi_N}(k)\}$. It follows that $\pi_{\chi_N}(jk) = 0, \wedge\{\pi_{\chi_N}(j), \pi_{\chi_N}(k)\} = 1$ and $\vartheta_{\chi_N}(jk) = 1$,
$\vee\, \{\vartheta_{\chi_N}(j), \vartheta_{\chi_N}(k)\} = 0$. This implies that $j,k \in N$ but $jk \notin N$, a contradicts to N.
Suppose that $\pi_{\chi_N}(jk) < \pi_{\chi_N}(k)$ and $\vartheta_{\chi_N}(jk) > \vartheta_{\chi_N}(k)$. It follows that $\pi_{\chi_N}(jk) = 0, \pi_{\chi_N}(k) = 1$ and $\vartheta_{\chi_N}(jk) = 1, \vartheta_{\chi_N}(k) = 0$. This implies that $k \in N$ but $jk \notin N$, a contradicts to N. This shows that $\chi_N = (\pi_{\chi_N}, \vartheta_{\chi_N})$ is a PFLI (resp. PFRI) of X.
Conversely, $\chi_N = (\pi_{\chi_N}, \vartheta_{\chi_N})$ is a PFLI(resp. PFRI) of X for any subset N of X.
Let $j-k, k \in N$ for any $j,k \in X$ then $\pi_{\chi_N}(j-k) = \pi_{\chi_N}(k) = 1$ and $\vartheta_{\chi_N}(j-k) = \vartheta_{\chi_N}(k) = 0$. Since χ_N is a PFLI (resp. PFRI) of X.
$\pi_{\chi_N}(j) \geq \wedge\{\pi_{\chi_N}(j-k), \pi_{\chi_N}(k)\} \geq \wedge\{1,1\} = 1$ and $\vartheta_{\chi_N}(j) \leq \vee\{\vartheta_{\chi_N}(j-k), \vartheta_{\chi_N}(k)\} \leq \wedge\{0,0\} = 0$. This implies that $j \in N$.
Let $j,k \in N$ for any $j,k \in X$ then $\pi_{\chi_N}(j) = \pi_{\chi_N}(k) = 1$ and $\vartheta_{\chi_N}(j) = \vartheta_{\chi_N}(k) = 0$.

Since χ_N is a PFLI (resp. PFRI) of X.
$\pi_{\chi_N}(jk) \geq \wedge\{\pi_{\chi_N}(j), \pi_{\chi_N}(k)\} \geq \wedge\{1,1\} = 1$ and $\vartheta_{\chi_N}(jk) \leq \vee\{\vartheta_{\chi_N}(j), \vartheta_{\chi_N}(k)\} \leq \wedge\{0,0\} = 0$. This implies that $jk \in N$.
Let $k \in N$ and $j \in X$ then $\pi_{\chi_N}(k) = 1$ and $\vartheta_{\chi_N}(k) = 0$. $\pi_{\chi_N}(jk) \geq \pi_{\chi_N}(k) \geq 1$ and $\vartheta_{\chi_N}(jk) \leq \vartheta_{\chi_N}(k) \geq 0$. This gives that $jk \in N$.
Hence, N is a LI (RI) of X. □

4. Pythagorean fuzzy ideal of near-subtraction semigroup

Definition 4.1. Let X be a near-subtraction semigroup(NSS), (X, D) be a PFI. A PFS $D = (\pi, \vartheta)$ is called a PFI of X, if
(1) $\pi(j-k) \geq \wedge\{\pi(j), \pi(k)\}$ and $\vartheta(j-k) \leq \vee\{\vartheta(j), \vartheta(k)\}$
(2) $\pi(xj - x(y-j)) \geq \pi(j)$ and $\vartheta(xj - x(y-j)) \leq \vartheta(j)$
(3) $\pi(jk) \geq \pi(j)$ and $\vartheta(jk) \leq \vartheta(j)$ $\forall j, k, l, x, y \in X$.
If $D = (\pi, \vartheta)$ is a PFLI of X if it satisfies (1),(2) and if $D = (\pi, \vartheta)$ is a PFRI of X if it satisfies (1) and (3).

Theorem 4.1. *If $D_i = (\pi_i, \vartheta_i | i \in \Omega)$ is a family of PFI of a NSS X then $\bigcap_{i\in\Omega} D_i = (\cap_{i\in\Omega}\pi_i, \cup_{i\in\Omega}\vartheta_i)$ is also a PFI of X where Ω is any IS(index set).*

Proof. If $\{D_i\}_{i\in\Omega}$ is a family of PFI of X.
Let $\cap\pi_i(x) = (\bigwedge \pi_i)(j) = \bigwedge \pi_i(j)$ and $\cup\vartheta_i(x) = (\bigvee \vartheta_i)(j) = \bigvee \vartheta_i(j)$. $\forall j, k \in X$, we have

$$\begin{aligned}(\cap_{i\in\Omega}\pi_i)(j-k) &= \bigwedge\{\pi_i(j-k) | i \in \Omega\}\\ &\geq \bigwedge \wedge\{\pi_i(j), \pi(k) | i \in \Omega\}\\ &= \wedge\{\bigwedge\{\pi_i(j) | i \in \Omega\}, \bigwedge\{\pi_i(k) | i \in \Omega\}\}\\ &= \wedge\{\cap_{i\in\Omega}\pi_i(j), \cap_{i\in\Omega}\pi_i(k)\}.\end{aligned}$$

$$\begin{aligned}(\cup_{i\in\Omega}\vartheta_i)(j-k) &= \bigvee\{\vartheta_i(j-k) | i \in \Omega\}\\ &\leq \bigvee \vee\{\vartheta_i(j), \vartheta(k) | i \in \Omega\}\\ &= \vee\{\bigvee\{\vartheta_i(j) | i \in \Omega\}, \bigvee\{\vartheta_i(k) | i \in \Omega\}\}\\ &= \vee\{\cup_{i\in\Omega}\vartheta_i(j), \cup_{i\in\Omega}\vartheta_i(k)\}.\end{aligned}$$

For all $j, x, y \in X$, we have

$$\begin{aligned}(\cap_{i\in\Omega}\pi_i)(xj - x(y-j)) &= \bigwedge\{\pi_i(xj - x(y-j)) | i \in \Omega\}\\ &\geq \bigwedge\{\pi_i(j) | i \in \Omega\}\\ &= \cap_{i\in\Omega}\pi_i(j).\end{aligned}$$

$$\begin{aligned}(\cup_{i\in\Omega}\vartheta_i)(xj - x(y-j)) &= \bigvee\{\vartheta_i(xj - x(y-j)) | i \in \Omega\}\\ &\leq \bigvee\{\vartheta_i(j) | i \in \Omega\}\\ &= \cup_{i\in\Omega}\vartheta_i(j).\end{aligned}$$

For all $j, k \in X$, we have

$$\begin{aligned}(\cap_{i\in\Omega}\pi_i)(jk) &= \bigwedge\{\pi_i(jk) | i \in \Omega\}\\ &\geq \bigwedge\{\pi_i(j) | i \in \Omega\}\\ &= \cap_{i\in\Omega}\pi_i(j).\end{aligned}$$

$$\begin{aligned}(\cup_{i\in\Omega}\vartheta_i)(jk) &= \bigvee\{\vartheta_i(jk) | i \in \Omega\}\\ &\leq \bigvee\{\vartheta_i(j) | i \in \Omega\}\\ &= \cup_{i\in\Omega}\vartheta_i(j).\end{aligned}$$

Hence, $\bigcap_{i\in\Omega} D_i = (\cap_{i\in\Omega}\pi_i, \cup_{i\in\Omega}\vartheta_i)$ is also a PFI of X. □

Definition 4.2. An PFS $D = (\pi, \vartheta)$ of X is said to be an PFBI of X if $\forall j, k \in X$
(i) $\pi(j-k) \geq \wedge\{\pi(j), \pi(k)\}$
(ii) $\vartheta(j-k) \leq \vee\{\vartheta(j), \vartheta(k)\}$
(iii) $(\pi \circ X \circ \pi) \cap (\pi \circ X) \star \pi \subseteq \pi$

(iv) $(\vartheta \circ X \circ \vartheta) \cup (\vartheta \circ X) \star \vartheta \supseteq \vartheta$.

Example 4.3. Let $X = \{0, j, k, l\}$ be a NSS with two binary operations $'-'$ and $'\cdot'$ is defined as follows.

$-$	0	j	k	l	$\cdot$	0	j	k	l
0	0	0	0	0	0	0	0	0	0
j	j	0	j	j	j	j	j	j	j
k	k	k	0	k	k	0	0	0	k
l	l	l	l	0	l	0	0	0	l

Define a PFS $D = (\pi, \vartheta)$ where $\pi : X \to [0, 1]$ by $\pi(0) = 0.8, \pi(j) = 0.6, \pi(k) = 0.3, \pi(l) = 0.2$. $(\pi \circ X \circ \pi)(0) = 0.8, (\pi \circ X \circ \pi)(j) = 0.7, (\pi \circ X \circ \pi)(k) = 0.5, (\pi \circ X \circ \pi)(l) = 0.2, (\pi \circ X) \star \pi(0) = 0.7, (\pi \circ X) \star \pi(j) = 0.6, (\pi \circ X) \star \pi(k) = 0.3, (\pi \circ X) \star \pi(l) = 0.1$ Then $\vartheta : X \to [0, 1]$ by $\vartheta(0) = 0.3, \vartheta(j) = 0.4, \vartheta(k) = 0.6, \vartheta(l) = 0.7$ $(\pi \circ X \circ \pi)(0) = 0.4, (\pi \circ X \circ \pi)(j) = 0.5, (\pi \circ X \circ \pi)(k) = 0.6, (\pi \circ X \circ \pi)(l) = 0.8, (\pi \circ X) \star \pi(0) = 0.3, (\pi \circ X) \star \pi(j) = 0.6, (\pi \circ X) \star \pi(k) = 0.7 and (\pi \circ X) \star \pi(l) = 0.8.$

Proposition 4.2. *Let $D = (\pi, \vartheta)$ be a PFS of X. If $D = (\pi, \vartheta)$ is a PFLI of X then $D = (\pi, \vartheta)$ is a PFBI of X.*

Proof. Let $j^{'} \in X$ be such that $j^{'} = xyz = jl - j(k - l)$, where $x, y, z, j, k, l \in X$. Then

$$((\pi X \pi) \cap (\pi X \star \pi))(j^{'}) = \wedge\{((\pi X \pi)(j^{'}), (\pi X \star \pi))(j^{'})\}$$
$$= \wedge\{ \bigvee_{j^{'}=xyz} \wedge\{\pi(x).\pi(y), \pi(z)\}\}, \bigvee_{j^{'}=jl-j(k-l)} \wedge\{(\pi X)(j), \pi(l)\}$$
$$= \wedge\{\bigvee\{\pi(x), \pi(z)\}, \bigvee\{(\pi X)(j), \pi(l)\}\}.$$

(Since $\pi X \subseteq X$ and π is a PFLI, then $\pi(jl - j(k - l)) \geq \pi(l)$)

$$\leq \wedge\{X(x), X(z), X(j), \pi(jl - j(k - l))\}$$
$$= \wedge\{1, 1, 1, \pi(jl - j(k - l))\}$$
$$= \pi(jl - j(k - l))$$
$$= \pi(j^{'}).$$

If $j^{'}$ is not expressible as $j^{'} = xyz = jl - j(k - l)$ then $(\pi X \pi \cap \pi X \star \pi)(j^{'}) = 0 \leq \pi(j^{'})$. Then $\pi X \pi \cap \pi X \star \pi \subseteq \pi$. Hence D is a PFBI of X.

And

$$((\vartheta X \vartheta) \cup (\vartheta X \star \vartheta))(j^{'}) = \vee\{((\vartheta X \vartheta)(j^{'}), (\vartheta X \star \vartheta))(j^{'})\}$$
$$= \vee\{ \bigwedge_{j^{'}=xyz} \vee\{\vartheta(x), \vartheta(y), \vartheta(z)\}\}, \bigwedge_{j^{'}=jl-j(k-l)} \vee\{(\vartheta X)(j), \vartheta(l)\}$$
$$= \vee\{\bigwedge\{\vartheta(x), \vartheta(z)\}, \bigwedge\{(\vartheta X)(j), \vartheta(l)\}\}.$$

(Since $\vartheta X \supseteq X$ and ϑ is a PFLI, then $\vartheta(jl - j(k - l)) \leq \vartheta(l)$)

$$\geq \vee\{X(x), X(z), X(j), \vartheta(jl - j(k - l))\}$$
$$= \vee\{0, 0, 0, \vartheta(jl - j(k - l))\}$$
$$= \vartheta(jl - j(k - l))$$
$$= \vartheta(j^{'}).$$

If $j^{'}$ is not expressible as $j^{'} = xyz = jl - j(k - l)$ then $(\vartheta X \vartheta \cup \vartheta X \star \vartheta)(j^{'}) = 0 \geq \vartheta(j^{'})$. Then $\vartheta X \vartheta \cup \vartheta X \star \vartheta \supseteq \vartheta$. Hence D is a PFBI of X. □

Proposition 4.3. *Let $D = (\pi, \vartheta)$ be a PFS of X. If $D = (\pi, \vartheta)$ is a PFRI of X then $D = (\pi, \vartheta)$ is a PFBI of X.*

Proof. Let $j^{'} \in X$ be $\ni\ j^{'} = xy = jl - j(k-l),\ x = x_1x_2$, where x, x_1, x_2, y, j, k and l are in X. Consider,

$((\pi X\pi) \cap (\pi X \star \pi))(j^{'}) = \wedge\{(\pi X\pi)(j^{'}), (\pi X \star \pi)(j^{'})\}$
$= \wedge\{\bigvee_{j^{'}=xy} \wedge\{(\pi X)(x), \pi(y)\}, (\pi X \star \pi)(jl - j(k-l))\}$
$= \wedge\{\bigvee_{j^{'}=xy} \wedge\{\bigvee_{x=x_1x_2} \wedge\{\pi(x_1), X(x_2)\}, \pi(y)\}, (\pi X\star\pi)(jl-j(k-l))\}$
$= \wedge\{\bigvee_{j^{'}=xy} \wedge\{\bigvee_{x=x_1x_2} \{\pi(a_1)\}, \pi(b)\}, (\pi X \star \pi)(jl - j(k-l))\}$
$= \wedge\{\pi(x_1), \pi(y), (\pi X \star \pi)(jl - j(k-l))\}$,
$(since\ D = (\pi, \vartheta) is\ a\ PFRI,\ we\ have\ \pi(xy) = \pi(x_1x_2y) = \pi(x_1(x_2y)) \geq \pi(x_1))$
$\leq \wedge\{\pi(xy), 1, 1\} = \pi(xy) = \pi(j^{'})$.

If $j^{'}$ is not expressible as $j^{'} = xyz = jl-j(k-l)$ then $(\pi X\pi \cap \pi X \star\pi)(j^{'}) = 0 \leq \pi(j^{'})$. Then $\pi X\pi \cap \pi X \star \pi \subseteq \pi$. Hence D is a PFBI of X.

And

$((\vartheta X\vartheta) \cup (\vartheta X \star \vartheta))(j^{'}) = \vee\{(\vartheta X\vartheta)(j^{'}), (\vartheta X \star \vartheta)(j^{'})\}$
$= \vee\{\bigwedge_{j^{'}=xy} \vee\{(\vartheta X)(x), \vartheta(y)\}, (\vartheta X \star \vartheta)(jl - j(k-l))\}$
$= \vee\{\bigwedge_{j^{'}=xy} \vee\{\bigwedge_{x=x_1x_2} \vee\{\vartheta(x_1), X(x_2)\}, \vartheta(y)\}, (\vartheta X\star\vartheta)(jl-j(k-l))\}$
$= \vee\{\bigwedge_{j^{'}=xy} \vee\{\bigwedge_{x=x_1x_2} \{\vartheta(x_1)\}, \vartheta(y)\}, (\vartheta X \star \vartheta)(jl - j(k-l))\}$
$= \vee\{\vartheta(x_1), \vartheta(y), (\vartheta X \star \vartheta)(jl - j(k-l))\}$,
$(since\ D = (\pi, \vartheta) is\ a\ PFRI,\ we\ have\ \vartheta(xy) = \vartheta(x_1x_2y) = \vartheta(x_1(x_2y)) \leq \vartheta(x_1))$
$\geq \vee\{\vartheta(xy), 1, 1\} = \vartheta(xy) = \vartheta(j^{'})$.

If $j^{'}$ is not expressible as $j^{'} = xyz = jl-j(k-l)$ then $(\vartheta X\vartheta \cup \vartheta X \star\vartheta)(j^{'}) = 0 \leq \vartheta(j^{'})$. Then $\vartheta X\vartheta \cup \vartheta X \star \vartheta \subseteq \vartheta$. Hence D is a PFBI of X. □

Theorem 4.4. *Let $D = (\pi, \vartheta)$ be a PFSA(Pythagorean fuzzy subalgebra) of X. If $DXD \subseteq D$, then D is a PFBI of X.*

Proof. Assume that π is a PFSA of X and $\pi X\pi \subseteq \pi$. Let $j \in X$. Then
$(\pi X\pi \cap \pi X \star \pi)(j) = \wedge\{(\pi X\pi)(j), (\pi X \star \pi)(j)\} \leq (\pi X\pi)(j) \leq \pi(j)$.
Thus $(\pi X\pi \cap \pi X \star \pi) \subseteq \pi$ and π is a PFBI of X and
assume that ϑ is a PFSA of X and $\vartheta X\vartheta \supseteq \vartheta$. Let $j \in X$. Then
$(\vartheta X\vartheta \cup \vartheta X \star \vartheta)(j) = \vee\{(\vartheta X\vartheta)(j), (\vartheta X \star \vartheta)(j)\} \geq (\vartheta X\vartheta)(j) \geq \vartheta(j)$.
Thus $(\vartheta X\vartheta \cup \vartheta X \star \vartheta) \supseteq \vartheta$ and ϑ is a PFBI of X. □

Theorem 4.5. *If X is a ZSNSS(zero symmetric near-subtraction semigroup) and D is a PFBI of X then $\pi X\pi \subseteq \pi$ and $\vartheta X\vartheta \supseteq \vartheta$.*

Proof. Let π be a PFBI of X. Then $\pi X\pi \cap \pi X \star \pi \subseteq \pi$. Clearly $\pi(0) \geq \pi(j)$. Thus $(\pi X)(0) \geq (\pi X)(j)\ \forall j \in X$. Since X is a ZSNSS, $\pi X\pi \subseteq \pi X \star \pi$. So $\pi X\pi \cap \pi X \star \pi = \pi X\pi \subseteq \pi$,
and let ϑ be a PFBI of X. Then $\vartheta X\vartheta \cup \vartheta X \star \vartheta \supseteq \vartheta$. Clearly $\vartheta(0) \leq \vartheta(j)$. Thus $(\vartheta X)(0) \leq (\vartheta X)(j)\ \forall j \in X$. Since X is a ZSNSS, $\vartheta X\vartheta \supseteq \vartheta X \star \vartheta$. So $\vartheta X\vartheta \cup \vartheta X \star \vartheta = \vartheta X\vartheta \supseteq \vartheta$. Which is the required result. □

Theorem 4.6. *Let $D = (\pi, \vartheta)$ be a PFBI of a ZSNSS X. Then $\pi(jkl) \geq \wedge\{\pi(j), \pi(l)\}$ and $\pi(jkl) \leq \vee\{\pi(j), \pi(l)\}$.*

Proof. Assume that π is a PFBI of ZSNSS X. BY Theorem 3.19 $\pi X\pi \subseteq \pi$. Let $j,k,l \in X$. Then

$$\begin{aligned}\pi(jkl) &\geq (\pi X\pi)(jkl)\\ &= \bigvee_{jkl=xy} \wedge\{(\pi X)(x), \pi(y)\}\\ &\geq \wedge\{(\pi X)(jk), \pi(l)\}\\ &\geq \wedge\{(\pi X)(j), X(k), \pi(l)\}\\ &= \wedge\{(\pi X)(j), 1, \pi(l)\}\\ &= \wedge\{(\pi X)(j), \pi(l)\}.\end{aligned}$$

Thus $\pi(pqr) \geq \wedge\{\pi(j), \pi(l)\}$.

And

$$\begin{aligned}\vartheta(jkl) &\geq (\vartheta X\vartheta)(jkl)\\ &= \bigwedge_{jkl=xy} \vee\{(\vartheta X)(x), \vartheta(y)\}\\ &\leq \vee\{(\vartheta X)(jk), \vartheta(l)\}\\ &\leq \vee\{(\vartheta X)(j), X(k), \vartheta(l)\}\\ &= \vee\{(\vartheta X)(j), 1, \vartheta(l)\}\\ &= \vee\{(\vartheta X)(j), \vartheta(l)\}.\end{aligned}$$

Thus $\vartheta(jkl) \leq \vee\{\vartheta(j), \vartheta(l)\}$. □

Theorem 4.7. *Let $D_1 = (\pi_1, \vartheta_1)$ and $D_2 = (\pi_2, \vartheta_2)$ be any two PFBI of X. Then $D_1 \cap D_2$ is also a PFBI of X.*

Proof. Let π_1 and π_2 be any two PFBI of X. Let $j,k \in X$. Then

$$\begin{aligned}(\pi_1 \cap \pi_2)(j-k) &= \wedge\{\pi_1(j-k), \pi_2(j-k)\}\\ &\geq \wedge\{\wedge\{\pi_1(j), \pi_1(k)\}, \wedge\{\pi_2(j), \pi_2(k)\}\}\\ &= \wedge\{\wedge\{\pi_1(j), \pi_2(j)\}, \wedge\{\pi_1(k), \pi_2(k)\}\}\\ &= \wedge\{(\pi_1 \cap \pi_2)(j), (\pi_1 \cap \pi_2)(k)\}.\end{aligned}$$

Let $j^{'} \in X$. Choose $x,y,j,k,l \in X$ such that $j^{'} = xyz = jl - j(k-l)$. Since π_1 and π_2 are PFBI of X, we have

$$\begin{aligned}&= \wedge\{\bigvee_{j^{'}=xyz} \wedge\{\pi_1(x), \pi_1(z)\}, \bigvee_{j^{'}=jl-j(k-l)} \pi_1\} \leq \pi_1(j)\\ &= \wedge\{\bigvee_{j^{'}=xyz} \wedge\{\pi_2(x), \pi_2(z)\}, \bigvee_{j^{'}=jl-j(k-l)} \pi_2\} \leq \pi_2(j).\end{aligned}$$

Now

$$\begin{aligned}&= \wedge\{((\pi_1 \cap \pi_2)X(\pi_1 \cap \pi_2))(j^{'}), ((\pi_1 \cap \pi_2)X \star (\pi_1 \cap \pi_2))(j^{'})\}\\ &= \wedge\{\bigvee_{j^{'}=xyz} \wedge\{(\pi_1 \cap \pi_1)(x), (\pi_1 \cap \pi_2)(z)\}, \bigvee_{j^{'}=jl-j(k-l)} (\pi_1 \cap \pi_2)(l)\}\\ &= \wedge\{\bigvee_{j^{'}=xyz} \wedge\{\wedge\{\pi_1(x), \pi_2(x)\}, \wedge\{\pi_1(z), \pi_2(z)\}\}, \bigvee_{j^{'}=jl-j(k-l)} \wedge\{\pi_1(l), \pi_2(l)\}\}\\ &= \wedge\{\wedge\{\bigvee_{j^{'}=xyz} \wedge\{\pi_1(x), \pi_1(z)\}, \bigvee_{j^{'}=jl-j(k-l)} \pi_1(l)\}, \wedge\{\bigvee_{j^{'}=xyz} \wedge\{\pi_2(x), \pi_2(z)\},\\ &\qquad \bigvee_{j^{'}=jl-j(k-l)} \pi_2(l)\}\}\\ &\leq \wedge\{\pi_1(j), \pi_2(j)\}\\ &= (\pi_1 \cap \pi_2)(j).\end{aligned}$$

Thus $\pi_1 \cap \pi_2$ is a PFBI of X. and

$$\begin{aligned}(\vartheta_1 \cup \vartheta_2)(j-k) &= \vee\{\vartheta_1(j-k), \vartheta_2(j-k)\}\\ &\leq \vee\{\vee\{\vartheta_1(j), \vartheta_1(k)\}, \vee\{\vartheta_2(j), \vartheta_2(k)\}\}\\ &= \vee\{\vee\{\vartheta_1(j), \vartheta_2(j)\}, \vee\{\vartheta_1(k), \vartheta_2(k)\}\}\\ &= \vee\{(\vartheta_1 \cup \vartheta_2)(j), (\vartheta_1 \cup \vartheta_2)(k)\}.\end{aligned}$$

Let $j^{'} \in X$. Choose $x, y, j, k, l \in X, \ni\ j^{'} = xyz = jl - j(k-l)$. Since ϑ_1 and ϑ_2 are PFBI of X, we have

$$= \vee\{\bigwedge_{j^{'}=xyz} \vee\{\vartheta_1(x), \vartheta_1(z)\}, \bigwedge_{j^{'}=jl-j(k-l)} \vartheta_1\} \geq \vartheta_1(j)$$
$$= \vee\{\bigwedge_{j^{'}=xyz} \vee\{\vartheta_2(x), \vartheta_2(z)\}, \bigwedge_{j^{'}=jl-j(k-l)} \vartheta_2\} \geq \vartheta_2(j).$$

Now

$$= \vee\{((\vartheta_1 \cup \vartheta_2)X(\vartheta_1 \cup \vartheta_2))(j^{'}), ((\vartheta_1 \cup \vartheta_2)X \star (\vartheta_1 \cup \vartheta_2))(j^{'})\}$$
$$= \vee\{\bigwedge_{j^{'}=xyz} \vee\{(\vartheta_1 \cup \vartheta_1)(x), (\vartheta_1 \cup \vartheta_2)(z)\}, \bigwedge_{j^{'}=jl-j(k-l)} (\vartheta_1 \cup \vartheta_2)(l)\}$$
$$= \vee\{\bigwedge_{j^{'}=xyz} \vee\{\vee\{\vartheta_1(x), \vartheta_2(x)\}, \vee\{\vartheta_1(z), \vartheta_2(z)\}\}, \bigwedge_{j^{'}=jl-j(k-l)} \vee\{\vartheta_1(l), \vartheta_2(l)\}\}$$
$$= \vee\{\vee\{\bigwedge_{j^{'}=xyz} \vee\{\vartheta_1(x), \vartheta_1(z)\}, \bigwedge_{j^{'}=jl-j(k-l)} \vartheta_1(l)\}, \vee\{\bigwedge_{j^{'}=xyz} \vee\{\vartheta_2(x), \vartheta_2(z)\}, \bigwedge_{j^{'}=jl-j(k-l)} \vartheta_2(l)\}\}$$
$$\geq \vee\{\vartheta_1(j), \vartheta_2(j)\}$$
$$= (\vartheta_1 \cup \vartheta_2)(j).$$

Thus $\vartheta_1 \cup \vartheta_2$ is a PFBI of X. □

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