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ON m-EXPANSIVE AND m-CONTRACTIVE TUPLE OF OPERATORS IN HILBERT SPACES

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ABSTRACT. In this paper we introduce and studied the concept of joint m-expansive and joint m-contractive tuples of commuting operators of a Hilbert space.

1. INTRODUCTION

In this paper $\mathcal{H}$ will denote a infinite-dimensional Hilbert space on $\mathbb{K} = \mathbb{C}$ (the complex plane). $\mathbb{N}$ is the set of positive integers and $\mathbb{Z}_+ = \mathbb{N} \cup \{0\}$. Let $\mathcal{B}(\mathcal{H})$ be the set of bounded linear operators from $\mathcal{H}$ into itself. An operator $S \in \mathcal{B}(\mathcal{H})$ we denote by $\mathcal{N}(S)$ and $\mathcal{R}(S)$ the null space and the range of S respectively. For $S \in \mathcal{B}(\mathcal{H})$, we set

$$\theta_m(S) := \sum_{0 \leq j \leq m} (-1)^j \binom{m}{j} S^{*j} S^j. \tag{1.1}$$

J. Agler and M. Stankus introduced the class of m-isometry on Hilbert space [1, 2, 3]. An operator $S \in \mathcal{B}(\mathcal{H})$ is said to be m-isometric operator for some integer $m \geq 1$ if it satisfies the operator equation $\theta_m(S) = 0$.

Notice that the defining property $\theta_m(S) = 0$ of an m-isometric operator is equivalently formulated that

$$\sum_{0 \leq k \leq m} (-1)^k \binom{m}{k} \|S^k x\|^2 = 0 \qquad (\forall\ x \in \mathcal{H}).$$

The Concept of m-isometric operators on Hilbert and Banach spaces has attracted much attention of various authors (see [8, 9, 10, 11, 12, 13, 17, 21, 22, 19]. A generalization of m-isometries to m-expansive and m-contractive operators on Hilbert spaces spaces has been presented by several authors. We refer the reader to [4, 5, 6, 7, 14, 15, 18, 20, 23] for recent articles about these subjects.

Definition 1.1. ([14]) An operator $S \in \mathcal{B}(\mathcal{H})$ is said to be

(i) m-expansive $(m \geq 1)$ if $\theta_m(S) \leq 0$.

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(ii) m-hyperexpensive $(m \geq 1)$, if $\theta_k(S) \leq 0$ for $k = 1, 2, ..., m$.

(iii) Completely hyperexpansive if $\theta_m(S) \leq 0$ for all m.

Definition 1.2. ([15]) An operator $S \in \mathcal{B}(\mathcal{H})$ is said to be

(i) m-contractive $(m \geq 1)$ if $\theta_m(S) \geq 0$.

(ii) m-hypercontractive $(m \geq 1)$, if $\theta_k(S) \geq 0$ for $k = 1, 2, ..., m$.

(iii) Completely hypercontractive if $\theta_m(S) \geq 0$ for all m.

The study of tuple of commuting operators on Hilbert space has consider by many authors in the recent years. In [16] the authors introduced the concept of m-isomeric tuple of commuting operators as follows: Let $\mathbf{S} = (S_1, \cdots, S_d) \in \mathcal{B}(\mathcal{H})^d$ be tuple of commuting operators. $\mathbf{S}$ is said to be m-isometric tuple if

$$\sum_{0 \leq k \leq m} (-1)^k \bigg(\sum_{|\alpha|=k} \frac{k!}{\beta!} \mathbf{S}^{*\beta} \mathbf{S}^{\beta} \bigg) = 0,$$

where $\beta = (\beta_1, \cdots, \beta_d) \in \mathbb{Z}_+^d$ and $\beta! = \beta_1! \cdots \beta_d!$.

2. Main results

In this section, we introduce and study the concepts on m-expansive and m-contractive tuples of operators on a Hilbert space.

Let $\mathbf{S} = (S_1, \cdots, S_d) \in \mathcal{B}(\mathcal{H})^d$ and set

$$\Psi_l(\mathbf{S}) = \sum_{0 \leq k \leq l} (-1)^k \bigg(\sum_{|\alpha|=k} \frac{k!}{\beta!} \mathbf{S}^{*\beta} \mathbf{S}^{\beta} \bigg), \tag{2.1}$$

and

$$\mathbf{Q}_l(\mathbf{S}, x) := \langle \mathbf{S}x,\ x \rangle = \sum_{0 \leq k \leq l} (-1)^k \binom{m}{k} \bigg(\sum_{|\beta|=k} \frac{k!}{\beta!} \|\mathbf{S}^{\beta} x\|^2 \bigg). \tag{2.2}$$

Clearly,

$$\Psi_l(\mathbf{S}) \geq 0 \Longleftrightarrow \mathbf{Q}_l(\mathbf{S}, x) \geq 0 \quad \forall\, x \in \mathcal{H},$$

and

$$\Psi_l(\mathbf{S}) \leq 0 \Longleftrightarrow \mathbf{Q}_l(\mathbf{S}, x) \leq 0 \quad \forall\, x \in \mathcal{H},$$

Definition 2.1. Let $\mathbf{S} = (S_1, \cdots, S_d) \in \mathcal{B}(\mathcal{H})^d$ be a commuting tuple of operators and $m \in \mathbb{N}$. We said that

(1) $\mathbf{S}$ is joint m-expansive if $\Psi_m(\mathbf{S}) \leq 0$ for some integer m.

(2) $\mathbf{S}$ is joint m-hyperexpansive tuple if $\Psi_k(\mathbf{S}) \leq 0$ for each $k = 1, 2, \cdots, m$.

(3) $\mathbf{S}$ is joint completely hyperexpansive tuple if $\Psi_k(\mathbf{S}) \leq 0$ for all $k \in \mathbb{N}$.

(4) $\mathbf{S}$ is joint m-contractive if $\Psi_m(\mathbf{S}) \geq 0$ for some integer m.

(5) $\mathbf{S}$ is joint m-hypercontractive if $\Psi_k(\mathbf{S}) \geq 0$ for each $k = 1, 2, \cdots, m$.

(6) $\mathbf{S}$ is joint completely hypercontractive if $\mathbf{S}$ is joint k-contractive for all positive integer k.

When $d = 1$, Definition 2.1 coincides with Definition 1.1 and Definition 1.2.

Remark. *Observe that*

$$\langle \Psi_l(\mathbf{S})x,\ x\rangle = \sum_{0\leq k\leq l}(-1)^k\bigg(\sum_{|\beta|=k}\frac{k!}{\beta!}\|\mathbf{S}^\beta x\|^2\bigg),\ \ \forall\, x\in\mathcal{H}.$$

Then ;

$$(1)\ \mathbf{S}\ \textit{is}\ m-\textit{expansive tuple}\iff \sum_{0\leq k\leq l}(-1)^k\bigg(\sum_{|\beta|=k}\frac{k!}{\beta!}\|\mathbf{S}^\beta x\|^2\bigg)\leq 0,\ \ \forall\, x\in\mathcal{H},$$

and

$$(2)\ \mathbf{S}\ \textit{is}\ m-\textit{contractive tuple}\iff \sum_{0\leq k\leq l}(-1)^k\bigg(\sum_{|\beta|=k}\frac{k!}{\beta!}\|\mathbf{S}^\beta x\|^2\bigg)\geq 0,\ \ \forall\, x\in\mathcal{H}$$

Remark. (i) Let $\mathbf{S} = (S_1,\cdots,S_d)\in\mathcal{B}(\mathcal{H})^d$ be a commuting tuple of operators. Then $\mathbf{S}$ is a joint expansive tuple if

$$\|x\|^2\leq \sum_{1\leq j\leq d}\|S_jx\|^2,\qquad (\forall\, x\in\mathcal{H}) \tag{2.3}$$

and it is a joint -contractive tuple if

$$\|x\|^2\geq \sum_{1\leq j\leq d}\|S_jx\|^2,\qquad (\forall\, x\in\mathcal{X}). \tag{2.4}$$

(ii) If $d = 2$, let $\mathbf{S} = (S_1,S_2)\in\mathcal{B}(\mathcal{H})^2$ be a commuting pair of operators. Then $\mathbf{S}$ is a joint 2-expansive pair if

$$\|x\|^2\leq 2(\|S_1x\|^2+\|S_2x\|^2)-(\|S_1^2x\|^p+\|S_2^2x\|^2+2\|S_1S_2x\|^2)\ \ (\forall\, x\in\mathcal{H}), \tag{2.5}$$

and it is a joint $(2,p)$-contractive pair if

$$\|x\|^p\geq 2(\|S_1x\|^2+\|S_2x\|^p)-(\|S_1^2x\|^p+\|S_2^2x\|^2+2\|S_1S_2x\|^2)\quad (\forall\, x\in\mathcal{H}). \tag{2.6}$$

(iii) Let $\mathbf{S} = (S_1,\cdots,S_d)\in\mathcal{B}(\mathcal{H})^d$ be a commuting tuple of operators. Then $\mathbf{S}$ is a joint 2-expansive tuple if

$$\|x\|^2\leq 2\sum_{1\leq j\leq d}\|S_jx\|^2-\bigg(\sum_{1\leq j\leq d}\|S_j^2x\|^2+2\sum_{1\leq j<k\leq d}\|S_jS_kx\|^2\bigg)\quad\forall\ \ x\in\mathcal{H}, \tag{2.7}$$

and it a is joint 2-contractive tuple if

$$\|x\|^2\geq 2\sum_{1\leq j\leq d}\|S_jx\|^2-\bigg(\sum_{1\leq j\leq d}\|S_j^2x\|^p+2\sum_{1\leq j\neq k\leq d}\|S_jS_kx\|^2\bigg)\quad\forall\ \ x\in\mathcal{H}. \tag{2.8}$$

Remark. *Since the operators $S_1,\cdots,S_d$ are commuting, every permutation of joint m-expansive tuple is also joint m-expansive tuple.*

The following examples show that there exists a joint (m,p)-expansive (resp. joint (m,p)-contractive) operator which is not (m,p)-isometric tuple for some positive integer m.

Example 2.2. Let $\mathcal{H} = \mathbb{C}^3$ be equipped with the norm

$$\|(x,y,z)\|_2 = \bigg(|x|^2+|y|^2+|z|^2\bigg)^{\frac{1}{2}}$$

and consider

$$S_1 = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \in \mathcal{B}(\mathbb{C}^3) \text{ and } S_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \in \mathcal{B}(\mathbb{C}^3).$$

Then the pair $\mathbf{S} = (S_1, S_2)$ is a joint 2-expansive pair on $(\mathcal{H} = \mathbb{C}^3, \|.\|_2)$.

In fact, obviously $S_1 S_2 = S_2 S_1$ and a simple computation shows that

$$\begin{aligned} & 2\left\|S_1 \begin{pmatrix} u \\ v \\ w \end{pmatrix}\right\|_2^p + 2\left\|S_2 \begin{pmatrix} u \\ v \\ w \end{pmatrix}\right\|_2^2 - \left(\left\|S_1^2 \begin{pmatrix} u \\ v \\ w \end{pmatrix}\right\|_2^2 + \left\|S_2^2 \begin{pmatrix} u \\ v \\ w \end{pmatrix}\right\|_2^2 + 2\left\|S_1 S_2 \begin{pmatrix} u \\ v \\ w \end{pmatrix}\right\|_2^2\right) \\ = & \left(9^2 - 1 + 3^2\right)\left|u + v + w\right|^2 \\ \geq & \left\|\begin{pmatrix} u \\ v \\ w \end{pmatrix}\right\|_2^2. \end{aligned}$$

However $\mathbf{S}$ is not a 2-isometric tuple.

Proposition 2.1. *Let $\mathbf{S} = (S_1, \cdots, S_d) \in \mathcal{B}(\mathcal{H})^d$ be a tuple of commuting operators. Then for all positive integer m, and$x \in \mathcal{H}$, we have*

$$\mathbf{Q}_{m+1}(\mathbf{S},\ x) = \mathbf{Q}_m(\mathbf{S},\ x) - \sum_{1 \leq k \leq n} \mathbf{Q}_m\big(\mathbf{S}\ , S_k x\big). \tag{2.9}$$

Proof. By taking into account Equation (2.4), a straightforward calculation shows that

$$\begin{aligned} & \mathbf{Q}_{m+1}(,\mathbf{S}, x) \\ = & \sum_{0 \leq k \leq m+1} (-1)^k \binom{m+1}{k} \sum_{|\beta| = k} \frac{k!}{\beta!} \|\mathbf{S}^\beta x\|^2 \\ = & \|x\|^2 + \sum_{1 \leq k \leq m} (-1)^k \left[\binom{m}{k} + \binom{m}{k-1}\right] \sum_{|\beta| = k} \frac{k!}{\beta!} \|\mathbf{S}^\beta x\|^2 \\ & +(-1)^{m+1} \sum_{|\beta| = m+1} \frac{(m+1)!}{\beta!} \|\mathbf{S}^\beta x\|^2 \\ = & \mathbf{Q}_m(\mathbf{S}, x) - \sum_{0 \leq k \leq m-1} (-1)^k \binom{m}{k} \sum_{|\beta| = k+1} \frac{(k+1)!}{\beta!} \|\mathbf{S}^\beta x\|^2 \\ + & (-1)^{m+1} \sum_{|\beta| = m+1} \frac{(m+1)!}{\beta!} \|\mathbf{S}^\beta x\|^2 \\ = & \mathbf{Q}_m(\mathbf{S}, x) - \sum_{0 \leq k \leq m-1} (-1)^{m-k} \binom{m}{k} \sum_{|\beta| = k+1} \frac{k!(\beta_1 + \cdots + \beta_n)}{\beta_1! \cdot \beta_2! \cdots \beta_n!} \|\mathbf{S}^\beta x\|^2 \\ & +(-1)^{m+1} \sum_{|\beta| = m+1} \frac{m!(\beta_1 + \cdots + \beta_n)}{\beta_1! \cdot \beta_2! \cdots \beta_n!} \|(\mathbf{S}^\beta x\|^2 \end{aligned}$$

$$\begin{aligned}
&= \mathbf{Q}_m(\mathbf{S},x) \\
&\quad - \sum_{1\le j\le n}\sum_{0\le k\le m-1}(-1)^k\binom{m}{k}\sum_{|\beta|=k+1}\frac{k!\beta_j}{\beta_1!\cdot\beta_2!\cdots\beta_n!} \\
&\quad \cdot\|S_j\big(S_1^{\beta_1}\cdots S_j^{\beta_j-1}S_{j+1}^{\beta_{j+1}}\cdots S_n^{\beta_n}x\|^2 \\
&\quad +(-1)^{m+1}\sum_{1\le j\le d}\sum_{|\beta|=m+1}\frac{m!\beta_j}{\beta_1!\cdot\beta_2!\cdots\beta_d!} \\
&\quad \cdot\|S_jS_1^{\beta_1}\cdots S_j^{\beta_j-1}S_{j+1}^{\beta_{j+1}}\cdots S_n^{\beta_d}x\|^2 \\
&= \mathbf{Q}_m(\mathbf{S}\ ,x) \\
&\quad - \sum_{1\le j\le d}\sum_{0\le k\le m-1}(-1)^k\binom{m}{k}\sum_{|\beta|=k}\frac{k!}{\beta!}\|\mathbf{S}^\alpha S_jx\|^2 \\
&\quad +(-1)^{m+1}\sum_{1\le j\le d}\sum_{|\beta|=m}\frac{m!}{\beta!}\|\mathbf{S}^\alpha S_jx\|^2 \\
&= \mathbf{Q}_m(\mathbf{S},\ x)-\sum_{1\le j\le n}\left(\sum_{0\le k\le m}(-1)^k\binom{m}{k}\sum_{|\beta|=k}\frac{k!}{\beta!}\|\mathbf{S}^\alpha S_jx\|^2\right. \\
&= \mathbf{Q}_m(\mathbf{S},\ x)-\sum_{1\le j\le d}Q_m(\mathbf{S};\ S_jx),
\end{aligned}$$

and so the equality (2.5) is satisfied . □

Example 2.3. Let $\mathcal{H}$ be an hilbert space and $I_\mathcal{H}$ the identity operator. Then $(5I_\mathcal{H}, I_\mathcal{H}, I_\mathcal{H}) \in \mathcal{B}(\mathcal{H})^3$ is a joint 2-contractive of operators which is not a 2-isometric tuple..

It is well-known that the class of m-isometric tuple is a subset of the class of $(m+1)$-isometric tuple. The following example shows that the class of joint m-expansive tuple and joint $m+1$-expansive tuple are independent.

Example 2.4. Let $\mathbf{S} = (I_\mathcal{H}, I_\mathcal{H}, I_\mathcal{H}) \in \mathcal{B}(\mathcal{H})^3$. A simple computation shows that

(1) $\mathbf{S}$ is a joint 1-expansive tuple but not a joint 2-expansive tuple.

(2) $\mathbf{S}$ is a joint 2-contractive tuple but not a joint 1-contractive tuple.

The following theorem gives some sufficient conditions under which a joint m-expansive tuple of operators in Hilbert space is a joint m-hyperexpansive tuple for $m \ge 2$.

Theorem 2.2. *Let* $\mathbf{S} = (S_1, \cdots, S_d) \in \mathcal{B}(\mathcal{H})^d$ *be a commuting tuple of operators. If* $\mathbf{S}$ *is a joint* m*-expansive tuple and satisfies the following conditions*

(1) $S_j^{k_j} \to 0$ *strongly for all* $j \in \{1, \cdots, d\}$,

(2) $S_j^*\Psi_k(\mathbf{S})S_j \le S_j^{*2}\Psi_k(\mathbf{S})S_j^2$ *for all* $j \in \{1, \cdots, d\}$ *and* $k \in \{1, \cdots, m-1\}$, *then* $\mathbf{S}$ *is a joint* m*-hyperexpansive tuple.*

Proof. As $\mathbf{S}$ is a joint m-expansive tuple, it follows that

$$\sum_{0\le k\le m}(-1)^k\binom{m}{k}\left(\sum_{|\beta|=k}\frac{k!}{\beta!}\mathbf{S}^{*\beta}\mathbf{S}^\beta\right)\le 0.$$

A computation shows that

$$\begin{aligned}
0 &\geq I_{\mathcal{H}} + \sum_{1\leq k\leq m-1} (-1)^k \binom{m}{k} \Big(\sum_{|\beta|=k} \frac{k!}{\beta!} \mathbf{S}^{*\beta}\mathbf{S}^{\beta} \Big) + (-1)^m \sum_{|\beta|=m} \frac{m!}{\beta!} \mathbf{S}^{*\beta}\mathbf{S}^{\beta} \\
&= I_{\mathcal{H}} + \sum_{1\leq k\leq m-1} (-1)^k \left(\binom{m-1}{k} + \binom{m-1}{k-1} \right) \sum_{|\alpha|=k} \frac{k!}{\alpha!} \mathbf{S}^{*\beta}\mathbf{S}^{\beta} + (-1)^m \sum_{|\beta|=m} \frac{m!}{\beta!} \mathbf{S}^{*\beta}\mathbf{S}^{\beta} \\
&= \underbrace{\sum_{0\leq k\leq m-1} (-1)^k \binom{m-1}{k} \sum_{|\beta|=k} \frac{k!}{\beta!} \mathbf{S}^{*\beta}\mathbf{S}^{\beta}}_{\Psi_{m-1}(\mathbf{S})} - \sum_{0\leq k\leq m-2} (-1)^k \binom{m-1}{k} \sum_{|\beta|=k+1} \frac{(k+1)!}{\beta!} \mathbf{S}^{*\beta}\mathbf{S}^{\beta} \\
&\quad + (-1)^m \sum_{|\beta|=m} \frac{m!}{\beta!} \mathbf{S}^{*\beta}\mathbf{S}^{\beta}
\end{aligned}$$

In view of the statement (2), we obtain

$$\begin{aligned}
\Psi_{m-1}(\mathbf{S}) &\leq \sum_{1\leq j\leq d} S_j^* \Big(\sum_{0\leq k\leq m-1} (-1)^k \binom{m-1}{k} \sum_{|\beta|=k} \frac{k!}{\beta!} \mathbf{S}^{*\beta}\mathbf{S}^{\beta} \Big) S_j \\
&= \sum_{1\leq j\leq d} S_j^* \Psi_{m-1}(\mathbf{S}) S_j \\
&\leq \sum_{1\leq j\leq d} S_j^{*2} \Psi_{m-1}(\mathbf{S}) S_j^2 \\
&\leq \cdots \\
&\leq \sum_{1\leq j\leq d} S_j^{*k_j} \Psi_{m-1}(\mathbf{S}) S_j^{k_j},
\end{aligned}$$

for every $k = (k_1, \cdots, k_d) \in \mathbb{Z}_+^d$. By the assumption in the statement (1) it follows that $\Psi_{m-1}(\mathbf{S}) \leq 0$. Repeating the process as above we can prove that

$$\Psi_l(\mathbf{S}) \leq 0 \ \ \forall \ l \in \{1, 2, \cdots, m\}.$$

This yields the conclusion that the operator $\mathbf{S}$ is a joint m-hyperexpansive tuple. □

A similar argument to the one above proves the following theorem:

Theorem 2.3. *Let $\mathbf{S} = (S_1, \cdots, S_d) \in \mathcal{B}(\mathcal{H})^d$ be a commuting tuple of operators. If $\mathbf{S}$ is a joint m-contractive tuple and satisfies the following conditions*

(1) $S_j^{k_j} \to 0$ *strongly for all* $j \in \{1, \cdots, d\}$,

(2) $S_j^* \Psi_k(\mathbf{S}) S_j \geq S_j^{*2} \Psi_k(\mathbf{S}) S_j^2$ *for all* $j \in \{1, 2, \cdots, d\}$ *and* $k \in \{1, \cdots, m-1\}$, *then* $\mathbf{T}$ *is* m*-hypercontractive tuple.*

Proposition 2.4. *Let $\mathbf{S} = (S_1, \cdots, S_d) \in \mathcal{B}(\mathcal{H})^d$ be commuting tuple of operators such that $\mathbf{S}_{/\overline{\mathcal{R}(\mathbf{S})}} := (S_{1/\overline{\mathcal{R}(\mathbf{S})}}, \cdots, S_{d/\overline{\mathcal{R}(\mathbf{S})}})$ is a joint 1-isometric tuple. The following properties hold.*

(1) *If $\mathbf{S}$ is joint m-expansive tuple, then $\mathbf{S}$ is m-hyperexpansive tuple.*

(2) *If $\mathbf{S}$ is joint m-contractive tuple, then $\mathbf{S}$ is a joint m-hypercontractive tuple.*

Proof. By (2.9) we have for all $k \in \{1, 2, \cdots, m\}$

$$Q_k(\mathbf{S}\ , x) = \mathbf{Q}_{k-1}(\mathbf{S}\ , x) - \sum_{1\leq j\leq d} \mathbf{Q}_{k-1}(\mathbf{S},\ S_jx),\ \ \forall\, x \in \mathcal{H}.$$

If $\mathbf{S}$ is a joint isometric tuple on $\overline{\mathcal{R}(\mathbf{S})}$, it is well known that $\mathbf{S}$ is an k-isometric tuple on $\overline{\mathcal{R}(\mathbf{S})}$ for $k = 1, \cdots, m$. Consequently,

$$\mathbf{Q}_1(\mathbf{S},\ x) = \mathbf{Q}_2(\mathbf{S},\ x) = ... = \mathbf{Q}_{m-1}(\mathbf{S},\ x) = \mathbf{Q}_m(\mathbf{S},\ x) = 0.$$

If $\mathbf{S}$ is a joint m-expansive tuple, then it is a joint $(m-1)$-expansive tuple. By repeating this process we get obtain that $\mathbf{S}$ is joint k-expansive tuple for $k = 1, 2\cdots, m$. Consequently, $\mathbf{S}$ is joint m-hyperexpansive.

By the same argument as above, (2) is obtained. □

A point $\mu = (\mu_1, \cdots, \mu_d) \in \mathbb{C}^d$ is said to be a joint approximate point of $\mathbf{S} = (S_1, \cdots, S_d))$ if there exists a unit sequence of vectors $(\xi_n)_n \subset \mathcal{H}$ such that

$$\lim_{n\to\infty} \|(S_j - \mu_j)\xi_n)\| = 0 \quad \text{for}\ \ j = 1, \cdots, d.$$

The joint approximate point spectrum of $\mathbf{S}$, denoted by $\sigma_{ap}(\mathbf{S})$ is defined by

$$\sigma_{ap}(\mathbf{S}) := \{\mu = (\mu_1, \cdots, \mu_d) \in \mathbb{C}^d\ /\ \mu \text{ is a joint approximate point spectrum of }\ \mathbf{S}\ \}.$$

We denote by

$$\mathbb{B}(\mathbb{C}^d) := \{\mu = (\mu_1, \cdots, \mu_d) \in \mathbb{C}^d\ /\ \|\mu\|_2 = \left(\sum_{1\leq j\leq d} |\mu_j|^2\right)^{\frac{1}{2}} < 1\ \}$$

and

$$\partial\mathbb{B}(\mathbb{C}^d) := \{\mu = (\mu_1, \cdots, \mu_d) \in \mathbb{C}^d\ /\ \|\mu\|_2 = \left(\sum_{1\leq j\leq d} |\mu_j|^2\right)^{\frac{1}{2}} = 1\ \}.$$

Proposition 2.5. *Let* $\mathbf{S} = (S_1, \cdots, S_d) \in \mathcal{B}(\mathcal{H})^d$ *be an* m*-expansive tuple for some positive integer* m*. The following properties hold.*

(1) *If* m *is even, then the approximate point spectrum of* $\mathbf{S}$ *lies in the boundary of the unit ball of* $(\mathbb{C}^d, \|.\|_2)$.

(2) *If* m *is odd, then* $\sigma_{ap}(\mathbf{S}) \subset \mathbb{C}^d \setminus \mathbb{B}(\mathbb{C}^d)$ *and* $\mathcal{N}(\mathbf{S}) := \bigcap_{1\leq j\leq d} \mathcal{N}(S_j) = \{0\}$.

Proof. Let $\mu = (\mu_1, \cdots, \mu_d) \in \mathbb{C}^d$ is in the approximate point spectrum of $\mathbf{S}$, then there exists a sequence $(\xi_n)_n \subset \mathcal{X}$ such that for all n, $||\xi_n|| = 1$ and $\|(S_j - \mu_j)\xi_n\| \to 0$ as $n \to +\infty$. Thus for each integer $\beta_j \in \mathbb{N}$, $\lim_{n\to+\infty} \|(S_j^{\beta_j} - \mu_j^{\beta_j})\xi_n\| = 0$ and so that

$$\lim_{n\to+\infty} \|\big(\mathbf{S}^\beta - \mu^\beta\big)\xi_n\| = 0\ \ \forall\ \ \beta = (\beta_1, \cdots, \beta_d) \in \mathbb{Z}_+^d.$$

Now it is easy to see that

$$\|\mathbf{S}^\beta\xi_n\|^2 \longrightarrow |\mu|^{2\beta} \quad \text{as}\ \ n \longrightarrow \infty.$$

Since $\mathbf{S}$ is an (m,p)-expansive, it follows that

$$0 \geq \sum_{0\leq k\leq m}(-1)^k\binom{m}{k}\bigg(\sum_{|\alpha|=k}\frac{k!}{\alpha!}\|(\mathbf{S})^\alpha\xi_n\|^2\bigg)$$

By taking $n\to\infty$ in the last inequality, we obtain

$$\begin{aligned} 0 &\geq \sum_{0\leq k\leq m}(-1)^k\binom{m}{k}\sum_{|\alpha|=k}\frac{k!}{\beta!}|\mu^\beta|^2 \\ &\geq \sum_{0\leq k\leq m}(-1)^k\binom{m}{k}\sum_{|\beta|=k}\frac{k!}{\beta!}|\mu|^{\beta 2} \\ &\geq \big(1-\|\mu\|_2^2\big)^m. \end{aligned}$$

So,

$$\big(1-\|\mu\|_p^p\big)^m \leq 0 \tag{2.10}$$

Now, we have that

(1) If m is even , then $0\leq \big(1-\|\mu\|_p^p\big)^m\leq 0$ from (2.10), and so $\|\mu\|_2=1$. Thus means that $\sigma_{ap}(\mathbf{S})\subset\partial\mathbb{B}(\mathbb{C}^d)$ and so

$$\partial\sigma(\mathbf{S})\subset\sigma_{ap}(\mathbf{S})\subset\partial\mathbb{B}(\mathbb{C}^d).$$

(2) Assume that m is odd. If $\|\mu\|_2<1$, then $0<(1-\|\mu\|_2^2)^m\leq 0$ from (2.10), which is a contradiction. Hence,

$\sigma_{ap}(\mathbf{S})\subset\mathbb{C}^d\setminus\mathbb{B}(\mathbb{C}^d)$. In particular $(0,\cdots,0)\notin\sigma_{ap}(\mathbf{S})$. Thus $\mathcal{N}(\mathbf{S})=\{0\}$. □

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