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CERTAIN RESULTS FOR THE HERMITE AND CHEBYSHEV POLYNOMIALS OF 2-VARIABLES

WASEEM A. KHAN

ABSTRACT. In this paper, we introduce Hermite-Chebyshev polynomials of two variables and to give some properties of Hermite and Chebyshev polynomials of two variables. We derive series representation, operational identities, generating functions and power series method by Hermite, Chebyshev and Hermite-Chebyshev polynomials of two variables. Finally, we consider generalized Hermite-Chebyshev polynomials of two variables and explicit representation of Hermite-Chebyshev polynomials of two variables.

1. INTRODUCTION AND PRELIMINARIES

Special functions appear in statistics, Lie group theory and number theory. The Hermite polynomials of the associated generating functions is reformulated within the framework of an operational formalism [5-8]. In the case of generalized special functions, the use of operational techniques, combined with the principle of monomiality [3, 4, 9] has provided new means of analysis for the derivation of the solution of large classes of partial differential equations often encountered in physical problems [10, 14] offers a power tool to treat the relevant generating functions and the differential equations they satisfy. The results are interpreted in terms of single, several variables, single index, index two, three and in turn p-index in terms of Hermite polynomials defined by Srivastava [18, 19]. The reason of interest for this family of Hermite polynomials is due to their intrinsic mathematical importance and to the fact that these polynomials have applications in physics. We recall some definitions as follows.

The generalization of 2-variable Kampé de Fériet polynomials (or, Gould Hopper polynomials) introduced by Gould and Hopper (see [10,p.58,(6.2)])

$$H_n^{(p)}(x,y) = n!\sum_{r=0}^{[\frac{n}{p}]}\frac{y^r x^{n-pr}}{r!(n-pr)!}. \tag{1.1}$$

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These polynomials are usually defined by the generating function [10,p.58,(6.3)]

$$e^{xt+yt^p} = \sum_{n=0}^{\infty} H_n^{(p)}(x,y)\frac{t^n}{n!}, \tag{1.2}$$

and reduce to the ordinary Hermite polynomials $H_n(x)$ (see [1]) when $p = 2, y = -1$ and x is replaced by $2x$.

We draw attention to familiar generating relations given by

$$(1-2xt+t^2)^{-\frac{1}{2}} = \sum_{n=0}^{\infty} P_n(x)t^n, \tag{1.3}$$

where $P_n(x)$ is Legendre's polynomial of first kind.

$$(1-2xt+t^2)^{-1} = \sum_{n=0}^{\infty} U_n(x)t^n, \tag{1.4}$$

where $U_n(x)$ is Chebychev polynomial of second kind.

$$(1-2xt+t^2)^{-\nu} = \sum_{n=0}^{\infty} C_n^{\nu}(x)t^n, \tag{1.5}$$

where $C_n^{\nu}(x)$ is Gegenbauer's polynomial.

$$(1-mxt+t^m)^{-\nu} = \sum_{n=0}^{\infty} h_{n,m}^{\nu}(x)t^n, \tag{1.6}$$

$$h_{n,m}^{\nu}(x) = \sum_{k=0}^{[\frac{n}{m}]} \frac{(-1)^k(\nu)_{n+(1-m)k}(mx)^{n-mk}}{k!(n-mk)!},$$

where $h_{n,m}^{\nu}(x)$ is Humbert polynomial and m is a positive integer. The Pochammer symbol $(a)_n$ is defined by

$$(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)} = \left[\begin{array}{ll} 1 & \text{if } n=0 \\ a(a+1)(a+2)\cdots(a+n-1) & \text{if } n=1,2,3\cdots \end{array}\right.$$

In 1965, Gould [11] gave the following generating relation

$$(c-mxt+yt^m)^p = \sum_{n=0}^{\infty} P_n(m,x,y,p,c)t^n, \tag{1.7}$$

where m is a positive integer and other parameters are unrestricted in general. $P_n(m,x,y,p,c)$ is defined explicitly by [11, p.699]

$$P_n(m,x,y,p,c) = \sum_{k=0}^{[\frac{n}{m}]} \begin{pmatrix} p \\ k \end{pmatrix} \begin{pmatrix} p-k \\ n-mk \end{pmatrix} c^{p-n+(m-1)k}y^k(-mx)^{n-mk}. \tag{1.8}$$

In 1989, Sinha [20] gave the following generating relation

$$\left[1-2xt+t^2(2x-1)\right]^{-\nu} = \sum_{n=0}^{\infty} S_n^{\nu}(x)t^n, \tag{1.9}$$

where

$$S_n^{\nu}(x)=\sum_{k=0}^{[\frac{n}{2}]}\frac{(-1)^k(\nu)_{n-k}(2x)^{n-2k}(2x-1)^k}{k!(n-2k)!}, \tag{1.10}$$

$S_n^{\nu}(x)$ is the generalization of Shrestha polynomial $S_n(x)$ (see [15]).

In 1991, Milovanović and Djordjević [16](see also [17]) gave the following generating relation

$$(1-2xt+t^m)^{-\lambda}=\sum_{n=0}^{\infty}p_{n,m}^{\lambda}(x)t^n, \tag{1.11}$$

where $m\in\mathbb{N}$ and $\lambda>-\frac{1}{2}$ and

$$p_{n,m}^{\lambda}(x)=\sum_{k=0}^{[\frac{n}{m}]}\frac{(-1)^k(\lambda)_{n-(m-1)_k}(2x)^{n-mk}}{k!(n-mk)!}. \tag{1.12}$$

It is to be noted that the polynomials represented by $p_{n,1}^{\lambda}(x)$, $p_{n,2}^{\lambda}(x)$ and $p_{n,3}^{\lambda}(x)$ are known as Horadam polynomials [12], Gegenbauer polynomials and Horadam-Pethe polynomials [13], respectively.

In particular in 1997, Pathan and Khan [15,p.54] generalized these polynomials and gave the following generating relation

$$\begin{aligned}[c-axt+bt^m(2x-1)^d]^{-\nu}&=\sum_{n=0}^{\infty}p_{n,m,a,b,c,d}^{\nu}(x)t^n\\&=\sum_{n=0}^{\infty}\Theta_n(x)t^n,\end{aligned} \tag{1.13}$$

where

$$\Theta_n(x)=\sum_{k=0}^{[\frac{n}{m}]}\frac{(-1)^kc^{-\nu-n+(m-1)k}(\nu)_{n+(1-m)k}(ax)^{n-mk}[b(2x-1)^d]^k}{k!(n-mk)!}. \tag{1.14}$$

In this paper, we consider Chebyshev polynomials of 2-variables by means of the following generating function

$$(1-2xt-yt^2)^{-1}=\sum_{n=0}^{\infty}U_n(x,y)t^n, \tag{1.15}$$

which reduces to ordinary Chebyshev polynomials when $y=-1$ and x is replaced by $2x$ and has the series representation

$$U_n(x,y)=\sum_{s=0}^{[\frac{n}{2}]}\frac{y^s(n-s)!x^{n-2s}}{(n-2s)!s!}. \tag{1.16}$$

2. On Hermite and Chebyshev polynomials of 2-variables

This section gives some properties of Hermite and Chebyshev polynomials of 2 variables. We start with the following theorem.

Theorem 2.1. For $k \in \mathbb{N}$, we have

$$\sum_{s=0}^{[\frac{n}{m}]} \frac{(ky)^s (kx)^{n-ms}}{s!(n-ms)!} = \sum_{n_1+n_2+\cdots+n_k=n} \frac{H_{n_1}^m(x,y) H_{n_2}^m(x,y) \cdots H_{n_k}^m(x,y)}{n_1! n_2! \cdots n_k!}. \tag{2.1}$$

Proof. Consider the generating function (1.2), we have

$$e^{kxt+kyt^m} = \sum_{n=0}^{\infty} \frac{(kxt)^n}{n!} \sum_{s=0}^{\infty} \frac{(kyt^m)^s}{s!}$$

$$= \sum_{n=0}^{\infty} \sum_{s=0}^{[\frac{n}{m}]} \frac{(ky)^s (kx)^{n-ms}}{s!(n-ms)!} t^n. \tag{2.2}$$

On the other hand from (1.2), we get

$$[e^{xt+yt^m}]^k = \left[\sum_{n=0}^{\infty} H_n^m(x,y) \frac{t^n}{n!} \right]^k$$

$$= \sum_{n=0}^{\infty} \sum_{n_1+n_2+\cdots+n_k=n} \frac{H_{n_1}^m(x,y) H_{n_2}^m(x,y) \cdots H_{n_k}^m(x,y)}{n_1! n_2! \cdots n_k!} t^n. \tag{2.3}$$

Comparing the coefficients of t^n in equations (2.2) and (2.3), we arrive at the desired result (2.1). $\square$

Remark 2.1. On setting $x = 2x$, $y = -1$ and $m = 2$ in equation (2.1), the result reduces to known result of Batahan et al. [2,p.50.Eq.(2.1)].

Theorem 2.2. For $k \in \mathbb{N}$, we have

$$\sum_{s=0}^{[\frac{n}{m}]} \frac{(ky)^s (k(x_1 + x_2 \cdots x_k))^{n-ms}}{s!(n-ms)!} = \sum_{n_1+n_2+\cdots+n_k=n} \frac{H_{n_1}^m(x_1,y) H_{n_2}^m(x_2,y) \cdots H_{n_k}^m(x_k,y)}{n_1! n_2! \cdots n_k!}. \tag{2.4}$$

Proof. Consider the generating function (1.2), we have

$$e^{kk(x_1+x_2\cdots x_k)t+kyt^m} = \sum_{n=0}^{\infty} \frac{(k(x_1 + x_2 \cdots x_k)t)^n}{n!} \sum_{s=0}^{\infty} \frac{(kyt^m)^s}{s!}$$

$$= \sum_{n=0}^{\infty} \sum_{s=0}^{[\frac{n}{m}]} \frac{(ky)^s (k(x_1 + x_2 \cdots x_k))^{n-ms}}{s!(n-ms)!} t^n. \tag{2.5}$$

On the other hand from (1.2), we get

$$[e^{(x_1+x_2\cdots x_k)t+yt^m}]^k = \left[\sum_{n=0}^{\infty} H_n^m(x_1 + x_2 \cdots x_k, y) \frac{t^n}{n!} \right]^k$$

$$= \sum_{n=0}^{\infty} \sum_{n_1+n_2+\cdots+n_k=n} \frac{H_{n_1}^m(x_1,y) H_{n_2}^m(x_2,y) \cdots H_{n_k}^m(x_k,y)}{n_1! n_2! \cdots n_k!} t^n. \tag{2.6}$$

Combining (2.5) and (2.6) gives (2.4). $\square$

Remark 2.2. On Letting $x_1 + x_2 \cdots x_k = 2(x_1 + x_2 \cdots x_k)$, $y = -1$ and $m = 2$ in equation (2.4), the result reduces to known result of Batahan et al. [2,p.51.Eq.(2.4)].

Theorem 2.3. For $k \in \mathbb{N}$, we have

$$\sum_{s=0}^{[\frac{n}{m}]} \frac{(-y)^s (k)_{n-s} (2x)^{n-ms}}{s!(n-ms)!} = \sum_{n_1+n_2+\cdots+n_k=n} \frac{U_{n_1}^m(x,y) U_{n_2}^m(x,y) \cdots U_{n_k}^m(x,y)}{n_1! n_2! \cdots n_k!}. \tag{2.7}$$

Proof. Using the power series of $(1-2xt+yt^m)^{-k}$ and making the necessary arrangements gives

$$(1-2xt+yt^m)^{-k} = \sum_{n=0}^{\infty} \frac{(k)_n}{n!} (2xt - yt^m)^n$$

$$= \sum_{n=0}^{\infty} \sum_{s=0}^{[\frac{n}{m}]} \frac{(-y)^s (k)_{n-s} (2x)^{n-ms}}{s!(n-ms)!} t^n. \tag{2.8}$$

In addition to this, we can write

$$(1-2xt+yt^m)^{-k} = ((1-2xt+yt^m)^{-1})^k = \left[\sum_{n=0}^{\infty} U_n^m(x,y) t^n \right]^k$$

$$= \sum_{n=0}^{\infty} \sum_{n_1+n_2+\cdots+n_k=n} \frac{U_{n_1}^m(x,y) U_{n_2}^m(x,y) \cdots U_{n_k}^m(x,y)}{n_1! n_2! \cdots n_k!} t^n. \tag{2.9}$$

Combining (2.8) and (2.9) gives (2.7). □

Remark 2.3. On setting $y = 1$ and $m = 2$ in equation (2.7), the result reduces to known result of Batahan et al. [2,p.52.Eq.(2.7)].

3. Generalized Hermite-Chebyshev polynomials of 2-variables

Here, we consider the generalized Hermite-Chebyshev polynomials of 2 variables ${}_H U_n^m(x,y)$ by

$${}_H U_n^m(x,y) = \sum_{k=0}^{[\frac{n}{m}]} \frac{y^k (n-mk+k)! (2)^{n-mk}}{k!(n-mk)!} H_{n-mk}^m(x,y) \tag{3.1}$$

From (3.1), we can write that

$$\sum_{n=0}^{\infty} {}_H U_n^m(x,y) t^n = \sum_{n=0}^{\infty} \sum_{k=0}^{[\frac{n}{m}]} \frac{y^k (n-mk+k)! (2)^{n-mk}}{k!(n-mk)!} H_{n-mk}^m(x,y) t^n.$$

Replacing n by $n+mk$, we get

$$\sum_{n=0}^{\infty} {}_H U_n^m(x,y) t^n = \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{y^k (n-k)! 2^n}{k! n!} H_n^m(x,y) t^{n+mk}.$$

Summing k-series, we get the following generating function for the generalized Hermite-Chebyshev polynomials of 2 variables

$$\sum_{n=0}^{\infty} {}_H U_n^m(x,y) t^n = (1-yt^m)^{-1} \sum_{n=0}^{\infty} \left(\frac{2t}{1-yt^m} \right)^n H_n^m(x,y). \tag{3.2}$$

In the above equation, we obtain explicit representation for the generating function of Hermite-Chebyshev polynomials of 2 variables in the form

$$\sum_{n=0}^{\infty} {}_HU_n(x)t^n = (1-4xt+yt^m)^{-1}2F_0\left[\frac{1}{2},1;-;\frac{16yt^m}{(1-4xt-yt^m)^2}\right]. \tag{3.3}$$

When $m = 2$, $y = -1$ and x is replaced by $2x$ then (3.2) gives a known result of Batahan and Shehata [2,p.53(3.2)].

$$\sum_{n=0}^{\infty} {}_HU_n(x)t^n = (1+yt^2)^{-1}\sum_{n=0}^{\infty}\left(\frac{2t}{1+yt^2}\right)^n H_n(x). \tag{3.4}$$

Remark 3.1. For $y = -1$, $m = 2$ in equation (3.4), the result reduces to known result of Batahan and Shehata [2,p.53(3.3)].

The following theorem presents a representation for the Hermite-Chebyshev polynomials of 2 variables and reduces to the operational rule.

Theorem 3.1. The Hermite-Chebyshev polynomials of 2 variables satisfy the following representation

$$ {}_HU_n^m(x,y) = \exp\left(\frac{y}{4}\frac{d^m}{dx^m}\right)U_n(2x). \tag{3.5}$$

Proof. From (3.1), we get

$$\begin{aligned} {}_HU_n^m(x,y) &= \sum_{k=0}^{[\frac{n}{m}]} \frac{y^k(n-mk+k)!(2)^{n-mk}}{k!(n-mk)!} H_{n-mk}^m(x,y) \\ &= \exp\left(\frac{y}{4}\frac{d^m}{dx^m}\right)U_n(2x). \end{aligned}$$

□

Remark 3.2. For $y = -1$ and $m = 2$, in equation (3.5), the result reduces to known result of Batahan et al[2,p.54,Eq.(3.5)].

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WASEEM A. KHAN
DEPARTMENT OF MATHEMATICS, FACULTY OF SCIENCE, INTEGRAL UNIVERSITY, LUCKNOW-226026,, INDIA
Email address: `waseem08_khan@rediffmail.com`