ANNALS OF COMMUNICATIONS IN MATHEMATICS
Volume 3, Number 1 (2020), 26-34
ISSN: 2582-0818

HESITANT INTUITIONISTIC FUZZY SOFT B-IDEALS OF BCK-ALGEBRAS

G. MUHIUDDIN AND M. BALAMURUGAN*

ABSTRACT. In this paper, we establish the concept of hesitant intuitionistic fuzzy soft ideals and hesitant intuitionistic fuzzy soft b-ideals are introduced and investigated. Also, the power-m, and ρ-multiply are defined in the hesitant intuitionistic fuzzy soft set theory to BCK-algebras. Finally, the newly introduced notions of the complement of a hesitant intuitionistic soft set (ideals and b-ideals) are discussed.

1. INTRODUCTION

Zadeh [19] introduced the concept of fuzzy sets. The idea of intuitionistic fuzzy sets was first exhibited by Atanassov [1] as an explanation of the notion of fuzzy sets. The notions of BCK/BCI-algebras [5] were exhibited by Imai and Iseki in 1996 explained the concept of set-theoretic difference and propositional calculus. Iskei [4] et. al initiated the conception of the ideal theory of BCK-algebras. Zhang et. al [20] presented on p-ideals of BCI-algebra.

Molodtsov [11] firstly proposed a new mathematical tool named soft set theory to deal with doubt and inaccurate. Jun [6] originated soft BCK-algebras. Maji et al. [10, 12] introduced intuitionistic fuzzy soft sets. Jun et. al [9] establish the conception of fuzzy soft set theory applied to BCK/BCI-algebras. Torra [18] popularized the concepts of hesitant fuzzy sets which are very useful to clear all human indecision in day to day life. The hesitant fuzzy set is very good devices to behave confused, which can be accurately and perfectly described in terms of the opinions of decision-makers.

Jun et al. [7] suggest the concept of hesitant fuzzy set theory applied to BCK/BCI-algebras. Babitha et al. [2] introduced hesitant fuzzy soft sets. Jun et al. [8] proposed hesitant fuzzy soft subalgebras and ideals in BCK/BCI-Algebras. Balamurugan et al. [3] introduced intuitionistic fuzzy soft ideals in BCK/BCI-algebras. Muhiuddin et al. [14, 17, 15, 16] introduced various concept are applied to BCK/BCI-algebras.

In this paper, we employ the notions of hesitant intuitionistic fuzzy soft sets to ideals and b-ideals in BCK/BCI- algebras. We brought the concept of hesitant intuitionistic fuzzy soft ideals and hesitant intuitionistic fuzzy soft b-ideals, and its related properties are examined. We provide conditions for a hesitant intuitionistic fuzzy soft b-ideal to be a hesitant

2010 *Mathematics Subject Classification.* 06D72, 06F35, 16D25, 94D05, 03G25, 03E72.
Key words and phrases. BCK-algebra; Hesitant intuitionistic fuzzy set; Hesitant intuitionistic fuzzy soft set; Hesitant intuitionistic fuzzy ideal; Hesitant intuitionistic fuzzy b-ideal.
Received: December 16, 2020. Accepted: March 15, 2020.
*Corresponding author.

intuitionistic fuzzy soft ideal. Finally, we focus complement of hesitant intuitionistic fuzzy soft b-ideals, and complement of hesitant intuitionistic fuzzy soft ideals.

2. Preliminaries

Definition 2.1. An algebra $(X;*,0)$ of type $(2,0)$ is said to be a BCK-algebra if for every $x,y,z \in X$ satisfies
(BCK-I) $((x*y)*(x*z))*(z*y)=0$,
(BCK-II) $(x*(x*y))*y)=0$,
(BCK-III) $x*x=0$,
(BCK-IV) $0*x=0$,
(BCK-V) $x*y=0$ and $y*x=0 \;\Rightarrow\; x=y$.
Let $(X;*,0)$ or simply X express a BCK-algebra unless is specified.

A BCK-algebra 'X' satisfies the following conditions for every $x,y,z \in X$,
(I) $x*0=x$;
(II) $(x*y)*z=(x*z)*y$;
(III) $x \leq y$ imply $x*z \leq y*z$ and $z*y \leq z*x$;
(IV) $(x*z)*(y*z) \leq x*y$;
where $x \leq y$ if and only if $x*y=0$.

Definition 2.2. [4]. A nonempty subset A is said to be an ideal of X if for every $x,y \in X$ satisfies
$(I_1) 0 \in A$,
$(I_2) x*y \in A$ and $y \in A$ imply $x \in A$.

Definition 2.3. Let A is a fuzzy ideal of X. Then the fuzzy set A^m with members operation $r_{R^m}$is defined by $r_{R^m}(x)=(r_R(x))^m$ for all $x \in X$.

Definition 2.4. [11]. A couple (F,A) is called a soft set over $X \Leftrightarrow F: A \to \mathcal{P}(X)$.

For every $\alpha \in A, F[\alpha]$ may be set of α -common elements of (F,A). Clearly, every set implies soft set but a soft set is not a set.

Definition 2.5. [2]. Let X be a fixed set and E act a set of factors. Let $HF(X)$ denote the set of all hesitant fuzzy soft set over X and $A \subset E$, where $\tilde{F}: A \to HF(X)$.

In general, for every $\alpha \in A$, $\tilde{F}[\alpha]$ is a hesitant fuzzy set in X and it is called a fuzzy value set of factor α. Clearly, $\tilde{F}[\alpha]$ can be written as a hesitant fuzzy set such that $\tilde{F}[\alpha] := \{(x, h_{\tilde{F}[\alpha]}(x) : x \in X$ and $\alpha \in A\}$ where $h_{\tilde{F}[\alpha]}(x)$ reveal the degree of members operation respectively. If for every $x \in X, h_{\tilde{F}[\alpha]}(x)$ is a crisp subset of X, then $\tilde{F}[\alpha]$ will be degraded to be a standard soft set.

3. Hesitant Intuitionistic Fuzzy Soft b-ideals

In this section, hesitant intuitionistic fuzzy soft ideals and hesitant intuitionistic fuzzy soft b-ideals of BCK-algebras are studied and a few of its results are given.

Definition 3.1. An ideal A is said to be a b-ideal of X if for every $x,y,z \in X$ satisfies
(BI-1) $0 \in A$,
(BI-2) $(x*z)*y \in A$ and $y \in A \Rightarrow x \in A$.

Definition 3.2. A hesitant intuitionistic fuzzy set on an initial universe set X is an object of the system $\tilde{F} := \{(x, h_{1_{\tilde{F}}}(x), h_{2_{\tilde{F}}}(x)) : x \in X\}$ where $h_{1_{\tilde{F}}} : X \to HIF(X)$ and $h_{2_{\tilde{F}}} : X \to HIF(X)$.

Definition 3.3. A hesitant intuitionistic fuzzy set $\tilde{F} := \{(x, h_{1_{\tilde{F}}}(x), h_{2_{\tilde{F}}}(x)) : x \in X\}$ is called a hesitant intuitionistic fuzzy ideal of Xif for every $x, y \in X$ satisfies
(HIFI-1) $h_{1_{\tilde{F}}}(0) \geq h_{1_{\tilde{F}}}(x)$ and $h_{2_{\tilde{F}}}(0) \leq h_{2_{\tilde{F}}}(x)$,
(HIFI-2) $h_{1_{\tilde{F}}}(x) \geq m\{h_{1_{\tilde{F}}}(x * y), h_{1_{\tilde{F}}}(y)\}$,
(HIFI-3) $h_{2_{\tilde{F}}}(x) \leq M\{h_{2_{\tilde{F}}}(x * y), h_{2_{\tilde{F}}}(y)\}$.

Definition 3.4. A hesitant intuitionistic fuzzy set $\tilde{F} := \{(x, h_{1_{\tilde{F}}}(x), h_{2_{\tilde{F}}}(x)) : x \in X\}$ is called a hesitant intuitionistic fuzzy b-ideal of X if for every $x, y, z \in X$ satisfies
(HIFBI-1) $h_{1_{\tilde{F}}}(0) \geq h_{1_{\tilde{F}}}(x)$ and $h_{2_{\tilde{F}}}(0) \leq h_{2_{\tilde{F}}}(x)$,
(HIFBI-2) $h_{1_{\tilde{F}}}(x) \geq m\{h_{1_{\tilde{F}}}((x * z) * y), h_{1_{\tilde{F}}}(y)\}$,
(HIFBI-3) $h_{2_{\tilde{F}}}(x) \leq M\{h_{2_{\tilde{F}}}((x * z) * y), h_{2_{\tilde{F}}}(y)\}$.

Definition 3.5. Let $HIF(X)$ be the set of all hesitant intuitionistic fuzzy soft set in X. A pair $(\tilde{F}, A)$ is called a hesitant intuitionistic fuzzy soft set if $\tilde{F} : A \to HIF(X)$, where $HIF(X)$ denotes the set of all hesitant intuitionistic fuzzy subsets of X.

Definition 3.6. Let $(\tilde{F}, A)$ be a hesitant intuitionistic fuzzy soft set over X and $A \subset E$. We say that $(\tilde{F}, A)$ is a hesitant intuitionistic fuzzy soft ideal if the hesitant intuitionistic fuzzy set $\tilde{F}[\alpha] := \{(x, h_{1_{\tilde{F}[\alpha]}}(x), h_{2_{\tilde{F}[\alpha]}}(x)) : x \in X \text{ and } \alpha \in A\}$ is a hesitant intuitionistic fuzzy ideal over X.

Definition 3.7. Let $(\tilde{F}, A)$ be a hesitant intuitionistic fuzzy soft set over X and $A \subset E$. We say that $(\tilde{F}, A)$ is a hesitant intuitionistic fuzzy soft b-ideal if the hesitant intuitionistic fuzzy set $\tilde{F}[\alpha] := \{(x, h_{1_{\tilde{F}[\alpha]}}(x), h_{2_{\tilde{F}[\alpha]}}(x)) : x \in X \text{ and } \alpha \in A\}$ is a hesitant intuitionistic fuzzy b-ideal over X.

Example 3.8. A set $X = \{0, a, b, c\}$ has the Cayley table given below:

*	0	1	2	3
0	0	0	0	0
1	1	0	1	0
2	2	2	0	0
3	3	2	1	0

Then X is a BCK-algebra. Consider act as a factors $A \subset E = \{\alpha_1, \alpha_2\}$. Let $(\tilde{F}, A)$ be a hesitant intuitionistic fuzzy soft set over X.

$\tilde{R}$	0	1	2	3
α_1	[0.7, 0.1]	[0.7, 0.1]	[0.3, 0.2]	[0.3, 0.2]
α_2	[0.5, 0.01]	[0.5, 0.01]	[0.2, 0.1]	[0.2, 0.1]

Then $\tilde{F}[\alpha_1]$ and $\tilde{F}[\alpha_2]$ are a hesitant intuitionistic fuzzy set over X. Therefore $\tilde{F}[\alpha_1]$ and $\tilde{F}[\alpha_2]$ are a hesitant intuitionistic fuzzy b-ideal based on factors α_1 and α_2 over X respectively. Hence $(\tilde{F}, A)$ is a hesitant intuitionistic fuzzy soft b-ideal over X.

Theorem 3.1. *If $(\tilde{F}, A)$ be hesitant intuitionistic fuzzy soft ideal over X, then $(\tilde{F}, A)$ is a hesitant intuitionistic fuzzy soft b-ideal over X.*

Proof. Let $(\tilde{F}, A)$ be a hesitant intuitionistic fuzzy soft ideal of X. Then
(i) For every $x \in X$ and $\alpha \in A$, we have

$$\begin{aligned} h_{1_{\tilde{F}[\alpha]}}(0) &= h_{1_{\tilde{F}[\alpha]}}(x * x) \\ &\geq m\{h_{1_{\tilde{F}[\alpha]}}(x), h_{1_{\tilde{F}[\alpha]}}(x)\} \\ &= h_{1_{\tilde{F}[\alpha]}}(x), \end{aligned}$$

$$h_{1_{\tilde{F}[\alpha]}}(0) \geq h_{1_{\tilde{F}[\alpha]}}(x)$$

and

$$\begin{aligned} h_{2_{\tilde{F}[\alpha]}}(0) &= h_{2_{\tilde{F}[\alpha]}}(x * x) \\ &\leq M\{h_{2_{\tilde{F}[\alpha]}}(x), h_{2_{\tilde{F}[\alpha]}}(x)\} \\ &= h_{2_{\tilde{F}[\alpha]}}(x), \end{aligned}$$

$$h_{2_{\tilde{F}[\alpha]}}(0) \leq h_{2_{\tilde{F}[\alpha]}}(x).$$

(ii) For every $x, y \in X$ and $\alpha \in A$, we have

$$h_{1_{\tilde{F}[\alpha]}}(x) \geq \mathrm{m}\{h_{1_{\tilde{F}[\alpha]}}(x * y), h_{1_{\tilde{F}[\alpha]}}(y)\}.$$

Since x imply $x * 0$ for every $x \in X$, we have

$$h_{1_{\tilde{F}[\alpha]}}(x) = \mathrm{m}\{h_{1_{\tilde{F}[\alpha]}}((x * 0) * y), h_{1_{\tilde{F}[\alpha]}}(y)\}.$$

If we substitute z for 0, we have

$$h_{1_{\tilde{F}[\alpha]}}(x) \geq \mathrm{m}\{h_{1_{\tilde{F}[\alpha]}}((x * z) * y), h_{1_{\tilde{F}[\alpha]}}(y)\}$$

for every $x, y, z \in X$ and $\alpha \in A$.
(iii) For every $x, y \in X$ and $\alpha \in A$, we have

$$h_{2_{\tilde{F}[\alpha]}}(x) \leq \mathrm{M}\{h_{2_{\tilde{F}[\alpha]}}(x * y), h_{2_{\tilde{F}[\alpha]}}(y)\}.$$

Since x imply $x * 0$ for every $x \in X$, we have

$$h_{2_{\tilde{F}[\alpha]}}(x) = M\{h_{2_{\tilde{F}[\alpha]}}((x * 0) * y), h_{2_{\tilde{F}[\alpha]}}(y)\}.$$

If we substitute z for 0, we have

$$h_{2_{\tilde{F}[\alpha]}}(x) \leq \mathrm{M}\{h_{2_{\tilde{F}[\alpha]}}((x * z) * y), h_{2_{\tilde{F}[\alpha]}}(y)\}$$

for every $x, y, z \in X$ and $\alpha \in A$.
Hence, $(\tilde{F}, A)$ is a hesitant intuitionistic fuzzy soft b-ideal over X. □

Definition 3.9. Let $(\tilde{F}, A)$ be hesitant intuitionistic fuzzy soft set over X. Then the "power-m" operation on hesitant intuitionistic fuzzy soft set is defined as follows:

$$\tilde{F}^m[\alpha] := \{(x, h_{1_{\tilde{F}^m[\alpha]}}(x), h_{2_{\tilde{F}^m[\alpha]}}(x) : x \in X \text{ and } \alpha \in A\}$$

where m is any non-negative integer.

Definition 3.10. Let $(\tilde{F}, A)^m$ be a hesitant intuitionistic fuzzy soft set over X and $A \subset E$. We say that $(\tilde{F}, A)^m$ is a hesitant intuitionistic fuzzy soft b-ideal if $\tilde{F}^m[\alpha] := \{(x, h_{1_{\tilde{F}^m[\alpha]}}(x), h_{2_{\tilde{F}^m[\alpha]}}(x) : x \in X \text{ and } \alpha \in A\}$ is a hesitant intuitionistic fuzzy b-ideal over X.

Theorem 3.2. *If $(\tilde{F}, A)$ is a hesitant intuitionistic fuzzy soft b-ideal over X based on factor α, then $(\tilde{F}, A)^m$ is an intuitionistic fuzzy soft b-ideal over X based in the similar manner.*

Proof. For every $x \in X$ *and* $\alpha \in A, \tilde{F}^m[\alpha]$ is a hesitant intuitionistic fuzzy set in X defined by $(\tilde{F}, A)^m = \{(x, h_{1_{\tilde{F}^m[\alpha]}}(x), h_{2_{\tilde{F}^m[\alpha]}}(x) : x \in X \text{ and } \alpha \in A\}$ where m is any non-negative integer.
Let $(\tilde{F}, A)$ be a hesitant intuitionistic fuzzy soft b-ideal of X. Then
(i) For every $x \in X$ and $\alpha \in A$. We have

$$\begin{aligned} h_{1_{\tilde{F}^m[\alpha]}}(0) &= [h_{1_{\tilde{F}[\alpha]}}(0)]^m \\ &= [h_{1_{\tilde{F}[\alpha]}}(x * x)]^m \\ &\geq m\{h_{1_{\tilde{F}[\alpha]}}(x), h_{1_{\tilde{F}[\alpha]}}(x)\}^m \end{aligned}$$

$$
\begin{aligned}
&= m\{h_{1_{\tilde{F}[\alpha]}}(x)^m, h_{1_{\tilde{F}[\alpha]}}(x)^m\}\\
&= m\{h_{1_{\tilde{F}^m[\alpha]}}(x), h_{1_{\tilde{F}^m[\alpha]}}(x)\}\\
h_{1_{\tilde{F}^m[\alpha]}}(0) &\geq h_{1_{\tilde{F}^m[\alpha]}}(x)
\end{aligned}
$$

and

$$
\begin{aligned}
h_{2_{\tilde{F}^m[\alpha]}}(0) &= [h_{2_{\tilde{F}[\alpha]}}(0)]^m\\
&= [h_{2_{\tilde{F}[\alpha]}}(x*x)]^m\\
&\leq M\{h_{2_{\tilde{F}[\alpha]}}(x), h_{2_{\tilde{F}[\alpha]}}(x)\}^m\\
&= M\{h_{2_{\tilde{F}[\alpha]}}(x)^m, h_{2_{\tilde{F}[\alpha]}}(x)^m\}\\
&= M\{h_{2_{\tilde{F}^m[\alpha]}}(x), h_{2_{\tilde{F}^m[\alpha]}}(x)\}\\
h_{2_{\tilde{F}^m[\alpha]}}(0) &\leq h_{2_{\tilde{F}^m[\alpha]}}(x).
\end{aligned}
$$

(ii) For every $x, y, z \in X$ and $\alpha \in A$, We have

$$
\begin{aligned}
h_{1_{\tilde{F}^m[\alpha]}}(x) &= [h_{1_{\tilde{F}[\alpha]}}(x)]^m\\
&\geq m\{h_{1_{\tilde{F}[\alpha]}}((x*z)*y), h_{1_{\tilde{F}[\alpha]}}(y)\}^m\\
&= m\{h_{1_{\tilde{F}[\alpha]}}((x*z)*y)^m, h_{1_{\tilde{F}[\alpha]}}(y)^m\}\\
h_{1_{\tilde{F}^m[\alpha]}}(x) &\geq m\{h_{1_{\tilde{F}^m[\alpha]}}((x*z)*y), h_{1_{\tilde{F}^m[\alpha]}}(y)\}.
\end{aligned}
$$

(iii) For every $x, y, z \in X$ *and* $\alpha \in A$, We have

$$
\begin{aligned}
h_{2_{\tilde{F}^m[\alpha]}}(x) &= [h_{2_{\tilde{F}[\alpha]}}(x)]^m\\
&\leq M\{h_{2_{\tilde{F}[\alpha]}}((x*z)*y), h_{2_{\tilde{F}[\alpha]}}(y)\}^m\\
&= M\{h_{2_{\tilde{F}[\alpha]}}((x*z)*y)^m, h_{2_{\tilde{F}[\alpha]}}(y)^m\}\\
h_{2_{\tilde{F}^m[\alpha]}}(x) &\leq M\{h_{2_{\tilde{F}^m[\alpha]}}((x*z)*y), h_{2_{\tilde{F}^m[\alpha]}}(y)\}.
\end{aligned}
$$

Hence, $(\tilde{F}, A)^m$ is a hesitant intuitionistic fuzzy soft b-ideal over X based on factor $\alpha \in A$. □

Theorem 3.3. *If $(\tilde{F}, A)$ is a hesitant intuitionistic fuzzy soft ideal over X, then $(\tilde{F}, A)^m$ is a hesitant intuitionistic fuzzy soft ideal over X.*

Proof. Straightforward. □

Theorem 3.4. *If $(\tilde{F}, A)^m$ is a hesitant intuitionistic fuzzy soft b-ideal over X, then $(\tilde{F}, A)^m$ is a hesitant intuitionistic fuzzy soft ideal over X.*

Proof. Let $(\tilde{F}, A)^m$ be a hesitant intuitionistic fuzzy soft b-ideal of X and let $\alpha \in A$ be a parameter. For every $x, y, z \in X$,

(i)
$$
\begin{aligned}
h_{1_{\tilde{F}^m[\alpha]}}(x) &\geq m\{h_{1_{\tilde{F}^m[\alpha]}}((x*z)*y), h_{1_{\tilde{F}^m[\alpha]}}(y)\}\\
&= m\{h_{1_{\tilde{F}^m[\alpha]}}((x*z)*(y*0)), h_{1_{\tilde{F}^m[\alpha]}}(y)\}.
\end{aligned}
$$

If we substitute z for 0, we have

$h_{1_{\tilde{F}^m[\alpha]}}(x) \geq m\{h_{1_{\tilde{F}^m[\alpha]}}((x*0)*(y*0)), h_{1_{\tilde{F}^m[\alpha]}}(y)\}$.

Since $x * 0$ imply x, we have

$h_{1_{\tilde{F}^m[\alpha]}}(x) \geq m\{h_{1_{\tilde{F}^m[\alpha]}}((x*y), h_{1_{\tilde{F}^m[\alpha]}}(y)\}$

for every $x, y \in X$ and $\alpha \in A$.

(ii)
$$
\begin{aligned}
h_{2_{\tilde{F}^m[\alpha]}}(x) &\leq M\{h_{2_{\tilde{F}^m[\alpha]}}((x*z)*y), h_{2_{\tilde{F}^m[\alpha]}}(y)\}\\
&= M\{h_{2_{\tilde{F}^m[\alpha]}}((x*z)*(y*0)), h_{2_{\tilde{F}^m[\alpha]}}(y)\}.
\end{aligned}
$$

If we substitute z for 0, we have

$h_{2_{\tilde{F}^m[\alpha]}}(x) \leq M\{h_{2_{\tilde{F}^m[\alpha]}}((x*0)*(y*0)), h_{2_{\tilde{F}^m[\alpha]}}(y)\}$.

Since $x * 0$ imply x, we have

$h_{2_{\tilde{F}^m[\alpha]}}(x) \leq M\{h_{2_{\tilde{F}^m[\alpha]}}((x*y), h_{2_{\tilde{F}^m[\alpha]}}(y)\}$

for every $x, y \in X$ and $\alpha \in A$.

Hence, $(\tilde{F}, A)^m$ is a hesitant intuitionistic fuzzy soft b-ideal over X. □

Theorem 3.5. *If $(\tilde{F}, A)^m$ is a hesitant intuitionistic fuzzy soft ideal over X, then $(\tilde{F}, A)^m$ is a hesitant intuitionistic fuzzy soft b-ideal over X.*

Proof. Straightforward. □

Definition 3.11. If $(\tilde{F}, A)$ be hesitant intuitionistic fuzzy soft set over X, then the "ρ-multiply" operation on hesitant intuitionistic fuzzy soft set is defined as follows:
$\rho\tilde{F}[\alpha] := \{(x, h_{1_{\rho\tilde{F}[\alpha]}}(x), h_{2_{\rho\tilde{F}[\alpha]}}(x) : x \in X$ and $\alpha \in A\}$.

Definition 3.12. Let $\rho(\tilde{F}, A)$ be a hesitant intuitionistic fuzzy soft set over X and $A \subset E$. We say that $\rho(\tilde{F}, A)$ is a hesitant intuitionistic fuzzy soft b-ideal if $\rho\tilde{F}[\alpha] := \{(x, h_{1_{\rho\tilde{F}[\alpha]}}(x), h_{2_{\rho\tilde{F}[\alpha]}}(x) : x \in X$ and $\alpha \in A\}$ is a hesitant intuitionistic fuzzy b-ideal over X.

Theorem 3.6. *If $(\tilde{F}, A)$ is a hesitant intuitionistic fuzzy soft b-ideal over X, then $\rho(\tilde{F}, A)$ is a hesitant intuitionistic fuzzy soft b-ideal over X.*

Proof. For every $x \in X$ and $\alpha \in A$, $\rho\tilde{F}[\alpha]$ is a hesitant intuitionistic fuzzy set in X defined by

$$\rho\tilde{F}[\alpha] = \{(x, h_{1_{\rho\tilde{F}[\alpha]}}(x), h_{2_{\rho\tilde{F}[\alpha]}}(x) : x \in X \text{ and } \alpha \in A\}.$$

Let $(\tilde{F}, A)$ be a hesitant intuitionistic fuzzy soft b-ideal of X. Then
(i) For any $x \in X$ and $\alpha \in A$, we have

$$\begin{aligned} h_{1_{\rho\tilde{F}[\alpha]}}(0) &= \rho h_{1_{\tilde{F}[\alpha]}}(0) \\ &= \rho h_{1_{\tilde{F}[\alpha]}}(x * x) \\ &\geq m\{\rho h_{1_{\tilde{F}[\alpha]}}(x), \rho h_{1_{\tilde{F}[\alpha]}}(x)\} \\ &= m\{h_{1_{\rho\tilde{F}[\alpha]}}(x), h_{1_{\rho\tilde{F}[\alpha]}}(x)\} \\ h_{1_{\rho\tilde{F}[\alpha]}}(0) &\geq h_{1_{\rho\tilde{F}[\alpha]}}(x) \end{aligned}$$

and

$$\begin{aligned} h_{2_{\rho\tilde{F}[\alpha]}}(0) &= \rho h_{2_{\tilde{F}[\alpha]}}(0) \\ &= \rho h_{2_{\tilde{F}[\alpha]}}(x * x) \\ &\leq M\{\rho h_{2_{\tilde{F}[\alpha]}}(x), \rho h_{2_{\tilde{F}[\alpha]}}(x)\} \\ &= M\{h_{2_{\rho\tilde{F}[\alpha]}}(x), h_{2_{\rho\tilde{F}[\alpha]}}(x)\} \\ h_{2_{\rho\tilde{F}[\alpha]}}(0) &\leq h_{2_{\rho\tilde{F}[\alpha]}}(x). \end{aligned}$$

(ii) For any $x, y, z \in X$ and $\alpha \in A$, we have

$$\begin{aligned} h_{1_{\rho\tilde{F}[\alpha]}}(x) &= [\rho h_{1_{\tilde{F}[\alpha]}}(x)] \\ &\geq m\{\rho h_{1_{\tilde{F}[\alpha]}}((x * z) * y), \rho h_{1_{\tilde{F}[\alpha]}}(y)\} \\ h_{1_{\rho\tilde{F}[\alpha]}}(x) &\geq m\{h_{1_{\rho\tilde{F}[\alpha]}}((x * z) * y), h_{1_{\rho\tilde{F}[\alpha]}}(y)\}. \end{aligned}$$

(iii) For any $x, y, z \in X$ and $\alpha \in A$, we have

$$\begin{aligned} h_{2_{\rho\tilde{F}[\alpha]}}(x) &= [\rho h_{2_{\tilde{F}[\alpha]}}(x)] \\ &\leq M\{\rho h_{2_{\tilde{F}[\alpha]}}((x * z) * y), \rho h_{2_{\tilde{F}[\alpha]}}(y)\} \\ h_{2_{\rho\tilde{F}[\alpha]}}(x) &\leq M\{h_{2_{\rho\tilde{F}[\alpha]}}((x * z) * y), h_{2_{\rho\tilde{F}[\alpha]}}(y)\}. \end{aligned}$$

Hence, $\rho(\tilde{F}, A)$ is a hesitant intuitionistic fuzzy soft b-ideal over X for every $\alpha \in A$. □

Definition 3.13. Let $\rho(\tilde{F}, A)$ be a hesitant intuitionistic fuzzy soft set over X and $A \subset E$. We say that $\rho(\tilde{F}, A)$ is a hesitant intuitionistic fuzzy soft ideal if $\rho\tilde{F}[\alpha] := \{(x, h_{1_{\rho\tilde{F}[\alpha]}}(x), h_{2_{\rho\tilde{F}[\alpha]}}(x) : x \in X$ and $\alpha \in A\}$ is a hesitant intuitionistic fuzzy ideal over X.

Theorem 3.7. *If $(\tilde{F}, A)$ is a hesitant intuitionistic fuzzy soft ideal over X, then $\rho(\tilde{F}, A)$ is a hesitant intuitionistic fuzzy soft ideal over X.*

Proof. Straightforward. □

4. Complement of Hesitant Intuitionistic Fuzzy Soft b-ideals of BCK-algebras

In this section, complement of hesitant intuitionistic fuzzy soft set (ideals and b-ideals) are defined and presented.

Definition 4.1. Let $(\tilde{F}, A)$ be a hesitant intuitionistic fuzzy soft set over X, then the complement of hesitant intuitionistic fuzzy soft set $(\tilde{F}, A)$ is defined by $(\tilde{F}, A)^C = (\tilde{F}^C, \neg A)$, where $\tilde{F}^C : \neg A \to HIF(X)$ is a mapping given by $\tilde{F}^C[\alpha] = \tilde{F}[\neg\alpha]$ for every $\alpha \in \neg A$.

Definition 4.2. Let $(\tilde{F}, A)^C$ be a hesitant intuitionistic fuzzy soft set over X and $A \subset E$. We say that $(\tilde{F}, A)^C$ is a hesitant intuitionistic fuzzy soft b-ideal if the hesitant intuitionistic fuzzy set $\tilde{F}^C[\alpha] := \{(x, h_{1_{\tilde{F}^C[\alpha]}}(x), h_{2_{\tilde{F}^C[\alpha]}}(x) : x \in X \text{ and } \alpha \in \neg A\}$ is a hesitant intuitionistic fuzzy b-ideal over X.

Theorem 4.1. *If $(\tilde{F}, A)$ is a hesitant intuitionistic fuzzy soft b-ideal over X, then $(\tilde{F}, A)^C$ is a hesitant intuitionistic fuzzy soft b-ideal over X.*

Proof. Let $(\tilde{F}, A)$ be a hesitant intuitionistic fuzzy soft b-ideal of X.
(i) Let $x \in X$ and $\alpha \in \neg A$. Then

$$\begin{aligned} h_{1_{\tilde{F}^C[\alpha]}}(0) &= [h_{1_{\tilde{F}[\neg\alpha]}}(0)] \\ &= [h_{1_{\tilde{F}[\neg\alpha]}}(x * x) \\ &\geq m\{h_{1_{\tilde{F}[\neg\alpha]}}(x), h_{1_{\tilde{F}[\neg\alpha]}}(x)\} \\ &= m\{h_{1_{\tilde{F}^C[\alpha]}}(x), h_{1_{\tilde{F}^C[\alpha]}}(x)\} \\ h_{1_{\tilde{F}^C[\alpha]}}(0) &\geq h_{1_{\tilde{F}^C[\alpha]}}(x) \end{aligned}$$

and

$$\begin{aligned} h_{2_{\tilde{F}^C[\alpha]}}(0) &= h_{2_{\tilde{F}[\neg\alpha]}}(0) \\ &= h_{2_{\tilde{F}[\neg\alpha]}}(x * x) \\ &\leq M\{h_{2_{\tilde{F}[\neg\alpha]}}(x), h_{2_{\tilde{F}[\neg\alpha]}}(x)\} \\ &= M\{h_{2_{\tilde{F}^C[\alpha]}}(x), h_{2_{\tilde{F}^C[\alpha]}}(x)\} \\ h_{2_{\tilde{F}^C[\alpha]}}(0) &\leq h_{2_{\tilde{F}^C[\alpha]}}(x). \end{aligned}$$

(ii) For every $x, y \in X$ and $\alpha \in \neg A$, we have

$$\begin{aligned} h_{1_{\tilde{F}^C[\alpha]}}(x) &= h_{1_{\tilde{F}[\neg\alpha]}}(x) \\ &\geq m\{h_{1_{\tilde{F}[\neg\alpha]}}((x * z) * y), h_{1_{\tilde{F}[\neg\alpha]}}(y)\} \\ h_{1_{\tilde{F}^C[\alpha]}}(x) &\geq m\{h_{1_{\tilde{F}^C[\alpha]}}((x * z) * y), h_{1_{\tilde{F}^C[\alpha]}}(y)\}. \end{aligned}$$

(iii) For every $x, y \in X$ and $\alpha \in \neg A$, we have

$$\begin{aligned} h_{2_{\tilde{F}^C[\alpha]}}(x) &= h_{2_{\tilde{F}[\neg\alpha]}}(x) \\ &\leq M\{h_{2_{\tilde{F}[\neg\alpha]}}((x * z) * y), h_{2_{\tilde{F}[\neg\alpha]}}(y)\} \\ h_{2_{\tilde{F}^C[\alpha]}}(x) &\leq M\{h_{2_{\tilde{F}^C[\alpha]}}((x * z) * y), h_{2_{\tilde{F}^C[\alpha]}}(y)\}. \end{aligned}$$

Hence, $(\tilde{F}, A)^C$ is a hesitant intuitionistic fuzzy soft b-ideal over X. □

Definition 4.3. Let $(\tilde{F}, A)^C$ be a hesitant intuitionistic fuzzy soft set over X and $A \subset E$. We say that $(\tilde{F}, A)^C$ is a hesitant intuitionistic fuzzy soft ideal if the hesitant intuitionistic fuzzy set $\tilde{F}^C[\alpha] := \{(x, h_{1_{\tilde{F}^C[\alpha]}}(x), h_{2_{\tilde{F}^C[\alpha]}}(x) : x \in X \text{ and } \alpha \in \neg A\}$ is a hesitant intuitionistic fuzzy ideal over X.

Theorem 4.2. *If $(\tilde{F}, A)$ is a hesitant intuitionistic fuzzy soft ideal over X, then $(\tilde{F}, A)^C$ is a hesitant intuitionistic fuzzy soft ideal over X.*

Proof. Straightforward. □

5. CONCLUSIONS

In this paper, we brought the notions of hesitant intuitionistic fuzzy soft ideal and hesitant intuitionistic fuzzy soft b-ideal of BCK-algebras. We have shown that the hesitant intuitionistic fuzzy soft ideal is a hesitant intuitionistic fuzzy soft b-ideal. Also, we have proved that the complement of hesitant intuitionistic fuzzy soft b-ideals. In our future, we will execute interval-valued intuitionistic fuzzy soft theory to BCK-algebras.

6. ACKNOWLEDGEMENTS

The authors would like to express their thanks to the referees for their valuable suggestions and helpful comments in improving this paper.

REFERENCES

[1] K. T. Atanassov. Intuitionistic fuzzy sets Fuzzy Sets and Systems, 20(1) (1986), 87-96.
[2] K. V. Babitha, S. J. John. Hesitant fuzzy soft sets. Journal of New Results in Science, 2(3) (2013), 98-107.
[3] M. Balamurugan, G. Balasubramanian, and C. Ragavan. Intuitionistic fuzzy soft ideals in BCK/BCI-algebras, Materials Today: Proceedings. 16 (2), 489-503.
[4] K. Iskei and S. Tanaka. Ideal theory of BCK-algebras Math. Japon. 21(4) (1976), 351-366.
[5] Y. Imai, K. Iseki. On axiom systems of propositional calculi. XIV. Proc. Japon Acad. 42 (1966), 19-22..
[6] Y. B. Jun. Soft BCK/BCI-algebras, Comput. Math. Appl. 56(5) (2008), 1408-1413.
[7] Y. B. Jun, S. S. Ahn. Hesitant fuzzy set theory applied to BCK/BCI-algebras, J. Comput. Anal. Appl. 20(4) (2016), 635-646.
[8] Y. B. Jun, S. S. Ahn, G. Muhiuddin. Hesitant fuzzy soft subalgebras and ideals in BCK/BCI-algebras, Scientific World Journal. vol.2014, Article ID763929.
[9] Y. B. Jun, K. J. Lee, C. H. Park. Fuzzy soft set theory applied to BCK/BCI-algebras, Comput. Math. Appl. 59(9) (2010), 3180-3192.
[10] P. K. Maji, R. Biswas, A. R. Roy. Intuitionistic fuzzy soft sets, J. Fuzzy Math. 9(3) (2001), 677-692.
[11] P. K. Maji, R. Biswas, A. R. Roy. Soft set theory, Comput. Math. Appl. 45(4-5) (2003), 555-562.
[12] P. K. Maji, R. Biswas, A. R. Roy. On intuitionistic fuzzy soft sets, J. Fuzzy Math. 12(3) (2004), 669-683.
[13] D. Molodstov. Soft set theory - first results, Global Optimization, Control, III, Comput. Math. Appli. 37 (4-5) (1999), 19-31.
[14] G. Muhiuddin. Hesitant fuzzy filters and hesitant fuzzy G-filters in residuated lattices, J. Comput. Anal. Appl. 21 (2) (2016), 394404.
[15] G. Muhiuddin and S. Aldhafeeri. Subalgebras and ideals in BCK/BCI- algebras based on uni-hesitant fuzzy set theory, Eur. J. Pure Appl. Math. 11 (2) (2018), 417430.
[16] G. Muhiuddin and S. Aldhafeeri. N-Soft p-ideal of BCI-algebras, Eur. J. Pure Appl. Math. 12 (1) (2019), 79-87.
[17] G. Muhiuddin, H. S. Kim, S. Z. Song and Y. B. Jun. Hesitant fuzzy translations and extensions of subalgebras and ideals in BCK/BCI-algebras, J. Intelli. Fuzzy Systems, 32 (1) (2017), 43-48, 2017.
[18] V. Torra. Hesitant fuzzy sets, International Journal of Intelligent Systems. 25(6), 529-539.
[19] L. A. Zadeh. Fuzzy sets, Information and Control. 8 (1965), 338-353.
[20] X. H. Zhang, H. Jiang, and S. A. Bhatti. On b-ideals of a BCI-algebra. Punjab Univ. J. Math. (Lahore). 27 (1994), 121-128.

G. MUHIUDDIN
DEPARTMENT OF MATHEMATICS, UNIVERSITY OF TABUK 71491, SAUDI ARABIA
Email address: chishtygm@gmail.com

M. BALAMURUGAN
DEPARTMENT OF MATHEMATICS, SRI VIDYA MANDIR ARTS AND SCIENCE COLLEGE, UTHANGARAI, TAMIL NADU-636902, INDIA
Email address: balamurugansvm@gmail.com