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p-SEMISIMPLE NEUTROSOPHIC QUADRUPLE BCI-ALGEBRAS AND NEUTROSOPHIC QUADRUPLE p-IDEALS

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ABSTRACT. Characterizations of neutrosophic quadruple BCI-algebra are considered. Conditions for the neutrosophic quadruple BCI-set to be a p-semisimple BCI-algebra are provided. A condition for a subalgebra to be an ideal in neutrosophic quadruple BCI-algebra is given. Conditions for the set $NQ(A, B)$ to be a neutrosophic quadruple closed ideal and neutrosophic quadruple p-ideal are discussed. Characterizations of neutrosophic quadruple p-ideal are considered.

1. INTRODUCTION

As a more general platform that extends the notions of classic set, (intuitionistic) fuzzy set and interval valued (intuitionistic) fuzzy set, the notion of neutrosophic set is developed by Smarandache ([16], [17] and [18]). Neutrosophic algebraic structures in BCK/BCI-algebras are discussed in the papers [3], [7], [8], [9], [10], [12], [14], [15] and [20]. Smarandache [19] considered an entry (i.e., a number, an idea, an object etc.) which is represented by a known part (a) and an unknown part (bT, cI, dF) where T, I, F have their usual neutrosophic logic meanings and a, b, c, d are real or complex numbers, and then he introduced the concept of neutrosophic quadruple numbers. Neutrosophic quadruple algebraic structures and hyperstructures are discussed in [1] and [2]. Jun et al. [11] used neutrosophic quadruple numbers based on a set, and constructed neutrosophic quadruple BCK/BCI-algebras. They investigated several properties, and considered ideal and positive implicative ideal in neutrosophic quadruple BCK-algebra, and closed ideal in neutrosophic quadruple BCI-algebra. Given subsets A and B of a neutrosophic quadruple BCK/BCI-algebra, they considered sets $NQ(A, B)$ which consists of neutrosophic quadruple BCK/BCI-numbers with a condition. They provided conditions for the set $NQ(A, B)$ to be a (positive implicative) ideal of a neutrosophic quadruple BCK-algebra, and the set $NQ(A, B)$ to be a (closed) ideal of a neutrosophic quadruple BCI-algebra. They gave an example to show that the set $\{\tilde{0}\}$ is not a positive implicative ideal in a neutrosophic quadruple BCK-algebra, and then they considered conditions for the set $\{\tilde{0}\}$ to be a positive implicative ideal in a neutrosophic quadruple BCK-algebra.

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In this paper, we consider characterizations of neutrosophic quadruple BCI-algebra, and give conditions for the neutrosophic quadruple BCI-set to be a p-semisimple BCI-algebra. We provide a condition for a subalgebra to be an ideal in neutrosophic quadruple BCI-algebra, and provide conditions for the set $NQ(A,B)$ to be a neutrosophic quadruple closed ideal and neutrosophic quadruple p-ideal. We disuss characterizations of neutrosophic quadruple p-ideal.

2. Preliminaries

A BCK/BCI-algebra is an important class of logical algebras introduced by K. Iséki (see [5] and [6]) and was extensively investigated by several researchers.

By a *BCI-algebra*, we mean a set S with a special element 0 and a binary operation $*$ that satisfies the following conditions:

(I) $(\forall x,y,z \in S)\ (((x*y)*(x*z))*(z*y)=0)$,
(II) $(\forall x,y \in S)\ ((x*(x*y))*y=0)$,
(III) $(\forall x \in S)\ (x*x=0)$,
(IV) $(\forall x,y \in S)\ (x*y=0,\ y*x=0\ \Rightarrow\ x=y)$.

If a BCI-algebra S satisfies the following identity:

(V) $(\forall x \in S)\ (0*x=0)$,

then S is called a *BCK-algebra.* Any BCK/BCI-algebra S satisfies the following conditions:

$$(\forall x \in S)\,(x*0=x)\,, \tag{2.1}$$
$$(\forall x,y,z \in S)\,(x \le y\ \Rightarrow\ x*z \le y*z,\ z*y \le z*x)\,, \tag{2.2}$$
$$(\forall x,y,z \in S)\,((x*y)*z=(x*z)*y)\,, \tag{2.3}$$
$$(\forall x,y,z \in S)\,((x*z)*(y*z) \le x*y) \tag{2.4}$$

where $x \le y$ if and only if $x*y=0$.

Any BCI-algebra S satisfies the following conditions (see [4]):

$$(\forall x,y \in S)(x*(x*(x*y))=x*y), \tag{2.5}$$
$$(\forall x,y \in S)(0*(x*y)=(0*x)*(0*y)), \tag{2.6}$$
$$(\forall x,y \in S)(0*(0*(x*y))=(0*y)*(0*x)). \tag{2.7}$$

A BCI-algebra S is said to be *p-semisimple* (see [4]) if $0*(0*x)=x$ for all $x \in S$.

Every p-semisimple BCI-algebra S satisfies (see [4]):

$$(\forall x,y,z \in S)((x*z)*(y*z)=x*y). \tag{2.8}$$

A BCI-algebra S is p-semisimple if and only if the following assertion is valid.

$$(\forall x,y \in S)(x*(x*y)=y). \tag{2.9}$$

An element a in a BCI-algebra S is said to be *minimal* (see [4]) if the following assertion is valid.

$$(\forall x \in S)(x*a=0\ \Rightarrow\ x=a). \tag{2.10}$$

A nonempty subset S of a BCK/BCI-algebra S is called a *subalgebra* of S if $x*y \in S$ for all $x,y \in S$. A subset I of a BCK/BCI-algebra S is called an *ideal* of S if it satisfies:

$$0 \in I, \tag{2.11}$$
$$(\forall x \in S)\,(\forall y \in I)\,(x*y \in I\ \Rightarrow\ x \in I)\,. \tag{2.12}$$

A subset I of a BCI-algebra S is called a *closed ideal* (see [4]) of S if it is an ideal of S which satisfies:

$$(\forall x \in S)(x \in I \;\Rightarrow\; 0 * x \in I). \tag{2.13}$$

A subset I of a BCI-algebra S is called a *p-ideal* (see [21]) of S if it satisfies (2.11) and

$$(\forall x, y, z \in S)(y \in I, (x * z) * (y * z) \in I \;\Rightarrow\; x \in I). \tag{2.14}$$

We refer the reader to the books [4, 13] for further information regarding BCK/BCI-algebras, and to the site "http://fs.gallup.unm.edu/neutrosophy.htm" for further information regarding neutrosophic set theory.

We consider neutrosophic quadruple numbers based on a set instead of real or complex numbers.

Definition 2.1 ([11])**.** Let S be a set. A *neutrosophic quadruple S-number* is an ordered quadruple (a, xT, yI, zF) where $a, x, y, z \in S$ and T, I, F have their usual neutrosophic logic meanings.

The set of all neutrosophic quadruple S-numbers is denoted by $NQ(S)$, that is,

$$NQ(S) := \{(a, xT, yI, zF) \mid a, x, y, z \in S\},$$

and it is called the *neutrosophic quadruple set* based on S. If S is a BCK/BCI-algebra, a neutrosophic quadruple S-number is called a *neutrosophic quadruple BCK/BCI-number* and we say that $NQ(S)$ is the *neutrosophic quadruple BCK/BCI-set*.

Let S be a BCK/BCI-algebra. We define a binary operation $\odot$ on $NQ(S)$ by

$$(a, xT, yI, zF) \odot (b, uT, vI, wF) = (a * b, (x * u)T, (y * v)I, (z * w)F)$$

for all (a, xT, yI, zF), $(b, uT, vI, wF) \in NQ(S)$. Given $a_1, a_2, a_3, a_4 \in S$, the neutrosophic quadruple BCK/BCI-number (a_1, a_2T, a_3I, a_4F) is denoted by $\tilde{a}$, that is,

$$\tilde{a} = (a_1, a_2T, a_3I, a_4F),$$

and the zero neutrosophic quadruple BCK/BCI-number $(0, 0T, 0I, 0F)$ is denoted by $\tilde{0}$, that is,

$$\tilde{0} = (0, 0T, 0I, 0F).$$

We define an order relation "$\ll$" and the equality "$=$" on $NQ(S)$ as follows:

$$\tilde{x} \ll \tilde{y} \Leftrightarrow x_i \leq y_i \text{ for } i = 1, 2, 3, 4,$$
$$\tilde{x} = \tilde{y} \Leftrightarrow x_i = y_i \text{ for } i = 1, 2, 3, 4$$

for all $\tilde{x}, \tilde{y} \in NQ(S)$. It is easy to verify that "$\ll$" is an equivalence relation on $NQ(S)$.

Theorem 2.1 ([11])**.** *If S is a BCK/BCI-algebra, then $(NQ(S); \odot, \tilde{0})$ is a BCK/BCI-algebra.*

We say that $(NQ(S); \odot, \tilde{0})$ is a *neutrosophic quadruple BCK/BCI-algebra,* and it is simply denoted by $NQ(S)$.

Let S be a BCK/BCI-algebra. Given $a, b \in S$ and nonempty subsets A and B of S, consider the sets

$$NQ(a, B) := \{(a, aT, yI, zF) \in NQ(S) \mid y, z \in B\},$$

$$NQ(A, b) := \{(a, xT, bI, bF) \in NQ(S) \mid a, x \in A\},$$

$$NQ(A, B) := \{(a, xT, yI, zF) \in NQ(S) \mid a, x \in A; y, z \in B\},$$

$$NQ(A^*, B) := \bigcup_{a\in A} NQ(a, B),$$

$$NQ(A, B^*) := \bigcup_{b\in B} NQ(A, b),$$

and

$$NQ(A \cup B) := NQ(A, 0) \cup NQ(0, B).$$

The set $NQ(A, A)$ is denoted by $NQ(A)$.

3. p-SEMISIMPLE NEUTROSOPHIC QUADRUPLE BCI-ALGEBRAS AND IDEALS

Definition 3.1. Given nonempty subsets A and B of S, if $NQ(A, B)$ is a (closed) ideal (resp., p-ideal) of a neutrosophic quadruple BCI-algebra $NQ(S)$, we say $NQ(A, B)$ is a *neutrosophic quadruple (closed) ideal* (resp., *neutrosophic quadruple* p*-ideal*) of $NQ(S)$.

Theorem 3.1. *Let $NQ(S)$ be the neutrosophic quadruple set based on a set S. Then $(NQ(S); \odot, \tilde{0})$ is a neutrosophic quadruple BCI-algebra if and only if the following assertions are valid.*

$$(\forall \tilde{x}, \tilde{y}, \tilde{z} \in NQ(S))\ (((\tilde{x} \odot \tilde{y}) \odot (\tilde{x} \odot \tilde{z})) \odot (\tilde{z} \odot \tilde{y}) = \tilde{0}), \tag{3.1}$$

$$(\forall \tilde{x}, \tilde{y} \in NQ(S))\ (\tilde{x} \odot \tilde{y} = \tilde{0},\ \tilde{y} \odot \tilde{x} = \tilde{0}\ \Rightarrow \tilde{x} = \tilde{y}), \tag{3.2}$$

$$(\forall \tilde{x} \in NQ(S))\ (\tilde{x} \odot \tilde{0} = \tilde{x}). \tag{3.3}$$

Proof. Assume that $(NQ(S); \odot, \tilde{0})$ is a neutrosophic quadruple BCI-algebra. Then two conditions (3.1) and (3.2) are clearly true. Note that

$$\tilde{x} \odot \tilde{x} = \tilde{0}, \tag{3.4}$$

$$(\tilde{x} \odot (\tilde{x} \odot \tilde{y})) \odot \tilde{y} = \tilde{0} \tag{3.5}$$

for all $\tilde{x}, \tilde{y} \in NQ(S)$. Hence

$$(\tilde{x} \odot (\tilde{x} \odot \tilde{0})) \odot \tilde{0} = \tilde{0} \tag{3.6}$$

for all $\tilde{x} \in NQ(S)$, and it follows from (3.1), (3.4) and (3.6) that

$$\begin{aligned}\tilde{0} &= ((\tilde{x} \odot (\tilde{x} \odot \tilde{0})) \odot (\tilde{x} \odot \tilde{x})) \odot (\tilde{x} \odot (\tilde{x} \odot \tilde{0}))\\ &= ((\tilde{x} \odot (\tilde{x} \odot \tilde{0})) \odot \tilde{0}) \odot (\tilde{x} \odot (\tilde{x} \odot \tilde{0}))\\ &= \tilde{0} \odot (\tilde{x} \odot (\tilde{x} \odot \tilde{0})).\end{aligned}$$

Using (3.2), we have $\tilde{x} \odot (\tilde{x} \odot \tilde{0}) = \tilde{0}$ for all $\tilde{x} \in NQ(S)$. Also we have $(\tilde{x} \odot \tilde{0}) \odot \tilde{x} = (\tilde{x} \odot (\tilde{x} \odot \tilde{x})) \odot \tilde{x} = \tilde{0}$ by (3.4) and (3.5). Therefore (3.3) is valid by using (3.2).

Conversely, suppose that the neutrosophic quadruple set $NQ(S)$ based on a set S satisfies three conditions (3.1), (3.2) and (3.3). It is sufficient to show that two conditions (3.4) and (3.5) are true. Let $\tilde{x}, \tilde{y} \in NQ(S)$. Using (3.3) and (3.1), we have

$$\tilde{x} \odot \tilde{x} = (\tilde{x} \odot \tilde{x}) \odot \tilde{0} = ((\tilde{x} \odot \tilde{0}) \odot (\tilde{x} \odot \tilde{0})) \odot (\tilde{0} \odot \tilde{0}) = \tilde{0}$$

and

$$(\tilde{x} \odot (\tilde{x} \odot \tilde{y})) \odot \tilde{y} = ((\tilde{x} \odot \tilde{0}) \odot (\tilde{x} \odot \tilde{y})) \odot (\tilde{y} \odot \tilde{0}) = \tilde{0}.$$

Therefore $(NQ(S); \odot, \tilde{0})$ is a neutrosophic quadruple BCI-algebra. □

We consider conditions for the neutrosophic quadruple BCI-set $NQ(S)$ to be a p-semisimple neutrosophic quadruple BCI-algebra.

Theorem 3.2. *If S is a p-semisimple BCI-algebra, then $(NQ(S); \odot, \tilde{0})$ is a p-semisimple neutrosophic quadruple BCI-algebra.*

Proof. Let S be a p-semisimple BCI-algebra. Then $(NQ(S); \odot, \tilde{0})$ is a neutrosophic quadruple BCI-algebra (see Theorem 2.1). For any $\tilde{x} = (x_1, x_2T, x_3I, x_4F) \in NQ(S)$, we have

$$\begin{aligned}\tilde{0} \odot (\tilde{0} \odot \tilde{x}) &= (0 * (0 * x_1), (0 * (0 * x_2))T, (0 * (0 * x_3))I, (0 * (0 * x_4))F) \\ &= (x_1, x_2T, x_3I, x_4F) = \tilde{x}.\end{aligned}$$

Hence $(NQ(S); \odot, \tilde{0})$ is a p-semisimple neutrosophic quadruple BCI-algebra. □

Theorem 3.3. *If the neutrosophic quadruple set $NQ(S)$ based on a BCI-algebra S satisfies the following assertion*

$$(\forall \tilde{x} \in NQ(S))(\tilde{0} \odot \tilde{x} = \tilde{0} \ \Rightarrow \ \tilde{x} = \tilde{0}), \tag{3.7}$$

then $(NQ(S); \odot, \tilde{0})$ is a p-semisimple neutrosophic quadruple BCI-algebra.

Proof. By Theorem 2.1, $(NQ(S); \odot, \tilde{0})$ is a neutrosophic quadruple BCI-algebra. Thus

$$\tilde{0} \odot (\tilde{x} \odot \tilde{y}) = (\tilde{0} \odot \tilde{x}) \odot (\tilde{0} \odot \tilde{y}) \tag{3.8}$$

$$\tilde{0} \odot (\tilde{0} \odot (\tilde{0} \odot \tilde{x})) = \tilde{0} \odot \tilde{x} \tag{3.9}$$

for all $\tilde{x}, \tilde{y} \in NQ(S)$. It follows from (3.4) that

$$\begin{aligned}\tilde{0} \odot (\tilde{x} \odot (\tilde{0} \odot (\tilde{0} \odot \tilde{x}))) &= (\tilde{0} \odot \tilde{x}) \odot (\tilde{0} \odot (\tilde{0} \odot (\tilde{0} \odot \tilde{x}))) \\ &= (\tilde{0} \odot \tilde{x}) \odot (\tilde{0} \odot \tilde{x}) = \tilde{0}.\end{aligned}$$

Hence $\tilde{x} \odot (\tilde{0} \odot (\tilde{0} \odot \tilde{x})) = \tilde{0}$ for all $\tilde{x} \in NQ(S)$ by (3.7). Since $(\tilde{0} \odot (\tilde{0} \odot \tilde{x})) \odot \tilde{x} = \tilde{0}$ for all $\tilde{x} \in NQ(S)$, it follows from (3.2) that $\tilde{0} \odot (\tilde{0} \odot \tilde{x}) = \tilde{x}$ for all $\tilde{x} \in NQ(S)$. Therefore $(NQ(S); \odot, \tilde{0})$ is a p-semisimple neutrosophic quadruple BCI-algebra. □

Corollary 3.4. *If the neutrosophic quadruple set $NQ(S)$ based on a BCI-algebra S satisfies the following assertion*

$$(\forall \tilde{x}, \tilde{y} \in NQ(S))(\tilde{x} \odot (\tilde{0} \odot \tilde{y}) = \tilde{y} \odot (\tilde{0} \odot \tilde{x})), \tag{3.10}$$

then $(NQ(S); \odot, \tilde{0})$ is a p-semisimple neutrosophic quadruple BCI-algebra.

Proof. By Theorem 2.1, $(NQ(S); \odot, \tilde{0})$ is a neutrosophic quadruple BCI-algebra. Let $\tilde{x} \in NQ(S)$ be such that $\tilde{0} \odot \tilde{x} = \tilde{0}$. Then

$$\tilde{x} = \tilde{x} \odot \tilde{0} = \tilde{x} \odot (\tilde{0} \odot \tilde{0}) = \tilde{0} \odot (\tilde{0} \odot \tilde{x}) = \tilde{0} \odot \tilde{0} = \tilde{0}$$

by (3.3), (3.4) and (3.10). It follows from Theorem 3.3 that $(NQ(S); \odot, \tilde{0})$ is a p-semisimple neutrosophic quadruple BCI-algebra. □

In a neutrosophic quadruple BCI-algebra, any subalgebra may not be an ideal as seen in the following example.

Example 3.2. Consider a BCI-algebra $S = \{0, 1, a\}$ with the binary operation $*$, which is given in Table 1.
Then the neutrosophic quadruple BCI-algebra $NQ(S)$ has 81 elements. If we take

$$B := \{\tilde{0}, \tilde{1}, \tilde{2}, \tilde{3}, \tilde{4}, \tilde{5}, \tilde{6}, \tilde{7}, \tilde{8}, \tilde{9}, \tilde{10}, \tilde{11}, \tilde{12}, \tilde{13}, \tilde{14}, \tilde{15}\}$$

where
$\tilde{0} = (0, 0T, 0I, 0F)$, $\tilde{1} = (0, 0T, 0I, aF)$, $\tilde{2} = (0, 0T, aI, 0F)$,

TABLE 1. Cayley table for the binary operation "$*$"

$*$	0	1	a
0	0	0	a
1	1	0	a
a	a	a	0

$\tilde{3} = (0, 0T, aI, aF)$, $\tilde{4} = (0, aT, 0I, 0F)$, $\tilde{5} = (0, aT, 0I, aF)$,
$\tilde{6} = (0, aT, aI, 0F)$, $\tilde{7} = (0, aT, aI, aF)$, $\tilde{8} = (a, 0T, 0I, 0F)$,
$\tilde{9} = (a, 0T, 0I, aF)$, $\tilde{10} = (a, 0T, aI, 0F)$, $\tilde{11} = (a, 0T, aI, aF)$,
$\tilde{12} = (a, aT, 0I, 0F)$, $\tilde{13} = (a, aT, 0I, aF)$,
$\tilde{14} = (a, aT, aI, 0F)$, $\tilde{15} = (a, aT, aI, aF)$.

Then B is a subalgebra of $NQ(S)$. But it is not an ideal of $NQ(S)$. In fact, if we take $\tilde{x} = (1, 1T, 0I, aF) \in NQ(S)$ then

$$\tilde{x} \odot \tilde{15} = (1, 1T, 0I, aF) \odot (a, aT, aI, aF) = (a, aT, aI, 0F) = \tilde{14} \in B$$

But $\tilde{x} = (1, 1T, 0I, aF) \notin B$.

We provide a condition for a subalgebra to be an ideal in neutrosophic quadruple BCI-algebra.

Theorem 3.5. *If $NQ(S)$ is a neutrosophic quadruple BCI-algebra based on a p-semisimple BCI-algebra S, then every subalgebra of $NQ(S)$ is an ideal of $NQ(S)$.*

Proof. If S is a p-semisimple BCI-algebra, then $(NQ(S); \odot, \tilde{0})$ is a p-semisimple neutrosophic quadruple BCI-algebra by Theorem 3.2. Let $NQ(S)$ be a subalgebra of $NQ(S)$. It is clear that $\tilde{0} \in NQ(S)$. Let $\tilde{x}, \tilde{y} \in NQ(S)$ be such that $\tilde{x} \odot \tilde{y} \in NQ(S)$ and $\tilde{y} \in NQ(S)$. Then $\tilde{0} \odot \tilde{y} \in NQ(S)$ and $(\tilde{x} \odot \tilde{y}) \odot (\tilde{0} \odot \tilde{y}) \in NQ(S)$. Note that

$$\begin{aligned} ((\tilde{x} \odot \tilde{y}) \odot (\tilde{0} \odot \tilde{y})) \odot \tilde{x} &= ((\tilde{x} \odot \tilde{y}) \odot \tilde{x}) \odot (\tilde{0} \odot \tilde{y}) \\ &= ((\tilde{x} \odot \tilde{y}) \odot (\tilde{x} \odot \tilde{0})) \odot (\tilde{0} \odot \tilde{y}) \\ &= \tilde{0}. \end{aligned}$$

Since $(NQ(S); \odot, \tilde{0})$ is p-semisimple, we have $\tilde{x} = (\tilde{x} \odot \tilde{y}) \odot (\tilde{0} \odot \tilde{y}) \in NQ(S)$ by (3.2). Therefore $NQ(S)$ is an ideal of $NQ(S)$. □

Lemma 3.6 ([11])**.** *If A and B are (closed) ideals of a BCI-algebra S, then the set $NQ(A, B)$ is a neutrosophic quadruple (closed) ideal of $NQ(S)$.*

Recall that there exist ideals A and B in a BCI-algebra S such that $NQ(A, B)$ is not a neutrosophic quadruple closed ideal of $NQ(S)$ (see [11, Example 3]).

We provide conditions for the set $NQ(A, B)$ to be a neutrosophic quadruple closed ideal of $NQ(S)$.

Theorem 3.7. *Let A and B be ideals of a BCI-algebra S. Then the set $NQ(A, B)$ is a neutrosophic quadruple closed ideal of $NQ(S)$ if and only if the following assertion is valid.*

$$(\forall a \in A, \forall b \in B)(0 * a \in A,\ 0 * b \in B). \tag{3.11}$$

Proof. Assume that $NQ(A, B)$ is a neutrosophic quadruple closed ideal of $NQ(S)$ for any ideals A and B of a BCI-algebra S. Let $a_1, a_2 \in A$ and $b_1, b_2 \in B$ be such that

$(a_1, a_2T, b_1I, b_2F) \in NQ(A, B)$. Then

$$(0 * a_1, (0 * a_2)T, (0 * b_1)I, (0 * b_2)F)$$
$$= (0, 0T, 0I, 0F) \odot (a_1, a_2T, b_1I, b_2F) \in NQ(A, B),$$

and so $0 * a_1$, $0 * a_2 \in A$ and $0 * b_1$, $0 * b_2 \in B$. Therefore (3.11) is valid.

Conversely, let A and B be ideals of a BCI-algebra S satisfying the condition (3.11). Then A and B are closed ideals of S. It follows from Lemma 3.6 that $NQ(A, B)$ is a neutrosophic quadruple closed ideal of $NQ(S)$. □

Corollary 3.8. *Given an ideal A of a BCI-algebra S, the set $NQ(A)$ is a neutrosophic quadruple closed ideal of $NQ(S)$ if and only if $0 * a \in A$ for all $a \in A$.*

Theorem 3.9. *For any ideals A and B of a BCI-algebra S, let $m(A)$ and $m(B)$ be the set of all minimal elements of A and B with $|m(A)| < \infty$ and $|m(B)| < \infty$, respectively. Then the set $NQ(A, B)$ is a neutrosophic quadruple closed ideal of $NQ(S)$.*

Proof. For any $a \in A$, $b \in B$ and $n, k \in \mathbb{N}$, let $a_n = 0*(0*a)^n$ and $b_k = 0*(0*b)^k$. Then $a_n \in m(A)$ and $b_k \in m(B)$. Using (2.6) repeatedly, we have $a_n = 0*(0*a)^n = 0*(0*a^n)$ and $b_k = 0 * (0 * b)^k = 0 * (0 * b^k)$. Hence

$$a_n * a^n = (0 * (0 * a^n)) * a^n = (0 * a^n) * (0 * a^n) = 0 \in A$$

and

$$b_k * b^k = (0 * (0 * b^k)) * b^k = (0 * b^k) * (0 * b^k) = 0 \in B.$$

Since A and B are ideals, it follows that $a_n \in A$ and $b_k \in B$. Since $|m(A)| < \infty$ and $|m(B)| < \infty$, there exist $p, q \in \mathbb{N}$ such that $a_{n+p} = a_n$ and $b_{k+q} = b_k$, that is, $a_n * (0 * a)^p = a_n$ and $b_k * (0 * b)^q = b_k$. It follows that

$$a_p = 0 * (0 * a)^p = (a_n * (0 * a)^p) * a_n = a_n * a_n = 0$$

and

$$b_q = 0 * (0 * b)^q = (b_k * (0 * b)^q) * b_k = b_k * b_k = 0.$$

Thus $a_{p-1}*(0*a) = 0$ and $b_{q-1}*(0*b) = 0$, and so $0*a = a_{p-1} \in A$ and $0*b = b_{q-1} \in B$. Hence A and B are closed ideals of S. Therefore $NQ(A, B)$ is a neutrosophic quadruple closed ideal of $NQ(S)$. by Lemma 3.6. □

4. Neutrosophic quadruple p-ideals

In what follows, let S be a BCI-algebra unless otherwise.

Question 1. *If A and B are ideals of S, then is $NQ(A, B)$ a neutrosophic quadruple p-ideal of $NQ(S)$?*

The following example shows that the answer to Question 1 is negative.

Example 4.1. Consider a BCI-algebra $S = \{0, 1, a, b\}$ with the binary operation $*$, which is given in Table 2.

Then the neutrosophic quadruple BCI-algebra $NQ(S)$ has 256 elements. Consider ideals $A = \{0, 1\}$ and $B = \{0, a\}$ of S. Note that $B = \{0, a\}$ is not a p-ideal of S. Then

$$NQ(A, B) = \{\tilde{0}, \tilde{1}, \tilde{2}, \tilde{3}, \tilde{4}, \tilde{5}, \tilde{6}, \tilde{7}, \tilde{8}, \tilde{9}, \tilde{10}, \tilde{11}, \tilde{12}, \tilde{13}, \tilde{14}, \tilde{15}\}$$

is a neutrosophic quadruple ideal of $NQ(S)$ where
$\tilde{0} = (0, 0T, 0I, 0F)$, $\tilde{1} = (0, 0T, 0I, aF)$, $\tilde{2} = (0, 0T, aI, 0F)$,
$\tilde{3} = (0, 0T, aI, aF)$, $\tilde{4} = (0, 1T, 0I, 0F)$, $\tilde{5} = (0, 1T, 0I, aF)$,

TABLE 2. Cayley table for the binary operation "$*$"

$*$	0	1	a	b
0	0	0	a	a
1	1	0	b	a
a	a	a	0	0
b	b	a	1	0

$\tilde{6} = (0, 1T, aI, 0F)$, $\tilde{7} = (0, 1T, aI, aF)$, $\tilde{8} = (1, 0T, 0I, 0F)$,
$\tilde{9} = (1, 0T, 0I, aF)$, $\tilde{10} = (1, 0T, aI, 0F)$, $\tilde{11} = (1, 0T, aI, aF)$,
$\tilde{12} = (1, 1T, 0I, 0F)$, $\tilde{13} = (1, 1T, 0I, aF)$,
$\tilde{14} = (1, 1T, aI, 0F)$, $\tilde{15} = (1, 1T, aI, aF)$.

If we take $\tilde{x} = (1, 1T, bI, bF) \in NQ(S)$ and $\tilde{z} = (b, bT, bI, bF) \in NQ(S)$, then

$$\begin{aligned}(\tilde{x} \odot \tilde{z}) \odot (\tilde{7} \odot \tilde{z}) &= (a, aT, 0I, 0F) \odot (a, aT, 0I, 0F) \\ &= (0, 0T, 0I, 0F) = \tilde{0} \in NQ(A, B).\end{aligned}$$

But $\tilde{x} \notin NQ(A, B)$, and so $NQ(A, B)$ is not a neutrosophic quadruple p-ideal of $NQ(S)$.

We provide a condition for the set $NQ(A, B)$ to be a neutrosophic quadruple p-ideal.

Theorem 4.1. *Let A and B be ideals of S. If S is p-semisimple, then $NQ(A, B)$ is a neutrosophic quadruple p-ideal of $NQ(S)$.*

Proof. If A and B are ideals of S, then $NQ(A, B)$ is an ideal of $NQ(S)$ (see Lemma 3.6), and so $\tilde{0} \in NQ(A, B)$. Let $\tilde{x}, \tilde{y}, \tilde{z} \in NQ(S)$ be such that $(\tilde{x} \odot \tilde{z}) \odot (\tilde{y} \odot \tilde{z}) \in NQ(A, B)$ and $\tilde{y} \in NQ(A, B)$. Since S is p-semisimple, it follows from (2.8) that

$$\begin{aligned}&(x_1 * y_1, (x_2 * y_2)T, (x_3 * y_3)I, (x_4 * y_4)F) \\ &= ((x_1 * z_1) * (y_1 * z_1), ((x_2 * z_2) * (y_2 * z_2))T, \\ &((x_3 * z_3) * (y_3 * z_3))I, ((x_4 * z_4) * (y_4 * z_4))F) \\ &= (x_1 * z_1, (x_2 * z_2)T, (x_3 * z_3)I, (x_4 * z_4)F) \odot \\ &(y_1 * z_1, (y_2 * z_2)T, (y_3 * z_3)I, (y_4 * z_4)F) \\ &= ((x_1, x_2T, x_3I, x_4F) \odot (z_1, z_2T, z_3I, z_4F)) \odot \\ &((y_1, y_2T, y_3I, y_4F) \odot (z_1, z_2T, z_3I, z_4F)) \\ &= (\tilde{x} \odot \tilde{z}) \odot (\tilde{y} \odot \tilde{z}) \in NQ(A, B).\end{aligned}$$

Hence $x_i * y_i \in A$ and $x_j * y_j \in B$ for $i = 1, 2$ and $j = 3, 4$. Since $y_1, y_2 \in A$ and $y_3, y_4 \in B$, we have $x_i \in A$ and $x_j \in B$ for $i = 1, 2$ and $j = 3, 4$. Thus $\tilde{x} = (x_1, x_2T, x_3I, x_4F) \in NQ(A, B)$. Therefore $NQ(A, B)$ is a neutrosophic quadruple p-ideal of $NQ(S)$. □

Corollary 4.2. *If A is an ideal of a p-semisimple BCI-algebra S, then $NQ(A)$ is a neutrosophic quadruple p-ideal of $NQ(S)$.*

Corollary 4.3. *If a BCI-algebra S satisfies:*

$$(\forall x, y, z \in S)((x * y) * z = x * (y * z)), \tag{4.1}$$

then $NQ(A, B)$ is a neutrosophic quadruple p-ideal of $NQ(S)$ for all ideals A and B of S.

Proof. Using (2.3) and (4.1), we have

$$\begin{aligned} y*(x*(x*y)) &= (y*x)*(x*y) = (y*(x*y))*x \\ &= ((y*x)*y)*x = (y*x)*(y*x) = 0 \end{aligned}$$

for all $x, y \in S$. It follows from (II) and (IV) that $x*(x*y) = y$. Hence S is p-semisimple, and therefore $NQ(A,B)$ is a neutrosophic quadruple p-ideal of $NQ(S)$ by Theorem 4.1. □

Theorem 4.4. *If A and B are p-ideals of S, then the set $NQ(A,B)$ is a neutrosophic quadruple p-ideal of $NQ(S)$.*

Proof. Assume that A and B are p-ideals of S. Obviously, $\tilde{0} \in NQ(A,B)$. Let $\tilde{x} = (x_1, x_2T, x_3I, x_4F)$, $\tilde{y} = (y_1, y_2T, y_3I, y_4F)$ and $\tilde{z} = (z_1, z_2T, z_3I, z_4F)$ be elements of $NQ(S)$ such that $(\tilde{x} \odot \tilde{z}) \odot (\tilde{y} \odot \tilde{z}) \in NQ(A,B)$ and $\tilde{y} \in NQ(A,B)$. Then

$$\begin{aligned} (\tilde{x} \odot \tilde{z}) \odot (\tilde{y} \odot \tilde{z}) = ((x_1*z_1)*(y_1*z_1), ((x_2*z_2)*(y_2*z_2))T, \\ ((x_3*z_3)*(y_3*z_3))I, ((x_4*z_4)*(y_4*z_4))F) \in NQ(A,B), \end{aligned}$$

which implies that $(x_1*z_1)*(y_1*z_1) \in A$, $(x_2*z_2)*(y_2*z_2) \in A$, $(x_3*z_3)*(y_3*z_3) \in B$ and $(x_4*z_4)*(y_4*z_4) \in B$. Since $\tilde{y} \in NQ(A,B)$, we have $y_1, y_2 \in A$ and $y_3, y_4 \in B$. It follows from (2.14) that $x_1, x_2 \in A$ and $x_3, x_4 \in B$. Hence $\tilde{x} = (x_1, x_2T, x_3I, x_4F) \in NQ(A,B)$, and therefore $NQ(A,B)$ is a neutrosophic quadruple p-ideal of $NQ(S)$. □

Corollary 4.5. *If A is a p-ideal of S, then $NQ(A)$ is a neutrosophic quadruple p-ideal of $NQ(S)$.*

Proposition 4.6. *For any p-ideals A and B of S, the set $NQ(A,B)$ satisfies the following implication.*

$$(\forall \tilde{x} \in NQ(S))(\tilde{0} \odot (\tilde{0} \odot \tilde{x}) \in NQ(A,B) \;\Rightarrow\; \tilde{x} \in NQ(A,B)). \tag{4.2}$$

Proof. If $\tilde{0} \odot (\tilde{0} \odot \tilde{x}) \in NQ(A,B)$, then

$$\begin{aligned} \tilde{0} \odot (\tilde{0} \odot \tilde{x}) &= (0, 0T, 0I, 0F) \odot ((0, 0T, 0I, 0F) \odot (x_1, x_2T, x_3I, x_4F)) \\ &= (0, 0T, 0I, 0F) \odot ((0*x_1), (0*x_2)T, (0*x_3)I, (0*x_4)F) \\ &= (0*(0*x_1), (0*(0*x_2))T, (0*(0*x_3))I, (0*(0*x_4))F) \\ &\in NQ(A,B). \end{aligned}$$

Hence $(x_1*x_1)*(0*x_1) = 0*(0*x_1) \in A$, $(x_2*x_2)*(0*x_2) = 0*(0*x_2) \in A$, $(x_3*x_3)*(0*x_3) = 0*(0*x_3) \in B$ and $(x_4*x_4)*(0*x_4) = 0*(0*x_4) \in B$. Since A and B are p-ideals of S, it follows from (2.14) that $x_1, x_2 \in A$ and $x_3, x_4 \in B$. Hence $\tilde{x} = (x_1, x_2T, x_3I, x_4F) \in NQ(A,B)$. □

Corollary 4.7. *For any p-ideal A of S, the set $NQ(A)$ satisfies the following implication.*

$$(\forall \tilde{x} \in NQ(S))(\tilde{0} \odot (\tilde{0} \odot \tilde{x}) \in NQ(A) \;\Rightarrow\; \tilde{x} \in NQ(A)). \tag{4.3}$$

Theorem 4.8. *Let A and B be ideals of S. Then $NQ(A,B)$ is a neutrosophic quadruple p-ideal of $NQ(S)$ if and only if the following assertion is valid.*

$$(\tilde{x} \odot \tilde{z}) \odot (\tilde{y} \odot \tilde{z}) \in NQ(A,B) \;\Rightarrow\; \tilde{x} \odot \tilde{y} \in NQ(A,B) \tag{4.4}$$

for all $\tilde{x}, \tilde{y}, \tilde{z} \in NQ(S)$.

Proof. Assume that $NQ(A,B)$ is a neutrosophic quadruple p-ideal of $NQ(S)$. Let $\tilde{x},\tilde{y},\tilde{z}\in NQ(S)$ be such that $(\tilde{x}\odot\tilde{z})\odot(\tilde{y}\odot\tilde{z})\in NQ(A,B)$. Then

$$\begin{aligned}&((\tilde{x}\odot\tilde{y})\odot(\tilde{x}\odot\tilde{y}))\odot(((\tilde{x}\odot\tilde{z})\odot(\tilde{y}\odot\tilde{z}))\odot(\tilde{x}\odot\tilde{y}))\\&=\tilde{0}\odot(((\tilde{x}\odot\tilde{z})\odot(\tilde{x}\odot\tilde{y}))\odot(\tilde{y}\odot\tilde{z}))\\&=\tilde{0}\odot\tilde{0}=\tilde{0}\in NQ(A,B),\end{aligned}$$

and so $\tilde{x}\odot\tilde{y}\in NQ(A,B)$ since $NQ(A,B)$ is a neutrosophic quadruple p-ideal of $NQ(S)$.

Conversely, let A and B be ideals of S such that the set $NQ(A,B)$ satisfies the condition (4.4). Then $NQ(A,B)$ is a neutrosophic quadruple ideal of $NQ(S)$ by Lemma 3.6, and so $\tilde{0}\in NQ(A,B)$. Let $\tilde{x},\tilde{y},\tilde{z}\in NQ(S)$ be such that $(\tilde{x}\odot\tilde{z})\odot(\tilde{y}\odot\tilde{z})\in NQ(A,B)$ and $\tilde{y}\in NQ(A,B)$. Then $\tilde{x}\odot\tilde{y}\in NQ(A,B)$ by (4.4), and thus $\tilde{x}\in NQ(A,B)$. Therefore $NQ(A,B)$ is a neutrosophic quadruple p-ideal of $NQ(S)$. □

Corollary 4.9. *Given an ideal A of S, the set $NQ(A)$ is a neutrosophic quadruple p-ideal of $NQ(S)$ if and only if the following assertion is valid.*

$$(\tilde{x}\odot\tilde{z})\odot(\tilde{y}\odot\tilde{z})\in NQ(A)\ \Rightarrow\ \tilde{x}\odot\tilde{y}\in NQ(A) \tag{4.5}$$

for all $\tilde{x},\tilde{y},\tilde{z}\in NQ(S)$.

Theorem 4.10. *Let A and B be ideals of S such that*

$$(\forall x\in S)(0*(0*x)\in A \text{ (resp., } B) \Rightarrow x\in A \text{ (resp., } B)). \tag{4.6}$$

Then $NQ(A,B)$ is a neutrosophic quadruple p-ideal of $NQ(S)$.

Proof. Let $x,y,z\in S$ be such that $(x*z)*(y*z)\in A$ (resp., B) and $y\in A$ (resp., B). Then

$$\begin{aligned}&(0*(0*((x*z)*(y*z))))*((x*z)*(y*z))\\&=(0*((x*z)*(y*z)))*(0*((x*z)*(y*z)))\\&=0\in A \text{ (resp., } B).\end{aligned}$$

and so $0*(0*((x*z)*(y*z)))\in A$ (resp., B) since A and B are ideals of S. Now we have

$$\begin{aligned}&0*(0*(x*y))=(0*y)*(0*x)=(((0*z)*(0*z))*y)*(0*x)\\&=(((0*(0*z))*z)*y)*(0*x)=(((0*y)*(0*z))*z)*(0*x)\\&=((0*(y*z))*z)*(0*x)=((0*z)*(0*x))*(y*z)\\&=((0*(0*(0*z)))*(0*x))*(y*z)\\&=((0*(0*x))*(0*(0*z)))*(y*z)\\&=(0*((0*x)*(0*z)))*(y*z)=(0*(0*(x*z)))*(y*z)\\&=(0*(y*z))*(0*(x*z))=(0*(0*(0*(y*z))))*(0*(x*z))\\&=(0*(0*(x*z)))*(0*(0*(y*z)))=0*((0*(x*z))*(0*(y*z)))\\&=0*(0*((x*z)*(y*z)))\in A \text{ (resp., } B).\end{aligned}$$

It follows from (4.6) that $x*y\in A$ (resp., B). Hence $x\in A$ (resp., B). This shows that A and B are p-ideals of S. Therefore $NQ(A,B)$ is a neutrosophic quadruple p-ideal of $NQ(S)$ by Theorem 4.4. □

Corollary 4.11. *Let A be an ideal of S such that*

$$(\forall x\in S)(0*(0*x)\in A\Rightarrow x\in A). \tag{4.7}$$

Then $NQ(A)$ is a neutrosophic quadruple p-ideal of $NQ(S)$.

5. Conclusion

In this paper, we consider characterizations of neutrosophic quadruple BCI-algebra, and give conditions for the neutrosophic quadruple BCI-set to be a p-semisimple BCI-algebra. Futhermore, we provide a condition for a subalgebra to be an ideal in neutrosophic quadruple BCI-algebra, and provide conditions for the set $NQ(A, B)$ to be a neutrosophic quadruple closed ideal and neutrosophic quadruple p-ideal. We hope that this work will provide a deep impact on the upcoming research in this field and other related areas to open up new horizons of interest and innovations. Indeed, this work may serve as a foundation for further study of neutrosophic subalgebras in BCK/BCI-algebras. To extend these results, one can further study the neutrosophic set theory of different algebras such as MTL-algerbas, BL-algebras, MV-algebras, EQ-algebras, R0-algebras and Q-algebras etc. One may also apply this concept to study some applications in many fields like decision making, knowledge base systems, medical diagnosis, data analysis and graph theory etc.

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